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CONSTRUCTIONS OF K-g-FUSION FRAMES AND THEIR DUAL IN HILBERT SPACES

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Abstract

Frames for operators or K-frames were recently considered by Găvruţa (2012) in connection with atomic systems. Also, generalized frames are important frames in the Hilbert space of bounded linear operators. Fusion frames, which are a special case of generalized frames have various applications. This paper introduces the concept of generalized fusion frames for operators also known as K-g-fusion frames and we get some results for characterization of these frames. We further discuss dual and Q-dual in connection with K-g-fusion frames. Also we obtain some useful identities for these frames. We also give several methods to construct K-g-fusion frames. The results of this paper can be used in sampling theory which are developed by g-frames and especially fusion frames. In the end, we discuss the stability of a more general perturbation for K-g-fusion frames.

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1 Introduction

Frames in Hilbert spaces were first proposed by Duffin and Schaeffer in the context of non-harmonic Fourier series [11]. Now, frames have been widely applied in signal processing, sampling, filter bank theory, system modeling, Quantum information, cryptography, etc ([3, 12, 15]). We can say that fusion frames are

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the generalization of conventional classical frames and special cases of g-frames in the field of frame theory. The fusion frames are in fact more susceptible due to complicated relations between the structure of the sequence of weighted subspaces and the local frames in the subspaces and due to the extreme sensitivity with respect to changes of the weights.

Frames for operators or K-frames have been introduced by Găvruţa in [17] to study the nature of atomic systems for a separable Hilbert space with respect to a bounded linear operator K. It is a well-known fact that K-frames are more general than the classical frames and due to higher generality of K-frames, many properties of frames may not hold for K-frames. Recently, we presented g-fusion frames in [22]. This paper presents K-g-fusion frames with respect to a bounded linear operator on a separable Hilbert space which are a generalization of g-fusion frames.

Throughout this paper, H is a separable Hilbert space and $\mathcal{B}(H)$ is the collection of all bounded linear operators of H into H. Also, π_V is the orthogonal projection from H onto a closed subspace $V \subset H$ and $\{H_j\}_{j\in\mathbb{J}}$ is a sequence of Hilbert spaces where $\mathbb{J}$ is a subset of $\mathbb{Z}$.

For the proof of the following lemma, we refer to [17].

Lemma 1. *Let $V \subseteq H$ be a closed subspace, and T be a linear bounded operator on H. Then*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

If T is unitary (i.e. $T^*T = Id_H$), then

$$\pi_{\overline{TV}} T = T \pi_V.$$

Definition 1. ***(K-frame)**[16]. Let $\{f_j\}_{j\in\mathbb{J}}$ be a sequence of members of H and $K \in \mathcal{B}(H)$. We say that $\{f_j\}_{j\in\mathbb{J}}$ is a K-frame for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,*

$$A\|K^*f\|^2 \leq \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \leq B\|f\|^2.$$

Definition 2. ***(g-fusion frame)**[22]. Let $W = \{W_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of H, $\{v_j\}_{j\in\mathbb{J}}$ be a family of weights, i.e. $v_j > 0$ and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in \mathbb{J}$. We say $\Lambda := (W_j, \Lambda_j, v_j)$ is a generalized fusion frame (or g-fusion frame) for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,*

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leq B\|f\|^2. \tag{1}$$

If an operator U has closed range, then there exists a right-inverse operator $U^\dagger$ (pseudo-inverse of U) in the following senses (see [8]).

Lemma 2. *Let $U \in \mathcal{B}(H_1, H_2)$ be a bounded operator with closed range $\mathcal{R}(U)$. Then there exists a bounded operator $U^\dagger \in \mathcal{B}(H_2, H_1)$ for which*

$$UU^\dagger x = x, \quad x \in \mathcal{R}(U).$$

Lemma 3. *Let $U \in \mathcal{B}(H_1, H_2)$. Then the following assertions hold:*

1. *$\mathcal{R}(U)$ is closed in H_2 if and only if $\mathcal{R}(U^*)$ is closed in H_1.*
2. *$(U^*)^\dagger = (U^\dagger)^*$.*
3. *The orthogonal projection of H_2 onto $\mathcal{R}(U)$ is given by $UU^\dagger$.*
4. *The orthogonal projection of H_1 onto $\mathcal{R}(U^\dagger)$ is given by $U^\dagger U$.*
5. *$\mathcal{N}(U^\dagger) = \mathcal{R}^\perp(U)$ and $\mathcal{R}(U^\dagger) = \mathcal{N}^\perp(U)$.*

Lemma 4. *([9]). Let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$ be on given Hilbert spaces. Then the following assertions are equivalent:*

1. *$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$;*
2. *$L_1 L_1^* \leq \lambda^2 L_2 L_2^*$ for some $\lambda > 0$;*
3. *there exists a mapping $U \in \mathcal{B}(H_1, H_2)$ such that $L_1 = L_2 U$.*

Moreover, if those conditions are valid, then there exists a unique operator U such that

(a) $\|U\|^2 = \inf\{\alpha > 0 \mid L_1 L_1^* \leq \alpha L_2 L_2^*\}$;

(b) $\mathcal{N}(L_1) = \mathcal{N}(U)$;

(c) $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2^*)}$.

2 K-g- Fusion Frames

In this section, we aim to define the notation of K-g-fusion frames and review their operators. First, we define the space $\mathscr{H}_2 := (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell_2}$ by

$$\mathscr{H}_2 = \big\{\{f_j\}_{j\in\mathbb{J}} \ : \ f_j \in H_j, \ \sum_{j\in\mathbb{J}} \|f_j\|^2 < \infty\big\} \tag{2}$$

with the inner product defined by

$$\langle \{f_j\}, \{g_j\} \rangle = \sum_{j\in\mathbb{J}} \langle f_j, g_j \rangle.$$

It is clear that $\mathscr{H}_2$ is a Hilbert space with pointwise operations.

