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# NEW FIXED POINT RESULTS ABOUT *F*-CONTRACTIONS IN A COMPLETE METRIC SPACE

**Alexandrina Maria PROCA**[*,1]

**Abstract**

In this paper, we extend the results of Wardowski by applying some new conditions on the self map on a complete metric space, concerning the $F$-contractions defined by Wardowski. We present some fixed point results of Wardowski type. An example is given to demonstrate the novelty of our work.

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## 1 Introduction and preliminaries

It is known that the contraction mapping principle formulated by Banach in 1920 in his Ph.D dissertation and published in 1922 in [1] is one of the most important theorems in clasical functional analysis. Because of its importance in mathematical theory many autors gave generalisations [2]-[15] of it in many directions. One of the most well-known generalisation of the Banach contraction principle is the Wardowski fixed point theorem [14].

Following this direction of research, in this paper, we well present some fixed point results of Wardowski type for self-mappings on complete metric spaces. Moreover, an example is given to illustrate the usability of these results.

**Definition 1.** *A self-map $T$ on a metric space $(X, d)$ is said to be an $F-$contraction, if there exists $F \in \mathcal{F}$ and $\tau \in (0, \infty)$ such that*

$$[d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F(d(x, y))], \quad (\forall)\, x, y \in X,$$

*where $\mathcal{F}$ is the family of all functions $F : (0, \infty) \to \mathbb{R}$ such that*

*$(F1)$ $F$ is strictly increasing, i.e. for all $x, y \in \mathbb{R}_+$, $x < y \Rightarrow F(x) < F(y)$;*

*$(F2)$ For each sequence $\{\alpha_n\}$ of positive numbers, $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;*

*$(F3)$ There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0_+} \alpha^k F\left(\alpha\right) = 0$.*

[1*] *Corresponding author*, Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: alexproca@unitbv.ro

Every $F$-contraction is contractive and necessarily continuous map.

**Theorem 1.** *[14] Let $(X,d)$ be a metric space and $T : X \to X$ be an $F-$contraction. Then $T$ has a unique fixed point $x^* \in X$ and $\{T^n x\} \to x^*$.*

Later, Wardowski and Van Dung [15] introduced the concept of an $F-$weak contraction as follows.

**Definition 2.** *Let $(X,d)$ be a metric space. A mapping $T : X \to X$ is said to be an $F-$ weak contraction on $(X,d)$ if there exists $F \in \mathfrak{F}$ and $\tau > 0$ such that, $(\forall)\, x, y \in X$*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F\left(d(Tx,Ty)\right) \leq F(M(x,y)),$$

*where*

$$M(x,y) = \max\left\{d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right), \frac{d\left(x,Ty\right)+d\left(y,Tx\right)}{2}\right\}.$$

**Theorem 2.** *[15] Let $(X,d)$ be a complete metric space and $T : X \to X$ be a $F-$ weak contraction. If $T$ or $F$ is continuous, then $T$ has a unique fixed point $x^* \in X$ and for all $x \in X$, the sequence $\{T^n x\}$ converges to $x^*$.*

Dung and Hang introduced the notationof a modified generalised $F$-contraction and proved new fixed point theorem. They generalised $F-$ weak contraction as folows.

**Definition 3.** *[5] Let $(X,d)$ be a metric space. A mapping $T : X \to X$ is said to be a generalised $F$-contraction on $(X,d)$ if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\forall x, y \in X, [d(Tx,Ty) > 0 \implies \tau + F(d(Tx,Ty)) \leq F(N(x,y))]$$

*where*

$$\begin{aligned} N(x,y) \;&=\; \max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ &\frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\Big\}\end{aligned}$$

In 2016 Piri and Kumam replaced condition $(F_3)$ in the definition of $F$-contraction introduced by Wardowski with

$(F_3')$ $F$ is continuous $(0,\infty)$.

