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TRIPLE POSITIVE SOLUTIONS FOR A CLASS OF BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract

In this paper, by using Leggett-Williams fixed-point theorem and *Hölder* inequality, we study the existence of three positive solutions for fourth-order differential equations with integral boundary conditions. The results are illustrated with an example.

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1 Introduction

In this paper, by using Leggett-Williams fixed-point theorem and *Hölder* inequality, we study the existence of three positive solutions for fourth-order two-point boundary value problem (BVP)

$$\begin{cases} u^{(4)}(t) = \omega(t) f(t, u(t)), \qquad t \in (0,1), \\ u(0) = u(1) = \int_0^1 g(s) u(s) ds, \\ u''(0) = \int_0^1 h(s) u''(s) ds, \quad u^{(3)}(1) = 0, \end{cases} \tag{1}$$

where $\omega(.)$ *is* $L^p - integrable$ for some $0 \leq p \leq +\infty$. In addition f and ω satisfy

(A_1) $\omega(t) \in L^P[0,1]$ for some $0 \leq p \leq +\infty$ and there exists $\lambda > 0$ such that $\omega(t) \geq \lambda$ *a.e. on* $[0,1]$,

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(A_2) $f \in C([0,1] \times [0,\infty), [0,\infty))$,
(A_3) g, $h \in L^1[0,1]$ are nonnegative and $\mu \in [0,1)$, $\nu \in [0,1)$, where

$$\nu = \int_0^1 g(t)\,dt, \quad \mu = \int_0^1 h(t)\,dt$$

The theory of multi-point boundary value problems for ordinary differential equations arises in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities that can be set up as a multi-point boundary value problem. Many problems in the theory of elastic stability can be handled as multi-point boundary value problems too. Higher-order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydromagnetic stability and astronomy, be a mandlong wave theory, induction motors, engineering and applied physics. The boundary value problems of higher-order have been examined due to their mathematical importance and applications in different areas of applied sciences. In particular, third-order, fourth-order and nth order were considred, see [3, 4, 5, 6, 7, 10, 11, 20, 23, 27, 34] and the references therein.

The existence of positive solutions for nonlinear fourth-order multi-point boundary value problems has been studied by many authors using nonlinear alternatives of Leray-Schauder, the fixed point theory, monotone iterative technique and the method of upper and lower solutions, see [1, 9, 12, 13, 14, 15, 21, 22, 25, 28, 35] and references therein. The multi-point boundary value problem is in fact a special case of the boundary value problem with integral boundary conditions.

Boundary value problem with integral boundary conditions is a typical nonlocal problem, which arises naturally in hydrodynamic problems [8], semiconductor problems [18], thermal conduction problems [29]. Such problems have been considered by many authors, see [16, 17, 24, 33].

Recently, Zhang and Ge [31] studied the existence of positive solutions by using the Krasnoselskii fixed point theorem of the following fourth-order boundary value problem with integral boundary conditions

$$\begin{cases} u^{(4)}(t) = \omega(t) f(t, u(t)), \ t \in (0,1), \\ u(0) = \int_0^1 g(s) u(s)\,ds, \quad u(1) = 0, \\ u''(0) = \int_0^1 h(s) u(s)\,ds, \ u''(1) = 0, \end{cases}$$

where $\omega(.) \in L^1[0,1]$, g, $h \in L^1[0,1]$, $g(s) \geq 0$, $h(s) \geq 0$.

Bai [2] studied the existence of positive solutions by using the Krasnoselskii fixed point theorem of nonlocal fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) + \beta u''(t) = \lambda f(t, u(t)), \ t \in (0,1), \\ u(0) = u(1) = \int_0^1 g(s) u(s)\,ds, \\ u''(0) = u''(1) = \int_0^1 h(s) u''(s)\,ds, \end{cases}$$

where $\lambda > 0$, $0 < \beta < \pi^2$, g, $h \in L^1[0,1]$, $g(s) \geq 0$, $h(s) \geq 0$ and $f \in C([0,1] \times [0,\infty) \times (-\infty,0], [0,\infty))$.