Definition 3. *Let $W = \{W_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of H, $\{v_j\}_{j\in\mathbb{J}}$ be a family of weights, i.e. $v_j > 0$, $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in \mathbb{J}$ and $K \in \mathcal{B}(H)$. We say $\Lambda := (W_j, \Lambda_j, v_j)$ is a K-g- fusion frame for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,*

$$A\|K^* f\|^2 \leq \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leq B\|f\|^2. \tag{3}$$

When $K = Id_H$, we get the g-fusion frame for H. Throughout this paper, Λ will be a triple (W_j, Λ_j, v_j) with $j \in \mathbb{J}$ unless otherwise noted. We say Λ is a Parseval K-g-fusion frame whenever

$$\sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 = \|K^* f\|^2.$$

The synthesis and the analysis operators of the K-g-fusion frames are defined by

$$T_\Lambda : \mathscr{H}_2 \longrightarrow H,$$
$$T_\Lambda(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} v_j \pi_{W_j} \Lambda_j^* f_j,$$

and

$$T_\Lambda^* : H \longrightarrow \mathscr{H}_2,$$
$$T_\Lambda^*(f) = \{v_j \Lambda_j \pi_{W_j} f\}_{j\in\mathbb{J}}.$$

Thus, the K-g-fusion frame operator is given by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f$$

and

$$\langle S_\Lambda f, f\rangle = \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2, \tag{4}$$

for all $f \in H$. Therefore,

$$\langle AKK^* f, f\rangle \leq \langle S_\Lambda f, f\rangle \leq \langle Bf, f\rangle \tag{5}$$

or

$$AKK^* \leq S_\Lambda \leq BId_H. \tag{6}$$

Hence, we conclude that:

Proposition 1. *Let Λ be a g-fusion Bessel sequence for H. Then Λ is a K-g-fusion frame for H if and only if there exists $A > 0$ such that $S_\Lambda \geq AKK^*$.*

Remark 1. *In K-g-fusion frames, like K-frames and K-fusion frames, the K-g-fusion frame operator is not invertible. But, if $K \in \mathcal{B}(H)$ has closed range, then operator S_Λ is an invertible operator on a subspace $\mathcal{R}(K) \subset H$. Indeed, suppose that $f \in \mathcal{R}(K)$, then*

$$\|f\|^2 = \|(K^\dagger|_{\mathcal{R}(k)})^* K^* f\|^2 \leq \|K^\dagger\|^2 \|K^* f\|^2.$$

Thus, we have

$$A\|K^\dagger\|^{-2}\|f\|^2 \leq \langle S_\Lambda f, f\rangle \leq B\|f\|^2,$$

which implies that $S_\Lambda : \mathcal{R}(K) \to S_\Lambda(\mathcal{R}(K))$ is a homeomorphism. Furthermore, for each $f \in S_\Lambda(\mathcal{R}(K))$ we have

$$B^{-1}\|f\|^2 \leq \langle (S_\Lambda|_{\mathcal{R}(K)})^{-1} f, f\rangle \leq A^{-1}\|K^\dagger\|^2\|f\|^2.$$

Remark 2. *Since $S_\Lambda \in \mathcal{B}(H)$ is positive and self-adjoint and $\mathcal{B}(H)$ is a C^*-algebra, then S_Λ^{-1} is positive and self-adjoint too whenever $K \in \mathcal{B}(H)$ has closed range. Now, for each $f \in S_\Lambda(\mathcal{R}(k))$ we can write*

$$\begin{aligned}\langle Kf, f\rangle &= \langle Kf, S_\Lambda S_\Lambda^{-1} f\rangle \\ &= \langle S_\Lambda(Kf), S_\Lambda^{-1} f\rangle \\ &= \langle \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}Kf, S_\Lambda^{-1} f\rangle \\ &= \sum_{j\in\mathbb{J}} v_j^2 \langle S_\Lambda^{-1}\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}Kf, f\rangle.\end{aligned}$$

Theorem 1. *Let $U \in \mathcal{B}(H)$ be an invertible operator on H and Λ be a K-g-fusion frame for H with bounds A and B. Then, $\Gamma := (UW_j, \Lambda_j\pi_{W_j}U^*, v_j)$ is a UK-g-fusion frame for H.*

Proof. By closed linear operator theorem, we obtain that UW_j is closed for any $j \in \mathbb{J}$. Let $f \in H$. Then by applying Lemma 1, with U instead of T, we have

$$\begin{aligned}\sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j\pi_{W_j}U^*\pi_{UW_j}f\|^2 &= \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j\pi_{W_j}U^*f\|^2 \\ &\le B\|U^*f\|^2 \\ &\le B\|U\|^2\|f\|^2.\end{aligned}$$

So, Γ is a g-fusion Bessel sequence for H. On the other hand,

$$\begin{aligned}\sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j\pi_{W_j}U^*\pi_{UW_j}f\|^2 &= \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j\pi_{W_j}U^*f\|^2 \\ &\ge A\|K^*U^*f\|^2\end{aligned}$$

and the proof is completed. □

Corollary 1. *If $U \in \mathcal{B}(H)$ is an invertible operator on Hilbert spaces, Λ is a K-g-fusion frame for H with bounds A, B and $KU = UK$, then $\Gamma := (UW_j, \Lambda_j\pi_{W_j}U^*, v_j)$ is a K-g-fusion frame for H with bounds $A\|U^{-1}\|^{-2}$ and $B\|U\|^2$.*

