Bulletin of the *Transilvania* University of Braşov • Vol 13(62), No. 2 - 2020
Series III: Mathematics, Informatics, Physics, 677-682
https://doi.org/10.31926/but.mif.2020.13.62.2.23

ON REPRESENTATION OF A FIRST DEGREE SPLINE FUNCTION

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Abstract

A presentation of related global representations for the first degree spline function is given, revisiting some known results. The given global representations are deduced one from another by simple transformations.

2000 *Mathematics Subject Classification:* 26A99, 41A15.
Key words: first degree spline, spline representation

1 Introduction

In this paper we give a presentation of related representations for the first degree spline function, revisiting some known results. A spline function may be represented by local and / or global representation. As an example, the local representation of a polynomial spline function is

$$s(x) = \sum_i s_i(x)\mathbf{1}_{[x_i,x_{i+1}]}(x),$$

where $s_i(x)$ is a polynomial giving the values of s in the interval $[x_i, x_{i+1}]$ and $\mathbf{1}_A$ is the indicator function of the set A. For the first degree interpolating spline

$$s_i(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i), \tag{1}$$

and s satisfies the interpolation constraints $s(x_i) = y_i$, for any i. A global representation formula does not take into account the position of the argument relative to nodes.

The given global representations are deduced one from another by simple transformations.

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2 First degree spline given by global expressions

Let be the mesh $x_0 < x_1 < \ldots < x_n$ and the real numbers $y_0, y_1, \ldots, y_n$. We shall use the notations

$$\begin{aligned} \Delta y_i &= y_{i+1} - y_i; \\ h_i &= x_{i+1} - x_i. \end{aligned}$$

The starting point is the following folkloric result:

Theorem 1. *The first degree spline function satisfying the interpolation conditions $s(x_i) = y_i$, for all $i \in \{0, 1, \ldots, n\}$ is given by*

$$s(x) = y_0 + \sum_{i=0}^{n-1} p_i (x - x_i)_+, \quad x \in \mathbb{R}, \tag{2}$$

with $p_0 = \frac{\Delta y_0}{h_0}$ and $p_i = \frac{\Delta y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}}$, $i \in \{1, 2, \ldots, n-1\}$.

Above, $(\cdot)_+$ is the usual first degree truncated polynomial, $(x)_+ = \begin{cases} x & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$. We give a short elementary proof.

Proof. Inductively we compute the coefficients $(p_i)_i$ so that the interpolation constraints to be satisfied and so that $s|_{[x_i, x_{i+1})}$ to be a first degree polynomial.

If $x \in [x_0, x_1)$ then

$$s(x) = y_0 + p_0(x - x_0).$$

The interpolation constrain $s(x_1) = y_1$ implies $p_0 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h_0}$.

If $x \in [x_1, x_2)$ then $s(x) = y_0 + \frac{\Delta y_0}{h_0}(x - x_0) + p_1(x - x_1)$. The interpolation constrain $s(x_2) = y_2$ implies

$$y_0 + \frac{\Delta y_0}{h_0}(x_2 - x_0) + p_1(x_2 - x_1) = y_2$$

and consequently

$$y_0 + \frac{\Delta y_0}{h_0}(h_0 + h_1) + p_1 h_1 = y_2 \quad \Leftrightarrow \quad y_1 + \frac{\Delta y_0}{h_0} h_1 + p_1 h_1 = y_2,$$

from where $p_1 = \frac{y_2 - y_1}{h_1} - \frac{\Delta y_0}{h_0} = \frac{\Delta y_1}{h_1} - \frac{\Delta y_0}{h_0}$. Also we find

$$s(x) = y_0 + \frac{\Delta y_0}{h_0}(x - x_0) + \left(\frac{\Delta y_1}{h_1} - \frac{\Delta y_0}{h_0} \right)(x - x_1) =$$

$$= y_0 + \frac{\Delta y_0}{h_0} h_0 + \frac{\Delta y_1}{h_1}(x - x_1) = y_1 + \frac{\Delta y_1}{h_1}(x - x_1), \qquad x \in [x_1, x_2),$$

which is consistent with (1), for $i = 1$.

We suppose that for any $j \in \{1, 2, \ldots, i-1\}$ the following relations are valid

$$\begin{aligned} p_{j-1} &= \frac{\Delta y_j}{h_j} - \frac{\Delta y_{j-1}}{h_{j-1}}, \\ s(x) &= y_{j-1} + \frac{\Delta y_{j-1}}{h_{j-1}}(x - x_{j-1}), \qquad x \in [x_{j-1}, x_j). \end{aligned}$$

If $x \in [x_i, x_{i+1})$ then $s(x) = y_{i-1} + \frac{\Delta y_{i-1}}{h_{i-1}}(x - x_{i-1}) + p_i(x - x_i)$. The interpolation constrain $s(x_{i+1}) = y_{i+1}$ implies

$$y_{i-1} + \frac{\Delta y_{i-1}}{h_{i-1}}(x_{i+1} - x_{i-1}) + p_i(x_{i+1} - x_i) = y_{i+1}$$

and then

$$y_i + \frac{\Delta y_{i-1}}{h_{i-1}} h_i + p_i h_i = y_{i+1} \quad \Rightarrow \quad p_i = \frac{y_{i+1} - y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}} = \frac{\Delta y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}}.$$

For $x \in [x_i, x_{i+1})$ the function $s(x)$ becomes

$$s(x) = y_{i-1} + \frac{\Delta y_{i-1}}{h_{i-1}}(x - x_{i-1}) + \left(\frac{\Delta y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}}\right)(x - x_i) = y_i + \frac{\Delta y_i}{h_i}(x - x_i). \qquad \square$$