Zhang et al. [32] studied the existence of positive solutions of the following fourth-order boundary value problem with integral boundary conditions

$$\begin{cases} u^{(4)}(t) - \lambda f(t, u(t)) = 0,\ t \in (0,1), \\ u(0) = u(1) = \int_0^1 g(s) u(s)\, ds, \\ u''(0) = u''(1) = \int_0^1 h(s) u(s)\, ds, \end{cases}$$

where λ, $0 < \beta < \pi^2$, g, $h \in L^1[0,1]$, $g(s) \geq 0$, $h(s) \geq 0$ and $f \in C([0,1] \times [0,\infty), [0,\infty))$.

In [30], Yan studied the existence of positive solutions by applying the Krein-Rutman theorem and fixed point index theory for the nonlinear problem

$$\begin{cases} u^{(4)}(t) + \beta u''(t) = \mu [u(t) - u''(t)],\ t \in (0,1), \\ u(0) = u(1) = \int_0^1 g(s) u(s)\, ds, \\ u''(0) = u''(1) = \int_0^1 h(s) u''(s)\, ds, \end{cases}$$

In [26], Shen and He used global bifurcation techniques to study the global structure of positive solutions of the singular problem

$$\begin{cases} u^{(4)}(t) - \lambda f(t, u(t)) = 0,\ t \in (0,1), \\ u(0) = \int_0^1 u(s)\, d\alpha(s), \\ u'(0) = u(1) = u'(1) = 0, \end{cases}$$

where $\lambda \in (0,\infty)$, $h(t)$ be singular at $t = 0$ and $t = 1$ and $\int_0^1 u(s)\, d\alpha(s)$ is a Stieltjes integral with $\alpha(t)$ being not a constant on $[0,1]$.

Motivated and inspired by the above-mentioned works, in this paper, we study the existence of three positive solutions for BVP (1). The argument are based upon a fixed point theorem due to Leggett and Williams which deals with fixed points of a cone-preserving operator defined on an ordered Banach space [19]. The current paper is organized as follows. In Section 2, we provide some lemmas that will be used to prove our main results of BVP (1). In Section 3, the main results of BVP (1) will be stated and proved, and we give an example to illustrate our results.

2 Preliminaries

We shall consider the Banach space $C[0,1]$ equipped with sup norm $\|u\| = \max\limits_{0 \leq t \leq 1} |u(t)|$. $C^+[0,1]$ is the cone of nonnegative functions in $C[0,1]$.

Definition 1. *Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that*
(i) $c_1 u + c_2 v \in P$ for all $c_1 \geq 0,\ c_2 \geq 0$, and
(ii) $u,\ -u \in P$ implies $u = 0$.
Every cone P induces an ordering in E given by $u \leq v$ if and only if $v - u \in P$.

Definition 2. *The map β is said to be nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and*

$$\beta\left(tx + (1-t)\,y\right) \geq t\beta\left(x\right) + (1-t)\,\beta\left(y\right),\ for\ all\ x, y \in P\ and\ \ 0 \leq t \leq 1.$$

We will reduce BVP (1) to an integral equation. To do this goal, firstly by means of the transformation

$$u''\left(t\right) = -v\left(t\right),$$

we convert problem (1) into

$$\begin{cases} v''\left(t\right) + \omega\left(t\right) f\left(t, u\left(t\right)\right) = 0, & 0 < t < 1, \\ v\left(0\right) = \int_0^1 h\left(t\right) v\left(t\right) dt, & v'\left(1\right) = 0, \end{cases} \tag{2}$$

and

$$\begin{cases} -u''\left(t\right) = v\left(t\right), & 0 < t < 1, \\ u\left(0\right) = u\left(1\right) = \int_0^1 g\left(t\right) u\left(t\right) dt. \end{cases} \tag{3}$$

In arriving our results, we need the following six preliminary lemmas. The first and third lemmas are well known.

Lemma 1. *Assume $(H_1) - (H_3)$ hold. Then problem (2) has a unique solution given by*

$$v\left(t\right) = \int_0^1 H\left(t, s\right) \omega\left(s\right) f\left(s, u\left(s\right)\right) ds, \tag{4}$$

where

$$H\left(t, s\right) = G\left(t, s\right) + \frac{1}{1-\mu} \int_0^1 G\left(s, \tau\right) h\left(\tau\right) d\tau, \tag{5}$$

and

$$G\left(t, s\right) = \begin{cases} t, & 0 \leq t \leq s \leq 1, \\ s, & 0 \leq s \leq t \leq 1. \end{cases} \tag{6}$$

The functions $G\left(t, s\right)$ and $H\left(t, s\right)$ have the following properties.