Proof. Just notice that

$$\|K^*f\|^2 = \|(U^{-1})^*U^*K^*f\|^2 \le \|U^{-1}\|^2\|K^*U^*f\|^2$$

and by Theorem 1 the proof is obtained. □

Theorem 2. *Let $U \in \mathcal{B}(H)$ be an invertible and unitary operator on H and Λ be a K-g-fusion frame for H with bounds A and B. Then, $(UW_j, \Lambda_jU^{-1}, v_j)$ is a $(U^{-1})^*K$-g-fusion frame for H.*

Proof. Using Lemma 1, we can write for any $f \in H$,

$$A\|K^*U^{-1}f\|^2 \leq \sum_{j\in\mathbb{J}} \upsilon_j^2 \|\Lambda_j U^{-1}\pi_{UW_j} f\|^2 \leq B\|U^{-1}\|^2\|f\|^2.$$

□

Corollary 2. *If $U \in \mathcal{B}(H)$ is an invertible and unitary operator on Hilbert spaces, Λ is a K-g-fusion frame for H with bounds A, B and $K^*U = UK^*$, then $(UW_j, \Lambda_j U^{-1}, \upsilon_j)$ is a K-g-fusion frame for H.*

Theorem 3. *If $U \in \mathcal{B}(H)$, Λ is a K-g-fusion frame for H with bounds A, B and $\mathcal{R}(U) \subseteq \mathcal{R}(K)$, then Λ is a U-g-fusion frame for H.*

Proof. Via Lemma 4, there exists $\lambda > 0$ such that $UU^* \leq \lambda^2 KK^*$. Thus, for each $f \in H$ we have

$$\|U^*f\|^2 = \langle UU^*f, f\rangle \leq \lambda^2\langle KK^*f, f\rangle = \lambda^2\|K^*f\|^2.$$

It follows that

$$\frac{A}{\lambda^2}\|U^*f\|^2 \leq \sum_{j\in\mathbb{J}} \upsilon_j^2 \|\Lambda_j \pi_{W_j} f\|^2.$$

□

In the following it is showed that condition $\mathcal{R}(U) \subseteq \mathcal{R}(K)$ in Theorem 3 is necessary.

Example 1. *Let $H = \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ be an orthonormal basis for H. We define two operators K and U on H by*

$$Ke_1 = e_2, \quad Ke_2 = e_3, \quad Ke_3 = e_3;$$
$$Ue_1 = 0, \quad Ue_2 = e_1, \quad Ue_3 = e_2.$$

Suppose that $W_j = H_j := span\{e_j\}$ where $j = 1, 2, 3$. Let

$$\Lambda_j f := \langle f, e_j\rangle e_j = f_j e_j,$$

where $f = (f_1, f_2, f_3)$ and $j = 1, 2, 3$. It is clear that $(W_j, \Lambda_j, 1)$ is a K-g-fusion frame for H with bounds $\frac{1}{2}$ and 1, respectively. Assume that $(W_j, \Lambda_j, 1)$ is a U-g-fusion frame for H. Then, by Proposition 1, there exists $A > 0$ such that $KK^* \geq AUU^*$. So, by Lemma 4, $\mathcal{R}(U) \subseteq \mathcal{R}(K)$. But, this is a contradiction with $\mathcal{R}(U) \nsubseteq \mathcal{R}(K)$, since $e_1 \in \mathcal{R}(U)$ but $e_1 \notin \mathcal{R}(K)$.

Theorem 4. *Let $K \in \mathcal{B}(H)$ be closed range, Λ be a K-g-fusion frame for H with bounds A, B and $U \in \mathcal{B}(H)$ where $\mathcal{R}(U) \subseteq \mathcal{R}(K)$. Then $(\overline{UW_j}, \Lambda_j \pi_{W_j} U^*, \upsilon_j)_{j\in\mathbb{J}}$ is a K-g-fusion frame for H if and only if there exists a $\delta > 0$ such that for every $f \in H$,*

$$\|U^*f\| \geq \delta\|K^*f\|.$$

Proof. Let $f \in H$ and $(\overline{UW_j}, \Lambda_j \pi_{W_j} U^*, v_j)_{j\in\mathbb{J}}$ be a g-fusion frame for H with the lower bound C and $U \in \mathcal{B}(H)$. So, by Lemma 1, we get

$$C\|K^*f\|^2 \leq \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} U^* \pi_{\overline{UW_j}} f\|^2 = \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} U^* f\|^2.$$

On the other hand, we have

$$\sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} U^* f\|^2 \leq B\|U^*f\|^2.$$

Thus, $\|U^*f\| \geq \sqrt{\frac{C}{B}}\|K^*f\|$. For the opposite implication, we can write for all $f \in H$,

$$\|U^*f\| = \|(K^\dagger)^* K^* U^* f\| \leq \|K^\dagger\|.\|K^*U^*f\|.$$

Therefore,

$$\begin{aligned}
A\delta^2\|K^\dagger\|^{-2}\|K^*f\|^2 &\leq A\|K^\dagger\|^{-2}\|U^*f\|^2 \\
&\leq A\|K^*U^*f\|^2 \\
&\leq \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} U^* f\|^2 \\
&= \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} U^* \pi_{\overline{UW_j}} f\|^2 \\
&\leq B\|U\|^2\|f\|^2.
\end{aligned}$$

So, $(\overline{UW_j}, \Lambda_j \pi_{W_j} U^*, v_j)_{j\in\mathbb{J}}$ is a g-fusion frame for H with frame bounds $A\delta^2\|K^\dagger\|^{-2}$ and $B\|U\|^2$. □