Using the family of all functions which satisfy conditions $(F_1,\ F_2,\ F_3')$, they proved some Wardowski and Suzuki type fixed point theorem. [9]

## 2 Main results

The aim of this paper is to give another type of generalisation for Wardowski fixed point theorem. We give an example to show that our result is a proper extension of clasical Wardowski fixed point theorem.

**Definition 4.** *Let $(X,d)$ be a metric space. $T : X \to X$ is a Picard operator if and only if:*
(1) $(\exists!)\, x^* \in X$ *a fixed point for* $T$,
(2) $\{T^n x\}$ *converges to* $x^*$, $(\forall)\, x \in X$.

**Theorem 3.** *Let $(X,d)$ be a complete metric space. Let $T : X \to X$ and $\tau > 0$ be such*

$$(\forall)\, x, y \in X,\ \ [d\,(Tx,Ty) > 0 \Rightarrow \tau + F\,(d\,(Tx,Ty)) \le F\,(M^*\,(x,y))] \qquad (1)$$

*where*

1. *$F : (0,\infty) \to \mathbb{R}$ is strictly increasing: $(\forall)\, x,y \in (0,\infty)\,,\ x < y \Rightarrow F(x) < F\,(y)$ ,and*

$$\begin{aligned} M^*\,(x,y) \;=\; & \max\{d(x,y) + |d(x,Tx) - d(y,Ty)|\,, \\ & d(x,Tx) + |d(y,Ty) - d(x,y)|\,, \\ & d(y,Ty) + |d(x,y) - d(x,Tx)|\,, \\ & \left.\frac{d(x,Ty) + d(y,Tx) + |d(x,Tx) - d(y,Ty)|}{2}\right\}. \end{aligned} \qquad (2)$$

2. *There exists $\tau > 0$, such that $\tau + \liminf_{u\to u_0} F(u) > \limsup_{u\to u_0} F(u)$, for every $u_0 > 0$.*

*Then $T$ is a Picard operator.*

*Proof.* Let $x_0 \in X$. Put $x_{n+1} = T^n x_0$, $x_0 \in X$, for all $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then $x_{n+1} = Tx_n = x_n$. That is $x_n$ is a fixed point of $T$.

Now, we suppose that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$. Then $d(x_n, x_{n+1}) > 0$, $(\forall)\, n \in \mathbb{N}$. We denote by

$$d_n = d\,(x_n, x_{n+1})\,, \quad (\forall)\, n \in \mathbb{N}. \qquad (3)$$

If we put $x = x_n$, $y = x_{n+1}$ in (1), we deduce:

$$\begin{aligned} \tau + F\,(d\,(Tx_n, Tx_{n+1})) &= \tau + F\,(d\,(x_{n+1}, x_{n+2})) \\ &= \tau + F\,(d_{n+1}) \le F\,(M^*\,(x_n, x_{n+1}))\,, \end{aligned} \qquad (4)$$

where

$$\begin{aligned} M^*(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}) + |d(x_n, Tx_n) - d(x_{n+1}, Tx_{n+1})|, \\ &\quad d(x_n, Tx_n) + |d(x_{n+1}, Tx_{n+1}) - d(x_n, x_{n+1})|, \\ &\quad d(x_{n+1}, Tx_{n+1}) + |d(x_n, x_{n+1}) - d(x_n, Tx_n)|, \\ &\quad \left.\frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n) + |d(x_n, Tx_n) - d(x_{n+1}, Tx_{n+1})|}{2}\right\} \\ &= \max\{d_n + |d_n - d_{n+1}|; d_n + |d_{n+1} - d_n|; d_{n+1} + |d_n - d_n|; \qquad (5) \\ &\quad \left.\frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1}) + |d_n - d_{n+1}|}{2}\right\} \\ &= \max\{d_n + |d_n - d_{n+1}|; d_{n+1}; \\ &\quad \left.\frac{d(x_n, x_{n+2}) + |d_n - d_{n+1}|}{2}\right\}. \end{aligned}$$