Lemma 2. *Let $\delta \in \left(0, \frac{1}{2}\right),\ J_\delta = [\delta, 1-\delta]$. If $\mu \in [0, 1)$, then, we have*

$$H\left(t, s\right) > 0, \quad G\left(t, s\right) > 0,\ \forall t,\ s \in (0, 1) \tag{7}$$

$$H(t,s) \geq 0, \quad G(t,s) \geq 0,\ \forall t,\ s \in J \tag{8}$$

$$e(t)\,e(s) \leq G(t,s) \leq G(t,t) = t = e(t) \leq 1,\ \forall t,\ s \in J \tag{9}$$

$$\rho e(t)\,e(s) \leq H(t,s) \leq \gamma s = \gamma e(s) \leq \gamma,\ \forall t,\ s \in J \tag{10}$$

$$G(t,s) \geq \delta G(s,s), \quad H(t,s) \geq \delta H(s,s),\ \forall t \in J_\delta,\ s \in J \tag{11}$$

where

$$e(t) = t,\ \gamma = \frac{1}{1-\mu},\ \rho = 1 + \frac{\int_0^1 sh(s)\,ds}{1-\mu} \tag{12}$$

Proof. The proof is evident, we omit it. □

Remark 1. *From (5) and (11), we can obtain*

$$H(t,s) \geq \delta s = \delta G(s,s)$$

Lemma 3. *If* (H_2) *and* (H_3) *hold, then problem (3) has a unique solution* u *given by*

$$u(t) = \int_0^1 H_1(t,s)\,v(s)\,ds, \tag{13}$$

where

$$H_1(t,s) = G_1(t,s) + \frac{1}{1-\nu}\int_0^1 G_1(s,\tau)\,g(\tau)\,d\tau, \tag{14}$$

and

$$G_1(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{15}$$

The functions $G_1(t,s)$ and $H_1(t,s)$ have the following properties.

Lemma 4. *Let* $\delta \in \left(0, \frac{1}{2}\right),\ J_\delta = [\delta, 1-\delta]$*. If* $v \in [0,1)$*,then, we have*

$$H_1(t,s) > 0, \quad G_1(t,s) > 0,\ \forall t,\ s \in (0,1) \tag{16}$$

$$H_1(t,s) \geq 0, \quad G_1(t,s) \geq 0,\ \forall t,\ s \in J \tag{17}$$

$$e_1(t)\,e_1(s) \leq G_1(t,s) \leq G_1(s,s) = e_1(s) \leq \frac{1}{4},\ \forall t,\ s \in J \tag{18}$$

$$e_1(t)\,H_1(s,s) \leq H_1(t,s) \leq H_1(s,s) \leq e_1(s),\ \forall t,\ s \in J \tag{19}$$

$$\rho_1 e_1(s)\,H_1(s,s) \leq H_1(t,s) \leq \gamma_1 G_1(s,s) = \gamma e_1(s) \leq \frac{1}{4}\gamma,\ \forall t,\ s \in J \tag{20}$$

$$G_1(t,s) \geq \delta^2 G(s,s), \quad H_1(t,s) \geq \delta^2 H_1(s,s) \geq \delta^2 e_1(s),\ \forall t \in J_\delta,\ s \in J \tag{21}$$

where

$$e_1(t) = t(1-t),\ \max_{0 \leq t \leq 1} e_1(t) = \frac{1}{4},\ \gamma_1 = \frac{1}{1-\nu},\ \rho_1 = \frac{\int_0^1 G_1(\tau,\tau)\,g(\tau)\,d\tau}{1-\nu}. \tag{22}$$