Theorem 5. *Let $\Lambda := (W_j, \Lambda_j, v_j)$ and $\Theta := (V_j, \Theta_j, w_j)$ be two g-fusion Bessel sequences for H with bounds B_1 and B_2, respectively. Suppose that T_Λ and T_Θ be their analysis operators such that $T_\Theta T_\Lambda^* = K^*$ where $K \in \mathcal{B}(H)$. Then, both Λ and Θ are K and K^*-g-fusion frames, respectively.*

Proof. For each $f \in H$ we have

$$\begin{aligned}
\|K^*f\|^4 &= \langle K^*f, K^*f\rangle^2 \\
&= \langle T_\Lambda^* f, T_\Theta^* K^* f\rangle^2 \\
&\leq \|T_\Lambda^* f\|^2 \|T_\Theta^* K^* f\|^2 \\
&= \Big(\sum_{j\in\mathbb{I}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2\Big)\Big(\sum_{j\in\mathbb{I}} w_j^2 \|\Theta_j \pi_{V_j} K^* f\|^2\Big) \\
&\leq \Big(\sum_{j\in\mathbb{I}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2\Big) B_2 \|K^*f\|^2,
\end{aligned}$$

thus, $B_2^{-1}\|K^*f\|^2 \leq \sum_{j\in\mathbb{I}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2$. This means that Λ is a K-g-fusion frame for H. Similarly, Θ is a K^*-g-fusion frame with the lower bound B_1^{-1}. □

3 Q-Duality of K-g-Fusion Frames

In this section, we shall define the duality of K-g-fusion frames and present some properties of them.

Definition 4. *Let $\Lambda = (W_j, \Lambda_j, v_j)$ be a K-g-fusion frame for H. A g-fusion Bessel sequence $\widetilde{\Lambda} := (\widetilde{W_j}, \widetilde{\Lambda_j}, \widetilde{v_j})$ is called Q-dual K-g-fusion frame (or brevity QK-gf dual) for Λ if there exists a bounded linear operator $Q : \mathscr{H}_2 \to \widetilde{\mathscr{H}_2}$ such that*

$$T_\Lambda Q^* T^*_{\widetilde{\Lambda}} = K, \tag{7}$$

where $\widetilde{\mathscr{H}_2} = (\sum_{j\in\mathbb{J}} \oplus \widetilde{H}_j)_{\ell_2}$.

Like K-frames, the following present equivalent conditions of the duality.

Proposition 2. *Let $\widetilde{\Lambda}$ be a QK-gf dual for Λ. The following conditions are equivalent:*

1. $T_\Lambda Q^* T^*_{\widetilde{\Lambda}} = K$;
2. $T_{\widetilde{\Lambda}} Q T^*_\Lambda = K^*$;
3. *for each $f, f' \in H$, we have*

$$\langle Kf, f'\rangle = \langle T^*_{\widetilde{\Lambda}} f, QT^*_\Lambda f'\rangle = \langle Q^* T^*_{\widetilde{\Lambda}} f, T^*_\Lambda f'\rangle.$$

Proof. Straightforward. □

Theorem 6. *If $\widetilde{\Lambda}$ is a QK-gf dual for Λ, then $\widetilde{\Lambda}$ is a K^*-g-fusion frame for H.*

Proof. Let $f \in H$ and B be an upper bound of Λ. Therefore,

$$\begin{aligned}
\|Kf\|^4 &= |\langle Kf, Kf\rangle|^2 \\
&= |\langle T_\Lambda Q^* T^*_{\widetilde{\Lambda}} f, Kf\rangle|^2 \\
&= |\langle T^*_{\widetilde{\Lambda}} f, QT^*_\Lambda Kf\rangle|^2 \\
&\leq \|T^*_{\widetilde{\Lambda}} f\|^2 \|Q\|^2 B \|Kf\|^2 \\
&= \|Q\|^2 B \|Kf\|^2 \sum_{j\in\mathbb{J}} \widetilde{v}_j^2 \|\widetilde{\Lambda}_j \pi_{\tilde{W}_j} f\|^2
\end{aligned}$$

and by definition, this completes the proof. □

Corollary 3. *Assume C_{op} and D_{op} are the optimal bounds of $\widetilde{\Lambda}$, respectively. Then*

$$C_{op} \geq B_{op}^{-1} \|Q\|^{-2} \quad \text{and} \quad D_{op} \geq A_{op}^{-1} \|Q\|^{-2}$$

where A_{op} and B_{op} are the optimal bounds of Λ, respectively.

Suppose that Λ is a K-g-fusion frame for H. Since $S_\Lambda \geq AKK^*$, then by Lemma 4, there exists an operator $U \in \mathcal{B}\big(H, \mathscr{H}_2\big)$ such that

$$T_\Lambda U = K. \tag{8}$$

Now, we define the j-th component of Uf by $U_j f = (Uf)_j$ for each $f \in H$. It is clear that $U_j \in \mathcal{B}(H, H_j)$. By this operator, we can construct some QK-gf dual for Λ.

