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NUMERICAL STUDY OF THE MOTION OF A HEAVY BALL SLIDING ON A ROTATING WIRE

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Abstract

The motion of a heavy ball slipping on a revolving wire was considered in this study. The importance of the system studied here arises from the fact that the motional energy in this situation a function of both the dynamical variable and its derivative. Our first step was constructing the classical Lagrangain. After that the Euler- Lagrange equation is derived. Finally, we solve the obtained Euler- Lagrange equation analytically and numerically using the ode45 code which is based on Runge-Kutta method.

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Key words: Lagrange of a heavy ball, revolving wire, Euler-Lagrange equation, numerical calculation, Runge-Kutta method, ode45.

1 Introduction

Differential equations (DEs) play an important role in many branches of physics such as: classical mechanics [10, 14, 6], electromagnetic theory [8], quantum mechanics [9], fluid mechanics [4], etc. In classical mechanics, we deal with ordinary differential equation (ODEs) either when using Newtonian mechanics, or when applying Lagrangian mechanics in studying many physical systems. The Lagrangian mechanics enables us to solve a variety of physical examples due to the fact that writing the Lagrangian depends only on scalar quantities (kinetic energy and potential energy). One can refer to some classical texts to show how Lagrangian can be built [10, 14, 6]. As a result of building the Lagrange equation of any system a DEs (called Euler- Lagrange equations) are obtained, and these equations have to be solved analytically or in some cases due to difficulties we

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seek numerical solutions [7]. Many numerical methods and techniques have been considered in solving DEs[16, 15, 2, 5]. Techniques and methods used in numerical are powerful because they help scientists in solving many kinds of differential equations without seeking for their analytical solutions. The rest of this paper is organized as follow. In Sec. 2 a physical description for the system is listed where Euler- Lagrange equation has been obtained. In Sec. 3, numerical method, and simulation results with discussion are presented. In section 4 we close the present work by a conclusion.

2 Description of the physical system

Consider a heavy ball slipping on a rotating wire, with angular frequency ω, as discussed in a well-known text in analytical mechanics [10]. The present system deals with a ball slipping without resistance on a thin wire revolving about a vertical axis by a mechanical external agent at a constant angular frequency ω as shown in Fig. 1 below. The wire deviated away from the vertical axis by an angle ψ . The ball is forced to move on the wire, and to define its motion we need just one variable which is r (i.e., the distance from origin). It is important to mentioned that the importance of this example comes from the fact that the kinetic energy in this case depends on both the dynamical variable and on its derivative, instead of on the time derivative alone.

The kinetic energy and the potential energy of the ball respectively read:

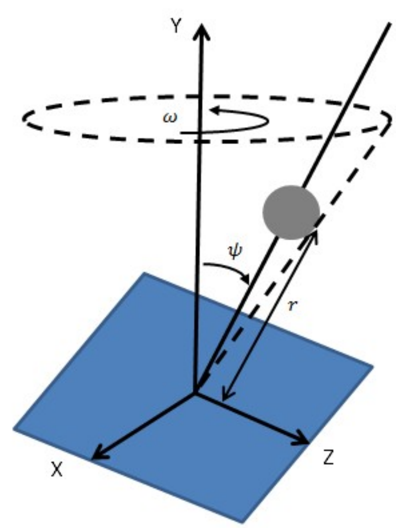


Figure 1: Heavy ball sliding on a rotating wire

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\sin^2\psi\right) \tag{1}$$

$$V = mgr\cos\psi \tag{2}$$

As a result, the classical Lagrangian takes the form

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2 \sin^2\psi\right) - mgr\cos\psi \tag{3}$$

The classical Euler-Lagrange equation (CELE) can be derived using

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \tag{4}$$

In view of Eqs. (4)- (5), the CELE reads:

$$\ddot{r} = r\omega^2 \sin^2\psi - g\cos\psi \tag{5}$$

The last equation obtained is a non- homogenous second order linear differential equation. We aim to solve this equation in the next section numerically for some given initial conditions.

3 Analytic solution of the problem

In this section we obtained the analytical solution for the motion of equation (5), the canonical system of Hamilton and Poisson parenthesis are deduced, the reader can refer to [17, 1, 12] for more details. Let us consider the differential equation of second degree non-homogeneous:

$$\ddot{r} - r\omega^2 \sin^2\psi = -g\cos\psi \tag{6}$$

The characteristic equations for the homogeneous differential equation of second degree is:

$$\lambda^2 - \omega^2 \sin^2\psi = 0 \tag{7}$$

having the real roots: $\lambda_{1,2} = \pm\omega\sin\psi$ Therefore the homogeneous solution is:

$$r_o = C_1 e^{\omega\sin\psi\cdot t} + C_2 e^{-\omega\sin\psi\cdot t} \tag{8}$$

Because the right term from the differential equation (6) is constant then the particular solution for (6) is

$$r_p = \frac{g}{\omega^2}\frac{\cos\psi}{\sin^2\psi} \tag{9}$$

In conclusion the general solution of (6) is:

$$r = r_o + r_p = C_1 e^{\omega\sin\psi\cdot t} + C_2 e^{-\omega\sin\psi\cdot t} + \frac{g}{\omega^2}\frac{\cos\psi}{\sin^2\psi} \tag{10}$$