Proof. Since the proof is similar, we need to prove (20). It follows from (14), (17) and (18) that

$$\begin{aligned}
H_1(t,s) &= G_1(t,s) + \frac{1}{1-\nu}\int_0^1 G_1(s,\tau)\, g(\tau)\, d\tau \\
&\geq \frac{1}{1-\nu}\int_0^1 G_1(s,\tau)\, g(\tau)\, d\tau \\
&\geq \frac{1}{1-\nu}\int_0^1 e_1(s)\, e_1(\tau)\, g(\tau)\, d\tau \\
&= \frac{\int_0^1 e_1(\tau)\, g(\tau)\, d\tau}{1-v} s(1-s) \\
&= \frac{\int_0^1 G_1(\tau,\tau)\, g(\tau)\, d\tau}{1-v} s(1-s) \\
&= \rho_1 e_1(s).
\end{aligned}$$

In addition, from (18), we have

$$\begin{aligned}
H_1(t,s) &= G_1(t,s) + \frac{1}{1-\nu}\int_0^1 G_1(s,\tau)\, g(\tau)\, d\tau \\
&\leq e_1(s) + \frac{1}{1-\nu}\int_0^1 e_1(s)\, g(\tau)\, d\tau \\
&\leq e_1(s)\left[1 + \frac{1}{1-\nu}\int_0^1 e_1(\tau)\, g(\tau)\, d\tau\right] \\
&= \frac{e_1(s)}{1-v} \\
&= \gamma_1 e_1(s).
\end{aligned}$$

The proof is complete. □

Remark 2. *Suppose that u is a solution of BVP (1). Then from Lemma 1 and Lemma 3, we have*

$$u(t) = \int_0^1 \int_0^1 H(t,s)\, H_1(s,\tau)\, \omega(\tau)\, f(\tau, u(\tau))\, d\tau ds.$$

Define a cone K by

$$K = \left\{ u \in C^{+}[0,1] : \ u(t) \geq 0, \ t \in [0,1] \right\}. \tag{23}$$

It is easy to see K is closed convex cone of $C^{+}[0,1]$.

Now, define an integral operator $T : K \to C^{+}[0,1]$ by

$$(Tu)(t) = \int_0^1 \int_0^1 H(t,s) H_1(s,\tau) \omega(\tau) f(\tau, u(\tau)) d\tau ds, \tag{24}$$

From (24), we know that u is a solution of PVB (1) if and only if u is a fixed point of operator T.

Lemma 5. *Assume (A_1) and (A_2) hold. Then $T(K) \subset K$ and $T : K \to K$ is completely continuous.*

Proof. Since the proof of completely continuous is standard, we need only to prove $T(K) \subset K$. In fact, for any $(t,s) \in [\delta, 1-\delta] \times [0,1]$ and since $f \geq 0$, $\omega \geq 0$ and $\int_0^1 \int_\delta^{1-\delta} s^2(1-s) > 0$, we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 \int_0^1 H(t,s) H_1(s,\tau) \omega(\tau) f(\tau, u(\tau)) d\tau ds, \\ &\geq \int_0^1 \int_\delta^{1-\delta} H(t,s) H_1(s,\tau) \omega(\tau) f(\tau, u(\tau)) d\tau ds \\ &\geq \delta^3 \int_0^1 \int_\delta^{1-\delta} s^2(1-s) \omega(\tau) f(\tau, u(\tau)) d\tau \geq 0. \end{aligned}$$

Therefore, $T(K) \subset K$. The proof is complete. □

Now, let $0 < l < r$ be given and let β be a nonnegative continuous concave functional on the cone K. Define the convex sets K_l and $K(\beta, l, r)$ by

$$K_l = \{ u \in K : \|u\| < l \},$$

and

$$K(\beta, l, r) = \{ u \in K : l \leq \beta(u), \|u\| \leq r \}.$$

The key tool in our approach is the following Leggett-Williams fixed point theorem.

Theorem 1. *[19] Let E be a Banach space and $K \subset E$ be a cone in E. $T : \bar{K}_c \to \bar{K}_c$ be a completely continuous and β be a nonnegative continuous concave functional on K with $\beta(u) \leq \|u\|$ for all $u \in K_c$. Suppose there exist $0 < d < l < r \leq c$ such that*

(i) $u \in \{K(\beta, l, r) : \beta(u) > l\} \neq \emptyset$ *and* $\beta(Tu) > l$ *for* $u \in K(\beta, l, r)$,
(ii) $\|Tu\| < d$ *for* $\|u\| \leq d$,
(iii) $\beta(Tu) > l$ *for* $u \in K(\beta, l, c)$ *with* $\|Tu\| > r$.
Then T *has at least three fixed points* u_1, u_2, u_3 *satisfying*

$$\|u_1\| < d, \quad l < \beta(u_2), \quad \|u_3\| > d \text{ and } \beta(u_3) < l.$$

We will employ *Hölder* inequality.