If $d_n < d_{n+1}$, $(\forall)\, n \in \mathbb{N}$,

$$M^*(x_n, x_{n+1}) = \max\left\{d_{n+1}; \frac{d(x_n, x_{n+2}) + d_{n+1} - d_n}{2}\right\}.$$

Using the triangle inequality

$$\frac{d(x_n, x_{n+2}) + d_{n+1} - d_n}{2} \leq \frac{d_n + d_{n+1} + d_{n+1} - d_n}{2} = d_{n+1}$$

so

$$M^*(x_n, x_{n+1}) = d_{n+1}.$$

From (4) we deduce

$$\tau + F(d_{n+1}) \leq F(d_{n+1}),$$

which is false for $\tau > 0$.

So, $d_n \geq d_{n+1}$, $(\forall)\, n \in \mathbb{N}$ and from (5)

$$\begin{aligned} M^*(x_n, x_{n+1}) &= \max\{d_n + d_n - d_{n+1}; d_{n+1}; \\ &\quad \left.\frac{d(x_n, x_{n+2}) + d_n - d_{n+1}}{2}\right\}. \end{aligned}$$

But, $d_n + d_n - d_{n+1} \geq d_n$, $d_{n+1} \leq d_n$ and

$$\frac{d(x_n, x_{n+2}) + d_n - d_{n+1}}{2} \leq \frac{d_n + d_{n+1} + d_n - d_{n+1}}{2} = d_n,$$

then

$$M^*(x_n, x_{n+1}) = 2d_n - d_{n+1}. \qquad (6)$$

By the assumption of the Theorem 3, specialy from relation (1) we have the following result:

$$\tau + F(d_{n+1}) \leq F(2d_n - d_{n+1}). \qquad (7)$$

We known that $d_n \geq d_{n+1}$ and $d_n > 0$, $(\forall)\, n \in \mathbb{N}$ so $\{d_n\}$ is convergent. Let now $d = \lim\limits_{n\to\infty} d_n$, and we suppose that $d > 0$. Taking the limit as $n \to \infty$, we get

$$d_{n+1} \searrow d$$

and

$$2d_n - d_{n+1} \searrow d.$$

Because $F$ is strictly increasing,

$$\tau + F\,(d+0) \leq F\,(d+0)\,. \tag{8}$$

The contradiction obtained shows that

$$d_n \searrow 0 \tag{9}$$

We claim now, that $\{x_n\}$ is a Cauchy sequence on $(X,d)$ wich is complete metric space. Suppose, on the contrary, that there exist $\varepsilon > 0$ and sequences $\{n(k)\}\,, \{m(k)\}$ of positive integers such that $n(k) > m(k) > k$, and

$$d\left(x_{n(k)}, x_{m(k)}\right) > \varepsilon, \;\; (\forall)\, k \geq 1 \tag{10}$$

$$d\left(x_{n(k)-1}, x_{m(k)}\right) \leq \varepsilon, \;\; (\forall)\, k \geq 1. \tag{11}$$

Using the triangle inequality and (10) and (11), we get

$$\begin{aligned} \varepsilon &< d\left(x_{n(k)}, x_{m(k)}\right) \\ &\leq d\left(x_{n(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{m(k)}\right). \end{aligned}$$

Taking the limit as $k \to \infty$ in this inequality and using (9) we deduce

$$d\left(x_{n(k)}, x_{m(k)}\right) \searrow \varepsilon. \tag{12}$$

But

$$d\left(x_{n(k)-1}, x_{m(k)-1}\right) \leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d\left(x_{m(k)}, x_{m(k)-1}\right)$$

so

$$\left|d(x_{n(k)-1}, x_{m(k)-1}) - d(x_{n(k)}, x_{m(k)})\right| \leq d(x_{n(k)-1}, x_{n(k)}) + d\left(x_{m(k)}, x_{m(k)-1}\right). \tag{13}$$