Theorem 7. *Let Λ be a K-g-fusion frame for H. If U is an operator as in (8) and $\widetilde{\Lambda} := (\widetilde{W_j}, \widetilde{\Lambda}_j, v_j)$ is a g-fusion Bessel sequence where $\widetilde{\Lambda}_j := \Lambda_j U^* U_j$ and $\widetilde{W_j} := U_j^* U W_j$, then $\widetilde{\Lambda}$ is a QK-gf dual for Λ.*

Proof. Define the mapping

$$\Phi_0 : \mathcal{R}(T^*_{\widetilde{\Lambda}}) \to \mathscr{H}_2,$$
$$\Phi_0(T^*_{\widetilde{\Lambda}} f) = Uf.$$

Then Φ_0 is well-defined. indeed, if $f_1, f_2 \in H$ and $T^*_{\widetilde{\Lambda}} f_1 = T^*_{\widetilde{\Lambda}} f_2$, then $\pi_{\widetilde{W_j}}(f_1 - f_2) = 0$. Therefore, for any $j \in \mathbb{J}$,

$$f_1 - f_2 \in (\widetilde{W_j})^\perp = \mathcal{R}(U_j^*)^\perp = \ker U_j.$$

Thus, $Uf_1 = Uf_2$. It is clear that Φ_0 is bounded and linear. Therefore, it has a unique linear extension (also denoted Φ_0) to $\overline{\mathcal{R}(T^*_{\widetilde{\Lambda}})}$. Define Φ on $\mathscr{H}_2$ by setting

$$\Phi = \begin{cases} \Phi_0, & \text{on } \overline{\mathcal{R}(T^*_{\widetilde{\Lambda}})}, \\ 0, & \text{on } \overline{\mathcal{R}(T^*_{\widetilde{\Lambda}})}^\perp \end{cases}$$

and let $Q := \Phi^*$. This implies that $Q^* \in \mathcal{B}(\mathscr{H}_2, \mathscr{H}_2)$ and

$$T_\Lambda Q^* T^*_{\widetilde{\Lambda}} = T_\Lambda \Phi T^*_{\widetilde{\Lambda}} = T_\Lambda U = K.$$

□

Proposition 3. *Let Λ be a K-g-fusion frame with optimal bounds of A_{op} and B_{op}, respectively and K has closed range. Then*

$$B_{op} = \|S_\Lambda\| = \|T_\Lambda\|^2 \quad , \quad A_{op} = \|U_0\|^{-2}$$

where U_0 is the unique solution of equation (8).

Proof. Via Lemma 4, equation (8) has a unique solution as U_0 such that

$$\begin{aligned} \|U_0\|^2 &= \inf\{\alpha > 0 \mid KK^* \leq \alpha T_\Lambda T_\Lambda^*\} \\ &= \inf\{\alpha > 0 \mid \|K^* f\|^2 \leq \alpha \|T_\Lambda^* f\|^2 \ , \ f \in H\}. \end{aligned}$$

Now, we have

$$\begin{aligned} A_{op} &= \sup\{A > 0 \mid A\|K^* f\|^2 \leq \|T_\Lambda^* f\|^2 \ , \ f \in H\} \\ &= \Big(\inf\{\alpha > 0 \mid \|K^* f\|^2 \leq \alpha \|T_\Lambda^* f\|^2 \ , \ f \in H\}\Big)^{-1} \\ &= \|U_0\|^{-2}. \end{aligned}$$

□

3.1 K-g-fusion dual and some identities

Definition 5. *Let Λ be a K-g-fusion frame for H. A g-fusion Bessel sequence $\widetilde{\Lambda}=(\widetilde{W_j},\widetilde{\Lambda_j},\widetilde{v_j})$ with $\widetilde{\Lambda_j}\in\mathcal{B}(H,H_j)$ is called a K-g-fusion dual of Λ if for each $f\in H$,*

$$Kf=\sum_{j\in\mathbb{J}} v_j\widetilde{v_j}\pi_{W_j}\Lambda_j^*\widetilde{\Lambda_j}\pi_{\widetilde{W_j}}f. \tag{9}$$

It is clear that a K-g-fusion dual is a QK-gf dual when Q is an identity. In this case, we can deduce that $\widetilde{\Lambda}$ is a K^*-g-fusion frame for H. Indeed, for each $f\in H$ we have

$$\begin{aligned}
\|Kf\|^4 &\leq \big|\langle\sum_{j\in\mathbb{J}} v_j\widetilde{v_j}\pi_{W_j}\Lambda_j^*\widetilde{\Lambda_j}\pi_{\tilde{W}_j}f,Kf\rangle\big|^2\\
&=\big|\sum_{j\in\mathbb{J}} v_j\widetilde{v_j}\langle\widetilde{\Lambda_j}\pi_{\widetilde{W_j}}f,\Lambda_j\pi_{W_j}Kf\rangle\big|^2\\
&\leq\Big(\sum_{j\in\mathbb{J}}\widetilde{v_j}^2\|\widetilde{\Lambda_j}\pi_{\widetilde{W_j}}f\|^2\Big)\Big(\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}Kf\|^2\Big)\\
&\leq B\|Kf\|^2\sum_{j\in\mathbb{J}}\widetilde{v_j}^2\|\widetilde{\Lambda_j}\pi_{\widetilde{W_j}}f\|^2,
\end{aligned}$$

where B is an upper bound of Λ.