Lemma 6. (*Hölder*). *Let* $f \in L^p[a,b]$ *with* $0 < a < b$ *and* $p > 1$, $g \in L^q[a,b]$ *with* $q > 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $fg \in L^1[a,b]$ *and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q .$$

Let $f \in L^1[a,b]$, $g \in L^\infty[a,b]$. *Then* $fg \in L^1[a,b]$ *and*

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty .$$

3 Existence results

In this section, we apply Theorem 1 and Lemma 6 to establish the existence of triple positive solutions for BVP (1). We consider the following three cases for $\omega(.) \in L^p[0,1]$: $p > 1$, $p = 1$ *and* $p = \infty$.

For convenience, we introduce the following notations:

$$D = \gamma\gamma_1 \|e_1\|_q \|\omega\|_p , \quad \delta^* = \frac{\delta^2}{\gamma_1},$$

and

$$f^\infty = \lim_{u\to\infty} \sup \max_{t\in[0,1]} \frac{f(t,u)}{u}.$$

Theorem 2. *Assume* $(A_1) - (A_3)$ *hold. Furthermore, suppose that there exist constants* $0 < d < l < \frac{l}{\delta^*} \leq c$ *such that*
(H_1) $f^\infty < \frac{1}{D}$,

(H_2) $f(t,u) > \frac{12l}{\delta^3(1-2\delta)\lambda}$, *for* $(t,u) \in [\delta, b-\delta] \times \left[l, \frac{l}{\delta^*}\right]$, $\delta \in \left[0, \frac{1}{2}\right]$, $\lambda \in \mathbb{N}^*$,

(H_3) $f(t,u) < \frac{d}{D}$, *for* $(t,u) \in [0,1] \times [0,d]$.

Then BVP (1) has at least three positive solutions u_1, u_2 *and* u_3 *such that*

$$\|u_1\| < d, \quad l < \beta(u_2), \quad u_3 > d \text{ with } \beta(u_3) < l.$$

Proof. Let $\beta(u) = \min\limits_{t\in[\delta,b-\delta]} u(t)$. Then $\beta(u)$ is nonnegative continuous concave functional on the cone K satisfying $\beta(u) \le \|u\|$ for all $u \in K$. We denote $r = \frac{l}{\delta^*}$. From (H_1), there exist $0 < \sigma < \frac{1}{D}$, and $l > 0$ such that

$$f(t,u) \le \sigma u, \ and\ u \ge l.$$

Let $\eta = \max\limits_{0\le u\le l,\, t\in[0,1]} f(t,u)$.
Then

$$f(t,u) \le \sigma u + \eta,\ t \in [0,1],\ 0 \le u \le +\infty. \tag{25}$$

Set

$$c > \max\left\{\frac{D\eta}{1 - D\sigma}, \frac{l}{\delta^*}\right\},$$

and

$$e_1(s) = s(1-s).$$

Then, for $u \in \bar{K}_c$, it follows from (10), (19) and (25) that

$$\begin{aligned}
(Tu)(t) &= \int_0^1\int_0^1 H(t,s) H_1(s,\tau)\omega(\tau) f(\tau, u(\tau))\, d\tau \\
&\le \int_0^1\int_0^1 \gamma\gamma_1 e_1(\tau)\omega(\tau)(\sigma u + \eta)\, d\tau ds \\
&\le \int_0^1\int_0^1 \gamma\gamma_1 e_1(\tau)\omega(\tau)(\sigma\|u\| + \eta)\, d\tau ds \\
&\le \gamma\gamma_1(\sigma c + \eta)\int_0^1 e_1(\tau)\omega(\tau)\, d\tau \\
&\le (\sigma c + \eta)\|\omega\|_p \|e\|_q \\
&< c,
\end{aligned}$$

which shows that $Tu \in K_c$. Hence, we have shown that if (H_1) holds, then T maps $\overline{K}_c$ into K_c.
We verify that $\{u/K(\beta,l,r) : \beta(u) > l\} \ne \emptyset$ and $\beta(Tu) > l$, for all $u \in K(\beta,l,r)$.
Take $\varphi_0(t) = \frac{\delta^*+1}{2\delta^*}l$, for $t \in [0,1]$. Then

$$\varphi_0 \in \left\{u/u \in K\left(\beta, l, \frac{l}{\delta^*}\right), \beta(u) > l\right\}.$$

This shows that $\{u/K(\beta,l,r) : \beta(u) > l\} \ne \emptyset$.