Taking the limit as $k \to \infty$ in (13) we deduce

$$\lim_{k\to\infty} d\left(x_{n(k)-1}, x_{m(k)-1}\right) = \varepsilon. \tag{14}$$

In relation (1), we put $x = x_{n(k)-1}$, $y = x_{m(k)-1}$ and obtain

$$\tau + F\left(d\left(x_{n(k)}, x_{m(k)}\right)\right) \leq F(M^*\left(x_{n(k)-1}, x_{m(k)-1}\right)), \tag{15}$$

where

$$M^*\left(x_{n(k)-1}, x_{m(k)-1}\right) =$$

$$\begin{aligned} = \quad & \max\left\{d(x_{n(k)-1},x_{m(k)-1})+\left|d\left(x_{n(k)-1},Tx_{n(k)-1}\right)-d\left(x_{m(k)-1},Tx_{m(k)-1}\right)\right|;\right.\\ & d(x_{n(k)-1},Tx_{n(k)-1})+\left|d\left(x_{m(k)-1},Tx_{m(k)-1}\right)-d\left(x_{n(k)-1},x_{m(k)-1}\right)\right|;\\ & d(x_{m(k)-1},Tx_{m(k)-1})+\left|d\left(x_{n(k)-1},x_{m(k)-1}\right)-d\left(x_{n(k)-1},Tx_{n(k)-1}\right)\right|;\\ & \frac{1}{2}\left[d(x_{n(k)-1},Tx_{m(k)-1})+d(x_{m(k)-1},Tx_{n(k)-1})+\right.\\ & \left.\left.+\left|d\left(x_{n(k)-1},Tx_{n(k)-1}\right)-d\left(x_{m(k)-1},Tx_{m(k)-1}\right)\right|\right]\right\} \qquad (16)\\ = \quad & \max\left\{d(x_{n(k)-1},x_{m(k)-1})+\left|d\left(x_{n(k)-1},x_{n(k)}\right)-d\left(x_{m(k)-1},x_{m(k)}\right)\right|;\right.\\ & d(x_{n(k)-1},x_{n(k)})+\left|d\left(x_{m(k)-1},x_{m(k)}\right)-d\left(x_{n(k)-1},x_{m(k)-1}\right)\right|;\\ & d(x_{m(k)-1},x_{m(k)})+\left|d\left(x_{n(k)-1},x_{m(k)-1}\right)-d\left(x_{n(k)-1},x_{n(k)}\right)\right|;\\ & \frac{1}{2}\left[d(x_{n(k)-1},x_{m(k)})-d(x_{m(k)-1},x_{n(k)})+\right.\\ & \left.\left.+\left|d\left(x_{n(k)-1},x_{n(k)}\right)-d\left(x_{m(k)-1},x_{m(k)}\right)\right|\right]\right\}. \end{aligned}$$

We will see that for $k\to\infty$, all the four terms which are in $M^*$ tends to $\varepsilon$, be taking into account relation (9).

Taking the limit $k\to\infty$ in (16) and using (12) and (14) we have $M^*(x_{n(k)-1},x_{m(k)-1})\to\varepsilon$, then it follows that

$$\limsup_{n\to\infty} F(M^*(x_{n(k)-1},x_{m(k)-1}))\le F(\varepsilon+0)$$

Also, taking the limit $k\to\infty$ in (15), we have

$$\tau+\liminf_{k\to\infty} F((x_{n(k)},x_{m(k)}))\le \liminf_{k\to\infty} F(M^*(x_{n(k)-1},x_{m(k)-1}))\le$$

$$\le \limsup_{k\to\infty} F(M^*(x_{n(k)-1},x_{m(k)-1}))\le F(\varepsilon+0),\ \text{so}$$

$$\tau+F(\varepsilon+0)\le F(\varepsilon+0) \qquad (17)$$

This is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence and $(X,d)$ is a complete metric space, so $\{x_n\}$ converges to some point $x^*\in X$.