Theorem 8. *Let Λ be a K-g-fusion frame for H with bounds A,B and K be with closed range. Then $\Big(K^*S_\Lambda^{-1}\pi_{S_\Lambda(\mathcal{R}(K))}W_j,\Lambda_j\pi_{W_j}\pi_{S_\Lambda(\mathcal{R}(K))}(S_\Lambda^{-1})^*K,v_j\Big)$ is a K-g-fusion dual of Λ (in this case, we say canonical K-g-fusion dual).*

Proof. We know that $S_\Lambda^{-1}S_\Lambda|_{\mathcal{R}(K)}=Id_{\mathcal{R}(K)}$. Then, we have for each $f\in H$,

$$\begin{aligned}
Kf&=S_\Lambda(S_\Lambda^{-1})^*Kf\\
&=S_\Lambda\pi_{S_\Lambda(\mathcal{R}(K))}(S_\Lambda^{-1})^*Kf\\
&=\sum_{j\in\mathbb{J}} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}\pi_{S_\Lambda(\mathcal{R}(K))}(S_\Lambda^{-1})^*Kf\\
&=\sum_{j\in\mathbb{J}} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}\pi_{S_\Lambda(\mathcal{R}(K))}(S_\Lambda^{-1})^*K\pi_{K^*S_\Lambda^{-1}\pi_{S_\Lambda(\mathcal{R}(K))}W_j}f.
\end{aligned}$$

On the other hand, we obtain by Remark 1, for all $f\in H$,

$$\begin{aligned}
&\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}\pi_{S_\Lambda(\mathcal{R}(K))}(S_\Lambda^{-1})^*K\pi_{K^*S_\Lambda^{-1}\pi_{S_\Lambda(\mathcal{R}(K))}W_j}f\|^2\\
&\quad=\Big\langle S_\Lambda\big((S_\Lambda^{-1})^*k\pi_{K^*S_\Lambda^{-1}\pi_{S_\Lambda(\mathcal{R}(K))}W_j}f\big),(S_\Lambda^{-1})^*K\pi_{K^*S_\Lambda^{-1}\pi_{S_\Lambda(\mathcal{R}(K))}W_j}f\Big\rangle\\
&\quad=\Big\langle K\pi_{K^*S_\Lambda^{-1}\pi_{S_\Lambda(\mathcal{R}(K))}W_j}f,(S_\Lambda^{-1})^*K\pi_{K^*S_\Lambda^{-1}\pi_{S_\Lambda(\mathcal{R}(K))}W_j}f\Big\rangle\\
&\quad\leq A^{-1}\|K^\dagger\|^2\|K\|^2\|f\|^2
\end{aligned}$$

and the proof is completed by Definition 5. $\square$

Remark 3. *If $K = Id_H$ in Theorem 8, we get a canonical g-fusion dual in [22].*

Let Λ be a K-g-fusion frame for H and $\widetilde{\Lambda}$ be a K-g-fusion dual of Λ. Suppose that $\mathbb{I}$ is a finite subset of $\mathbb{J}$ and we define

$$S_{\mathbb{I}} f = \sum_{j\in\mathbb{I}} v_j \widetilde{v_j} \pi_{W_j} \Lambda_j^* \widetilde{\Lambda_j} \pi_{\widetilde{W_j}} f, \qquad (\forall f \in H). \tag{10}$$

It is easy to check that $S_{\mathbb{I}} \in \mathcal{B}(H)$ is positive and

$$S_{\mathbb{I}} + S_{\mathbb{I}^c} = K.$$

Theorem 9. *Let $f \in H$, then*

$$\sum_{j\in\mathbb{I}} v_j \widetilde{v_j} \langle \widetilde{\Lambda}_j \pi_{\widetilde{W_j}} f, \Lambda_j \pi_{W_j} K f\rangle - \|S_{\mathbb{I}} f\|^2 = \sum_{j\in\mathbb{I}^c} v_j \widetilde{v_j} \overline{\langle \widetilde{\Lambda}_j \pi_{\widetilde{W_j}} f, \Lambda_j \pi_{W_j} K f\rangle} - \|S_{\mathbb{I}^c} f\|^2.$$

Proof. For each $f \in H$, we have

$$\begin{aligned}
\sum_{j\in\mathbb{I}} v_j \widetilde{v_j} \langle \widetilde{\Lambda}_j \pi_{\widetilde{W_j}} f, \Lambda_j \pi_{W_j} K f\rangle - \|S_{\mathbb{I}} f\|^2 &= \langle K^* S_{\mathbb{I}} f, f\rangle - \|S_{\mathbb{I}} f\|^2 \\
&= \langle K^* S_{\mathbb{I}} f, f\rangle - \langle S_{\mathbb{I}}^* S_{\mathbb{I}} f, f\rangle \\
&= \langle (K - S_{\mathbb{I}})^* S_{\mathbb{I}} f, f\rangle \\
&= \langle S_{\mathbb{I}^c}^* (K - S_{\mathbb{I}^c}) f, f\rangle \\
&= \langle S_{\mathbb{I}^c}^* K f, f\rangle - \langle S_{\mathbb{I}^c}^* S_{\mathbb{I}^c} f, f\rangle \\
&= \langle f, K^* S_{\mathbb{I}^c} f\rangle - \langle S_{\mathbb{I}^c} f, S_{\mathbb{I}^c} f\rangle \\
&= \sum_{j\in\mathbb{I}^c} v_j \widetilde{v_j} \overline{\langle \widetilde{\Lambda}_j \pi_{\widetilde{W_j}} f, \Lambda_j \pi_{W_j} k f\rangle} - \|S_{\mathbb{I}^c} f\|^2
\end{aligned}$$

and the proof is completed. □

Theorem 10. *Let Λ be a Parseval K-g-fusion frame for H. If $\mathbb{I} \subseteq \mathbb{J}$ and $E \subseteq \mathbb{I}^c$, then for each $f \in H$,*

$$\begin{aligned}
&\|\sum_{j\in\mathbb{I}\cup E} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 - \|\sum_{j\in\mathbb{I}^c\setminus E} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 \\
&\quad = \|\sum_{j\in\mathbb{I}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 - \|\sum_{j\in\mathbb{I}^c} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 \\
&\qquad + 2\,\mathrm{Re}\Big(\sum_{j\in E} v_j^2 \langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} K K^* f\rangle\Big).
\end{aligned}$$