Taking into consideration the initial conditions $r(0) = 5$ and $\dot{r}(0) = 0$ we obtain that $C_1 = C_2 = \frac{5}{2} - \frac{g}{2\omega^2}\frac{\cos\psi}{\sin^2\psi}$ and the solution of (6) will be

$$r(t) = \left(\frac{5}{2} - \frac{g}{2\omega^2}\frac{\cos\psi}{\sin^2\psi}\right)\left(e^{\omega\sin\psi\cdot t} + e^{-\omega\sin\psi\cdot t}\right) + \frac{g}{\omega^2}\frac{\cos\psi}{\sin^2\psi} \tag{11}$$

Next we want to deduce the canonical system of Hamilton in order to obtain the Poisson parenthesis.

We note by $p = \frac{\partial L}{\partial \dot{r}}$ where $p = p(t, r, \dot{r})$ is the generalized impulse. Let introduce the Hamilton function:

$$H = p\dot{r} - L \Leftrightarrow H = p\dot{r} - \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\sin^2\psi\right) + mgr\cos\psi \tag{12}$$

The Hamilton system will be:

$$\begin{cases} \dot{r} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial r} = mr\omega^2\sin^2\psi - mg\cos\psi \end{cases} \tag{13}$$

Based on the above system we may obtain the Poisson parenthesis. The symmetric system attached to the Hamilton system is:

$$\frac{dr}{\frac{\partial H}{\partial p}} = \frac{dp}{-\frac{\partial H}{\partial r}} \tag{14}$$

Left $F(t, r, p) = C$ be a prime integral of the symmetric system (14) such that $\frac{dF}{dt} = 0$. In this case we obtain the following equation:

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial r}\dot{r} + \frac{\partial F}{\partial p}\dot{p} = 0 \tag{15}$$

Taking into account the Hamilton system we have:

$$\frac{\partial F}{\partial t} + (F, H) = 0 \tag{16}$$

where

$$(F, H) = \frac{\partial F}{\partial r}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial r} \tag{17}$$

is the Poisson parenthesis.

4 Numerical Method and Simulation Results

4.1 Numerical Method

The `dsolve` command can be used to solve ordinary differential equations (ODE), such as first-order ODEs and high-order ODEs via MATLAB framework.

A first-order ODE is an relation that has the derivative of the dependent variable and it requires one initial condition [11, 7] If y is a variable that depends on t, then the relation between them can be written as [3].

$$\frac{dy}{dt} = f(t, y) \tag{18}$$

In order to obtain a particular solution of a first-order ODE, the `dsolve` method is used:

```
dsolve\left('eq','cond1','var'\right)
```
(19)

The general solution for a second- order ODE takes the form:

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}) \tag{20}$$

Note that here we requires two initial conditions [11, 7]. Again to obtain a particular solution of a second-order ODE, the dsolve command takes the form:

```
dsolve\left('eq','cond1','cond2',...,'var'\right)
```
(21)

General and particular solutions are achieved by `dsolve` method. In a particular solution, the constants are taken to have definite algebraic values such that the solution meets certain boundary [7]. Let y a function depending on a variable t , then the first derivative of y is written as:

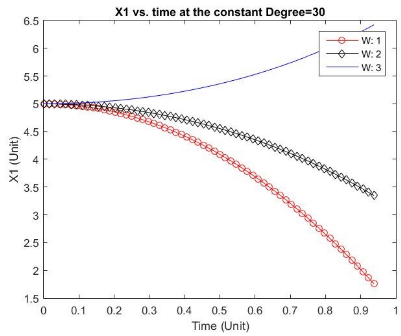
$$Dy = \frac{dy}{dt} \tag{22}$$

where D_i stands for the i-th derivatives. As an example, the equation $\frac{dy}{dt} + ay = b$ takes the following form in MATLAB `'Dy+a*y=b'` , where a and b are constants. On the other hand, the equation $\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = \sin(t)$ is written in MATLAB as `'D2y+a*Dy+b*y=sin(t)'` , again with a and b constants [3].

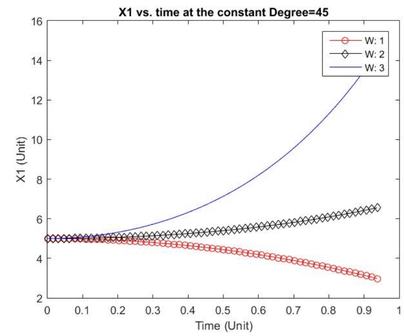
4.2 Simulation results and discussion

In this subsection, we propose the numerical solution for Eq. (5) using `diff(s,n)`, `dsolve\left('eq','cond1','var'\right)` , and loop structure. We consider some initial conditions, and the numerical solutions for these particular conditions have been obtained, and one has to note that in all figures obtained x_1 refers for r .