Using the relation $x_+ = \frac{1}{2}(|x| + x)$, from (2) we obtain the equality

$$s(x) = \alpha + \beta x + \frac{\Delta y_0}{h_0}|x - x_0| + \frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\Delta y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}}\right)|x - x_i|,$$

where

$$\alpha = y_0 - \frac{\Delta y_0}{h_0}x_0 - \frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\Delta y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}}\right)x_i;$$

and

$$\beta = \frac{\Delta y_0}{2h_0} + \frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\Delta y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}}\right) = \frac{\Delta y_{n-1}}{2h_{n-1}}.$$

The coefficient α may be further simplified

$$\alpha = y_0 - \frac{\Delta y_0}{h_0}x_0 - \frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\Delta y_i}{h_i}x_i - \frac{\Delta y_{i-1}}{h_{i-1}}(x_{i-1} + h_{i-1})\right) =$$

$$= \frac{1}{2}\left(y_0 + y_{n-1} - \frac{\Delta y_{n-1}}{h_{n-1}}x_{n-1}\right).$$

It results the global representation

$$s(x) = \frac{1}{2}\left(y_0 + y_{n-1} - \frac{\Delta y_{n-1}}{h_{n-1}}x_{n-1}\right) + \frac{\Delta y_{n-1}}{2h_{n-1}}x + \tag{3}$$

$$+\frac{\Delta y_0}{h_0}|x-x_0|+\frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\Delta y_i}{h_i}-\frac{\Delta y_{i-1}}{h_{i-1}}\right)|x-x_i|,\qquad x\in\mathbb{R}.$$

If we restrict to the interval $[x_0,x_n]$, then $|x-x_0|=x-x_0$ and consequently

$$s(x)=\frac{1}{2}\left(y_0+y_{n-1}-\frac{\Delta y_0}{h_0}x_0-\frac{\Delta y_{n-1}}{h_{n-1}}x_{n-1}\right)+\left(\frac{\Delta y_0}{2h_0}+\frac{\Delta y_{n-1}}{2h_{n-1}}\right)x+$$

$$+\frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\Delta y_i}{h_i}-\frac{\Delta y_{i-1}}{h_{i-1}}\right)|x-x_i|.$$

Simplifying the first term we obtain the result of KOLIHA, [1, Th. 1]

$$s(x)=\frac{y_0x_1-y_1x_0}{2h_0}+\frac{y_{n-1}x_n-y_nx_{n-1}}{2h_{n-1}}+\left(\frac{\Delta y_0}{2h_0}+\frac{\Delta y_{n-1}}{2h_{n-1}}\right)x+ \tag{4}$$

$$+\frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{\Delta y_i}{h_i}-\frac{\Delta y_{i-1}}{h_{i-1}}\right)|x-x_i|,\qquad x\in[x_0,x_n].$$

If the nodes are equidistant $x_i=x_0+ih,\ i\in\{0,1,\ldots,n\}$, where $h=\frac{x_n-x_0}{n}$, then from (4) we find

$$s(x)=\frac{1}{2h}(y_0x_1-y_1x_0+y_{n-1}x_n-y_nx_{n-1})+\frac{x}{2h}(y_1-y_0+y_n-y_{n-1})+$$

$$+\frac{1}{h}\sum_{i=1}^{n-1}\left(\frac{y_{i-1}+y_{i+1}}{2}-y_i\right)|x-x_i|.$$

Let $c\in\mathbb{R}$. Adding and subtracting the following two terms

$$\tfrac{y_0+ny_{n-1}-(n-1)y_n-2c}{2nh}|x-x_n|=\tfrac{y_0+ny_{n-1}-(n-1)y_n-2c}{2nh}(x_n-x)$$

$$\tfrac{y_n+ny_1-(n-1)y_0-2c}{2nh}|x-x_0|=\tfrac{y_n+ny_1-(n-1)y_0-2c}{2nh}(x-x_0)$$

after reducing the terms it results

$$s(x)=c+\frac{y_n+ny_1-(n-1)y_0-2c}{2nh}|x-x_0|+\frac{1}{h}\sum_{i=1}^{n-1}\left(\frac{y_{i-1}+y_{i+1}}{2}-y_i\right)|x-x_i|+$$

$$+\frac{y_0+ny_{n-1}-(n-1)y_n-2c}{2nh}|x-x_n|.$$

Denoting $\eta_0=\frac{y_n+ny_1-(n-1)y_0-2c}{2nh},\ \eta_n=\frac{y_0+ny_{n-1}-(n-1)y_n-2c}{2nh}$ and $\eta_i=\frac{1}{h}\left(\frac{y_{i-1}+y_{i+1}}{2}-y_i\right),\ i\in\{1,2,\ldots,n-1\}$, the above becomes the result of TODA, [3, Th. B]

$$s(x)=c+\sum_{i=0}^{n}\eta_i|x-x_i|,\qquad x\in[x_0,x_n]. \tag{5}$$

The parameter c matters only outside of the interval $[x_0, x_n]$.

It was observed in [3] that if $(y_i)_i$ are the values of a convex function f and if

$$c \leq \min\{y_n + ny_1 - (n-1)y_0, y_0 + ny_{n-1} - (n-1)y_n\}$$

then the coefficients η_i are nonnegative for any i. As a consequence the first degree spline function s given by (5) is convex in $\mathbb{R}$.

The convergence property $\lim_{n\to\infty} s(x) = f(x)$ is proven in [3], too.

In the case of a convex function the existence of a function expressed as in (5) and the corresponding convergence property is used in [2, Lm. 3.7] to characterize an entropy solution for scalar conservation laws.

References

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