Proof. Let

$$S_{\Lambda,\mathbb{I}} f := \sum_{j\in\mathbb{I}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f.$$

Therefore, $S_{\Lambda,\mathbb{I}} + S_{\Lambda,\mathbb{I}^c} = KK^*$. Hence,

$$\begin{aligned} S^2_{\Lambda,\mathbb{I}} - S^2_{\Lambda,\mathbb{I}^c} &= S^2_{\Lambda,\mathbb{I}} - (KK^* - S_{\Lambda,\mathbb{I}})^2 \\ &= KK^* S_{\Lambda,\mathbb{I}} + S_{\Lambda,\mathbb{I}} KK^* - (KK^*)^2 \\ &= KK^* S_{\Lambda,\mathbb{I}} - S_{\Lambda,\mathbb{I}^c} KK^*. \end{aligned}$$

Now, for each $f \in H$ we have

$$\|S^2_{\Lambda,\mathbb{I}} f\|^2 - \|S^2_{\Lambda,\mathbb{I}^c} f\|^2 = \langle KK^* S_{\Lambda,\mathbb{I}} f, f\rangle - \langle S_{\Lambda,\mathbb{I}^c} KK^* f, f\rangle.$$

Consequently, for $\mathbb{I} \cup E$ instead of $\mathbb{I}$:

$$\begin{aligned} &\|\sum_{j\in\mathbb{I}\cup E} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 - \|\sum_{j\in\mathbb{I}^c\setminus E} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 \\ &= \sum_{j\in\mathbb{I}\cup E} v_j^2 \langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle - \sum_{j\in\mathbb{I}^c\setminus E} v_j^2 \overline{\langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle} \\ &= \sum_{j\in\mathbb{I}} v_j^2 \langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle - \sum_{j\in\mathbb{I}^c} v_j^2 \overline{\langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle} \\ &\quad + 2\,\mathrm{Re}\Big(\sum_{j\in E} v_j^2 \langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle\Big) \\ &= \|\sum_{j\in\mathbb{I}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 - \|\sum_{j\in\mathbb{I}^c} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 \\ &\quad + 2\,\mathrm{Re}\Big(\sum_{j\in E} v_j^2 \langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle\Big). \end{aligned}$$

□

Theorem 11. *Let Λ be a Parseval K-g-fusion frame for H. If $\mathbb{I} \subseteq \mathbb{J}$ and $E \subseteq \mathbb{I}^c$, then for any $f \in H$,*

$$\begin{aligned} &\|\sum_{j\in\mathbb{I}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 + \mathrm{Re}\Big(\sum_{j\in\mathbb{I}^c} v_j^2 \langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle\Big) \\ &= \|\sum_{j\in\mathbb{I}^c} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f\|^2 + \mathrm{Re}\Big(\sum_{j\in\mathbb{I}} v_j^2 \langle \Lambda_j \pi_{W_j} f, \Lambda_j \pi_{W_j} KK^* f\rangle\Big) \geq \frac{3}{4}\|KK^* f\|^2. \end{aligned}$$

Proof. In Theorem 10, we showed that

$$S^2_{\Lambda,\mathbb{I}} - S^2_{\Lambda,\mathbb{I}^c} = KK^* S_{\Lambda,\mathbb{I}} - S_{\Lambda,\mathbb{I}^c} KK^*.$$

Therefore,

$$S^2_{\Lambda,\mathbb{I}} + S^2_{\Lambda,\mathbb{I}^c} = 2\Big(\frac{KK^*}{2} - S_{\Lambda,\mathbb{I}}\Big)^2 + \frac{(KK^*)^2}{2} \geq \frac{(KK^*)^2}{2}.$$

Thus,

$$\begin{aligned} KK^*S_{\Lambda,\mathbb{I}} + S^2_{\Lambda,\mathbb{I}^c} + (KK^*S_{\Lambda,\mathbb{I}} + S^2_{\Lambda,\mathbb{I}^c})^* &= KK^*S_{\Lambda,\mathbb{I}} + S^2_{\Lambda,\mathbb{I}^c} + S_{\Lambda,\mathbb{I}}KK^* + S^2_{\Lambda,\mathbb{I}^c} \\ &= KK^*(S_{\Lambda,\mathbb{I}} + S_{\Lambda,\mathbb{I}^c}) + S^2_{\Lambda,\mathbb{I}} + S^2_{\Lambda,\mathbb{I}^c} \\ &\geq \frac{3}{2}(kk^*)^2. \end{aligned}$$

Now, we obtain for any $f \in H$,

$$\begin{aligned} &\|\sum_{j\in\mathbb{I}} v_j^2 \pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j} f\|^2 + \operatorname{Re}\Big(\sum_{j\in\mathbb{I}^c} v_j^2 \langle \Lambda_j\pi_{W_j} f, \Lambda_j\pi_{W_j} KK^* f\rangle\Big) \\ &= \|\sum_{j\in\mathbb{I}^c} v_j^2 \pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j} f\|^2 + \operatorname{Re}\Big(\sum_{j\in\mathbb{I}} v_j^2 \langle \Lambda_j\pi_{W_j} f, \Lambda_j\pi_{W_j} KK^* f\rangle\Big) \\ &= \frac{1}{2}\big(\langle KK^*S_{\Lambda,\mathbb{I}} f, f\rangle + \langle S^2_{\Lambda,\mathbb{I}^c} f, f\rangle + \langle f, KK^*S_{\Lambda,\mathbb{I}} f\rangle + \langle f, S^2_{\Lambda,\mathbb{I}^c} f\rangle\big) \\ &\geq \frac{3}{4}\|KK^* f\|^2. \end{aligned}$$

□

4 Perturbation of K-g-Fusion Frames

Perturbation of frames has been discussed by Casazza and Christensen in [5]. In this section, we present some perturbation of K-g-fusion frames.