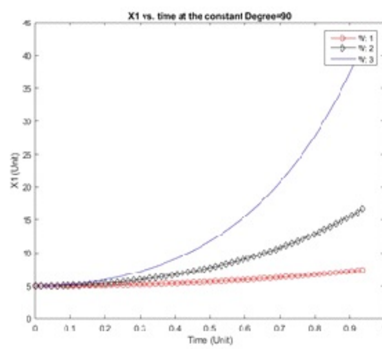
Here we consider the following initial condition $r(0) = 5$, and $\dot{r}(0) = 0$ for the following three different angles $\psi = \frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$, and for the following three different angular speeds $\omega = 1, 2$, and 3 . The obtained results that are shown in Fig. (2).



(a)



(b)

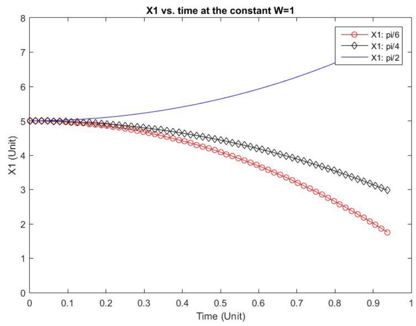


(c)

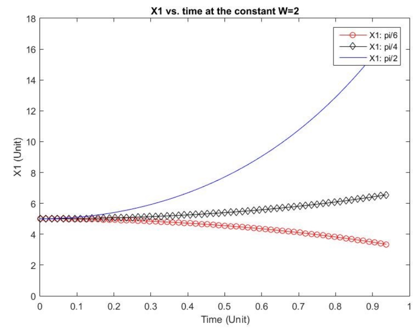
Figure 2: (a)The behaviour of the distance against time for $\psi = \frac{\pi}{6}$ and $\omega = 1, 2$ and 3; (b)The behaviour of the distance against time for $\psi = \frac{\pi}{4}$ and $\omega = 1, 2$ and 3; (c) The behaviour of the distance against time for $\psi = \frac{\pi}{2}$ and $\omega = 1, 2$ and 3;

In Fig. 2 shown above the behaviour of the distance against time is presented for different angular speed $\omega = 1, 2, 3$ where in each case we consider a specific angle, for example in Fig. (2a) $\psi = \frac{\pi}{6}$, while $\psi = \frac{\pi}{4}$ in Fig. (2b), and finally, we consider $\psi = \frac{\pi}{2}$ in Fig. (2c). On the other hand, in Fig. 3 below we show the behavior of the distance against time for different angles $\psi = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$ where in each case we consider a specific angular speed $\omega = 1$ for Fig. (3a), $\omega = 2$ for Fig. (3b), and in Fig. (3c) $\omega = 3$.

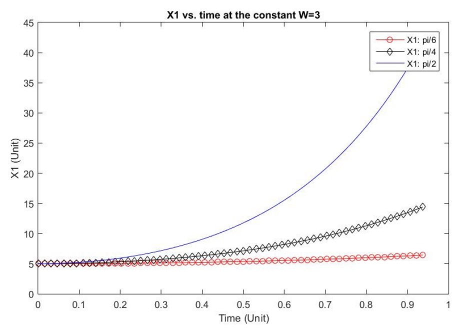
In all figures obtained above it is clear that the distance $r(t) = x_1(t)$ increases in all considered cases for the special case when $\psi = \frac{\pi}{2}$. This is due to the fact that in this case the wire rotates about the vertical axis in a horizontal plane, so the particle will move away from the origin (the right direction is considered to be positive). While for other chosen angles sometimes the heavy bead moves away from origin and sometimes moving towards the origin depending on the value of the angular speed ω at which the wire is rotating.



(a)



(b)



(c)

Figure 3: (a)The behaviour of the distance against time for $\omega = 1$ and $\psi = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$; (b)The behaviour of the distance against time for $\omega = 2$ and $\psi = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$; (c) The behaviour of the distance against time for $\omega = 3$ and $\psi = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}$;

5 Conclusion

In this paper the `ode45` code has been successfully applied to find a truthful numerical solution for the motion of a ball sliding on a rotating wire. The position of the bead is depicted for the time period $[0, 1]$. We examine the motion for different values of angle ψ and for different values of angular speed ω .

It is clear from the figures that when the angle $\psi = \frac{\pi}{2}$ the heavy ball moves away from the origin (to the right) in all considered cases because in this case the wire rotates on a horizontal plane, while for other consider angles the motion depends on the angular speed ω considered.

Furthermore, we believe that this method is effective for predicting analytical solutions in many branches of science and engineering problems.

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