Therefore, from (H_2),(11) and (21), we have

$$\begin{aligned}
\beta(Tu) &= \min_{t\in[\delta,1-\delta]}(Tu)(t) \\
&= \min_{t\in[\delta,1-\delta]}\int_0^1\int_0^1 H(t,s)\,H_1(s,\tau)\,\omega(\tau)\,f(\tau,u(\tau))\,d\tau ds \\
&\geq \delta\int_0^1\int_0^1 e(s)\,H_1(s,\tau)\,\omega(\tau)\,f(\tau,u(\tau))\,d\tau ds \\
&\geq \delta^3\int_0^1\int_\delta^{1-\delta} e(s)\,e_1(s)\,\omega(\tau)\,f(\tau,u(\tau))\,d\tau ds \\
&\geq \delta^3\int_0^1 s^2(1-s)\,ds\int_\delta^{1-\delta}\omega(\tau)\,f(\tau,u(\tau))\,d\tau \\
&\geq \frac{\delta^3}{12}\lambda(1-2\delta)\frac{12l}{\delta^3(1-2\delta)\lambda} = l.
\end{aligned}$$

If $u \in \overline{K}_d$, then it follows from (H_3) that

$$\begin{aligned}
(Tu)(t) &= \int_0^1\int_0^1 H(t,s)\,H_1(s,\tau)\,\omega(\tau)\,f(\tau,u(\tau))\,d\tau ds \\
&\leq \int_0^1\int_0^1 \gamma\gamma_1 e_1(\tau)\,\omega(\tau)\,f(\tau,u(\tau))\,d\tau ds \\
&\leq \int_0^1\int_0^1 \gamma\gamma_1 e_1(\tau)\,\omega(\tau)\,\frac{d}{D}\,d\tau ds \\
&\leq \left(\frac{d}{D}\right)\gamma\gamma_1\|\omega\|_p\|e_1\|_q = d.
\end{aligned}$$

Finally, we assert that if $u \in K(\beta,l,c)$ and $\|Tu\| > r$, then $\beta(Tu) > l$. Suppose $u \in K(\beta,l,c)$ and $\|Tu\| > r$, then it follows from (21) that

$$\begin{aligned}
\beta(Tu) &= \min_{t\in[\delta,1-\delta]}\int_0^1\int_0^1 H(t,s)\,H_1(s,\tau)\,\omega(\tau)\,f(\tau,u(\tau))\,d\tau ds \\
&\geq \delta^2\int_0^1\int_0^1 e_1(s)\,H(s,\tau)\,\omega(\tau)\,f(\tau,u(\tau))\,d\tau ds
\end{aligned}$$

$$\geq \frac{\delta^2}{\gamma_1} \int_0^1 \int_0^1 \gamma_1 e_1 (s) H (s, \tau) \omega (\tau) f (\tau, u (\tau)) d\tau ds$$
$$\geq \delta^* \|Tu\| > l$$

To sum up, the hypotheses of Theorem 1 hold. Therefore, BVP (1) has at least three positive solutions u_1, u_2 *and* u_3 such that

$$\|u_1\| < d, \quad l < \beta (u_2), \quad u_3 > d \; with \; \beta (u_3) < l.$$

□

The following Corollaries deals with $p = +\infty$.