Theorem 12. *Let Λ be a K-g-fusion frame for H with bounds A, B and $\{\Theta_j \in \mathcal{B}(H, H_j)\}_{j\in\mathbb{J}}$ be a sequence of operators such that for each $f \in H$,*

$$\|\big(v_j\Lambda_j\pi_{W_j} - z_j\Theta_j\pi_{Z_j}\big) f\| \leq \lambda_1\|v_j\Lambda_j\pi_{W_j} f\| + \lambda_2\|z_j\Theta_j\pi_{Z_j} f\| + \varepsilon v_j\|K^* f\|,$$

where $0 \leq \lambda_1, \lambda_2 < 1$ and $\varepsilon > 0$ such that

$$v^2 := \sum_{j\in\mathbb{J}} v_j^2 < \infty$$

and

$$\varepsilon < \frac{(1-\lambda_1)\sqrt{A}}{v}.$$

Then $\Theta := (Z_j, \Theta_j, z_j)$ is a k-g-fusion frame for H with bounds

$$\Big(\frac{(1-\lambda_1)\sqrt{A} - v\varepsilon}{1+\lambda_2}\Big)^2 \quad \textit{and} \quad \Big(\frac{(1+\lambda_1)\sqrt{B} + v\varepsilon\|K\|}{1-\lambda_2}\Big)^2.$$

Proof. Let $f \in H$. We can write

$$\begin{aligned}\Big(\sum_{j\in\mathbb{J}} z_j^2\|\Theta_j\pi_{Z_j}f\|^2\Big)^{\frac{1}{2}} &= \Big(\sum_{j\in\mathbb{J}} \|z_j\Theta_j\pi_{Z_j}f + v_j\Lambda_j\pi_{W_j}f - v_j\Lambda_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}}\\ &\le \Big(\sum_{j\in\mathbb{J}}(1+\lambda_1)\|v_j\Lambda_j\pi_{W_j}f\| + \lambda_2\|z_j\Theta_j\pi_{Z_j}f\| + \varepsilon v_j\|K^*f\|\Big)^{\frac{1}{2}}\\ &\le (1+\lambda_1)\Big(\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}} + \lambda_2\Big(\sum_{j\in\mathbb{J}} z_j^2\|\Theta_j\pi_{Z_j}f\|^2\Big)^{\frac{1}{2}}\\ &\quad + v\varepsilon\|K^*f\|.\end{aligned}$$

Therefore,

$$\sum_{j\in\mathbb{J}} z_j^2\|\Theta_j\pi_{Z_j}f\|^2 \le \Big(\frac{(1+\lambda_1)\sqrt{B} + v\varepsilon\|K\|}{1-\lambda_2}\Big)^2\|f\|^2.$$

For the lower bound, we have

$$\begin{aligned}\Big(\sum_{j\in\mathbb{J}} z_j^2\|\Theta_j\pi_{Z_j}f\|^2\Big)^{\frac{1}{2}} &= \Big(\sum_{j\in\mathbb{J}} \|z_j\Theta_j\pi_{Z_j}f + v_j\Lambda_j\pi_{W_j}f - v_j\Lambda_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}}\\ &\ge \Big(\sum_{j\in\mathbb{J}}(1-\lambda_1)\|v_j\Lambda_j\pi_{W_j}f\| - \lambda_2\|z_j\Theta_j\pi_{Z_j}f\| - \varepsilon v_j\|K^*f\|\Big)^{\frac{1}{2}}\\ &\ge (1-\lambda_1)\Big(\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}} - \lambda_2\Big(\sum_{j\in\mathbb{J}} z_j^2\|\Theta_j\pi_{Z_j}f\|^2\Big)^{\frac{1}{2}}\\ &\quad - v\varepsilon\|K^*f\|.\end{aligned}$$

Hence,

$$\sum_{j\in\mathbb{J}} z_j^2\|\Theta_j\pi_{Z_j}f\|^2 \ge \Big(\frac{(1-\lambda_1)\sqrt{A} - v\varepsilon}{1+\lambda_2}\Big)^2\|K^*f\|^2.$$

□

Theorem 13. *Let Λ be a K-g-fusion frame for H with bounds A, B and $\{\Theta_j \in \mathcal{B}(H, H_j)\}_{j\in\mathbb{J}}$ be a sequence of operators. If there exists a constant $0 < R < A$ such that*

$$\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f - \Theta_j\pi_{W_j}f\|^2 \le R\|K^*f\|^2$$

for all $f \in H$, then $\Theta := (W_j, \Theta_j, v_j)$ is a k-g-fusion frame for H with bounds

$$(\sqrt{A} - \sqrt{R})^2 \quad \textit{and} \quad (\|K\|\sqrt{R} + \sqrt{B})^2.$$

Proof. Let $f \in H$. By the triangle and Minkowski inequality, we can write

$$\begin{aligned}\Big(\sum_{j\in\mathbb{J}} v_j^2\|\Theta_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}} &\le \Big(\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f - \Theta_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}} + \Big(\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}}\\ &\le (\|K\|\sqrt{R} + \sqrt{B})\|f\|.\end{aligned}$$

Also

$$\Big(\sum_{j\in\mathbb{J}} v_j^2\|\Theta_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}} \geq \Big(\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}} - \Big(\sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f - \Theta_j\pi_{W_j}f\|^2\Big)^{\frac{1}{2}}$$
$$\geq (\sqrt{A} - \sqrt{R})\|K^*f\|.$$

Thus, these complete the proof. □

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