Corollary 1. *Assume* $(A_1) - (A_3)$, (H_1), (H_2) *and* (H_3) *hold. Then BVP (1) has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\|u_1\| < d, \quad l < \beta (u_2), \quad u_3 > d \; with \; \beta (u_3) < l.$$

Proof. Let $\|\omega\|_\infty \|e\|_1$ replace $\|\omega\|_p \|e\|_q$ and repeat the argument above. □

Finally we consider the case of $p = 1$.
Let
(H_4) $f^\infty < \frac{1}{D'}$,
(H_5) $f (t, u) \leq \frac{d}{D'}, \; for \; (t, u) \in [0, 1] \times [0, d]$,
where

$$D' = \gamma\gamma_1 \|\omega\|_1 .$$

Corollary 2. *Assume* $(A_1) - (A_3)$,(H_4) *and* (H_5) *hold. Then BVP (1) has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\|u_1\| < d, \quad l < \beta (u_2), \quad \|u_3\| > d \;\; with \;\; \beta (u_3) < l$$

Proof. Set

$$c' > \max \left\{ \frac{D\eta}{1 - D\sigma'}, \frac{l}{\delta^*} \right\},$$

and

$$e_1 (s) = s (1 - s),$$

where $0 < \sigma' < \frac{1}{D'}$. Then, for $u \in K_{c'}$, it follows from (10), (20) and (25) that

$$(Tu) (t) = \int_0^1 \int_0^1 H (t, s) H_1 (s, \tau) \omega (\tau) f (\tau, u (\tau)) d\tau ds$$

$$\leq \int_0^1 \int_0^1 \gamma\gamma_1 e_1 (\tau) \omega (\tau) f (\tau, u (\tau)) d\tau ds$$

$$\leq \int_0^1 \int_0^1 \gamma\gamma_1 e_1(\tau)\,\omega(\tau)\left(\sigma' u + \eta\right) d\tau ds$$

$$\leq \int_0^1 \int_0^1 \gamma\gamma_1 e_1(\tau)\,\omega(\tau)\left(\sigma' \|u\| + \eta\right) d\tau ds$$

$$\leq \gamma\gamma_1\left(\sigma' c' + \eta\right)\int_0^1 \omega(\tau)\, d\tau ds$$

$$\leq \gamma\gamma_1\left(\sigma' c' + \eta\right)\|\omega\|_1 = c',$$

which shows that $Tu \in K_{c'}$. Hence, we have shown that if (H_4) holds, then T maps $\overline{K}_{c'}$ into $K_{c'}$.

If $u \in \overline{K}_d$, then it follows from (H_5), (10) and (20) that

$$(Tu)(t) = \int_0^1 \int_0^1 H(t,s)\, H_1(s,\tau)\,\omega(\tau)\, f(\tau, u(\tau))\, d\tau ds$$

$$\leq \int_0^1 \int_0^1 \gamma\gamma_1 e_1(\tau)\,\omega(\tau)\, f(\tau, u(\tau))\, d\tau ds$$

$$\leq \int_0^1 \int_0^1 \frac{d}{D'}\gamma\gamma_1 e_1(\tau)\,\omega(\tau)\, d\tau ds$$

$$\leq \frac{d}{D'}\gamma\gamma_1 \int_0^1 \omega(\tau)\, d\tau$$

$$\leq \frac{d}{D'}\gamma\gamma_1 \|\omega\|_1 = d.$$

Finally, we assert that if $u \in K(\beta, l, c)$ and $\|Tu\| > r$, then $\beta(Tu) > l$.

Suppose $u \in K(\beta, l, c)$ and $\|Tu\| > r$, then it follows from

$$\beta(Tu) = \min_{t\in[\delta,1-\delta]} \int_0^1 \int_0^1 H(t,s)\, H_1(s,\tau)\,\omega(\tau)\, f(\tau, u(\tau))\, d\tau ds$$

$$\geq \delta^2 \int_0^1 \int_0^1 e_1(s)\, H(s,\tau)\,\omega(\tau)\, f(\tau, u(\tau))\, d\tau ds$$

$$\geq \frac{\delta^2}{\gamma_1} \int_0^1 \int_0^1 \gamma_1 e_1(s) H(s,\tau) \omega(\tau) f(\tau, u(\tau)) \, d\tau ds$$

$$\geq \delta^* \|Tu\| > l.$$

To sum up, the hypotheses of Theorem 1 hold. Therefore, BVP (1) has at least three positive solutions u_1, u_2 *and* u_3 such that

$$\|u_1\| < d, \quad l < \beta(u_2), \quad u_3 > d \text{ with } \beta(u_3) < l.$$

□

Remark 3. *We remark that the condition* (H_3) *in Theorem 2 can be replaced by the following condition:*
$(H_3)^{'}$ $f_0^d \leq \frac{1}{D}$, *where*

$$f_0^d \leq \max\left\{ \max_{t\in[0,1]} \frac{f(t,u)}{d} : u \in [0,d] \right\}.$$

$(H_3)^{''}$ $f^0 \leq \frac{1}{D}$.

Corollary 3. *If the condition* (H_3) *in Theorem 2 replaced by* $(H_3)^{'}$ *or* $(H_3)^{''}$, *respectively, then the conclusion of Theorem 2 also hold.*

We construct an example to illustrate the applicability of the results presented.

Example 1. *Let* $\delta = \frac{1}{4}$, $m = 3$, $a = 0$, $b = 1$ *and* $p = 1$. *It follows from* $p = 1$ *that* $q = \infty$. *Consider the following boundary value problem*

$$\begin{cases} u^{(4)}(t) = \omega(t) f(t, u(t)), & t \in (0,1), \\ u(0) = u(1) = \int_0^1 g(s) u(s) ds, \ u'(0) = 0, \\ u''(0) = \int_0^1 h(s) u''(s) ds, & u^{(3)}(1) = 0, \end{cases} \tag{26}$$

where

$$\omega(t) = \frac{1}{5}t + \frac{1}{10}, \ h(t) = g(t) = t,$$

and

$$f(t,u) = \begin{cases} d, & (t,u) \in [0,1] \times [0,d], \\ u + (10)^3 \left(\frac{l}{\delta^*}\right)\left(\frac{u-d}{l-d}\right), & (t,u) \in [0,1] \times [d,l], \\ (10)^3 \left(\frac{l}{\delta^*}\right), & (t,u) \in [0,1] \times \left[l, \frac{l}{\delta^*}\right], \\ (10)^3 \left(\frac{l}{\delta^*}\right) + \left(u - \frac{l}{\delta^*}\right) t, & (t,u) \in [0,1] \times \left[\frac{l}{\delta^*}, \infty\right). \end{cases}$$

It is easy to see by calculating that $\omega(t) \geq \lambda = \frac{1}{10}$, *for a.e.* $t \in [0,1]$.

By simple calculation, we obtain

$$\mu = \int_0^1 h(t)\,dt = \frac{1}{2},\ \nu = \int_0^1 g(t) = \frac{1}{2}$$

$$\gamma = \frac{1}{1-\mu} = 2,\ \gamma_1 = \frac{1}{1-\nu} = 2,\ \delta = \frac{1}{4}\ \textit{and}\ \delta^* = \frac{\delta^2}{\gamma_1} = \frac{1}{32}.$$

It follows from $\omega(t) = \frac{1}{5}t + \frac{1}{10}$ *and* $e(t) = t(1-t)$ *that*

$$\|\omega\|_1 = \int_0^1 \left(\frac{1}{5}t + \frac{1}{10}\right) dt = \frac{1}{10},$$

and

$$\|e\|_q = \|e\|_\infty = \lim_{q\to\infty} \left(\int_0^1 (t)^q (1-t)^q\,dt\right)^{\frac{1}{q}} = 1.$$

It is easy to verify that

$$D = \gamma\gamma_1 \|e\|_q \|\omega\|_1 = \frac{2}{5},\ \textit{and}\quad f^\infty = 1.$$

Choosing $0 < d < l < \frac{l}{\delta^*} \leq c$*, we have*

$$f^\infty = 1 < \frac{5}{2} = \frac{1}{D},$$

$$f(t,u) = (10)^3 \frac{l}{\delta^*} = 32(10)^3 l > l = 64(400) = \frac{20l}{\delta^3(1-2\delta)\lambda},$$

$$\forall (t,u) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[l, \frac{l}{\delta^*}\right],$$

and

$$f(t,u) = d < \frac{d}{D'} = \frac{5d}{2},\ \forall (t,u) \in [0,1]\times[0,d],$$

which shows (H_1), (H_2) *and* (H_3) *hold.*

Thus all assumptions and conditions of Theorem 2 are satisfied. Hence, Corollary 1 implies that BVP (26) has at least three positive solutions u_1, u_2 *and* u_3 *such that*

$$\|u_1\| < d,\quad l < \beta(u_2),\quad u_3 > d \textit{ with } \beta(u_3) < l.$$

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