We shall prove that $x^*$ is a fixed point of $T$. If there exist a sequence $\{l(n)\}_{n\in\mathbb{N}}$ of a natural numbers such that $x_{l(n)+1}=Tx_{l(n)}=Tx^*$, then $\lim_{n\to\infty}x_{l(n)+1}=x^*$, so $Tx^*=x^*$. Otherwise, there exist $N\in\mathbb{N}$, such that $x_{n+1}=Tx_n\ne Tx^*,\forall n\ge N$. Assume that $Tx^*\ne x^*$. For the assumption of the Theorem 3, for $x=x_n, y=x^*$ we obtain

$$\tau+F(d(x_{n+1},Tx^*))\le F(M^*(x_n,x^*)) \qquad (18)$$

where

$$\begin{aligned} M^*(x_n,x^*) \quad = \quad & \max\{d(x_n,x^*)+|d(x_n,x_{n+1})-d(x^*,Tx^*)|;\\ & d(x_n,x_{n+1})+|d(x^*,Tx^*)-d(x_n,x^*)|;\\ & d(x^*,Tx^*)+|d(x_n,x^*)-d(x_n,x_{n+1})|;\\ & \frac{1}{2}[d(x_n,Tx^*)+d(x^*,x_{n+1})+ \qquad (19)\\ & +|d(x_n,x_{n+1})-d(x^*,Tx^*)|]\} \end{aligned}$$

For $x_n \to x^*$ it clear that $M^*(x_n, x^*) \to d(x^*, Tx^*)$, and

$$\tau + \liminf_{n\to\infty} F(d(x_{n+1}, Tx^*)) \le \liminf_{n\to\infty} F(M^*(x_n, x^*)) \le \limsup_{n\to\infty} F(M^*(x_n, x^*))$$

For $n \to \infty$, we have $d(x_{n+1}, Tx^*) \to d(x^*, Tx^*)$, and $M(x_n, x^*) \to d(x^*, Tx^*)$, hence

$$\tau + \liminf_{u\to d(x^*,Tx^*)} F(u) \le \liminf_{u\to d(x^*,Tx^*)} F(u) \le \limsup_{u\to d(x^*,Tx^*)} F(u)$$

wich contradicts the second condition of the Theorem 3. Hence $Tx^* = x^*$.

Let now, $x^*, y^*$ be two fixed points of $T$ and suppose that $x^* \neq y^*$. It folows that $d(Tx^*, Ty^*) > 0$ and from the hypotesis of the Theorem 3:

$$\tau + F\left(d\left(Tx^*, Ty^*\right)\right) \le F\left(M^*\left(x^*, y^*\right)\right) \tag{20}$$

where

$$\begin{aligned} M^*\left(x^*, y^*\right) &= \max\{d(x^*, y^*) + \left|d(x^*, Tx^*) - d\left(y^*, Ty^*\right)\right|; \\ &\quad d(x^*, Tx^*) + \left|d(y^*, Ty^*) - d\left(x^*, y^*\right)\right|; \\ &\quad d(y^*, Ty^*) + \left|d(x^*, y^*) - d\left(x^*, Tx^*\right)\right|; \\ &\quad \frac{1}{2}\left[d(x^*, Ty^*) + d\left(y^*, Tx^*\right) + \right. \\ &\quad \left. + \left|d(x^*, Tx^*) - d\left(y^*, Ty^*\right)\right|\right]\} \\ &= d(x^*, y^*). \end{aligned} \tag{21}$$

We obtain a contradiction, which shows that condition $x^* \neq y^*$ is false:

$$\tau + F\left(d(x^*, y^*)\right) \le F\left(d(x^*, y^*)\right)$$

so,

$$d(x^*, y^*) = 0 \Rightarrow x^* = y^*.$$

This proves than the fixed point of $T$ is unique. □

**Example 1.** *Let* $X = [0,1]$, $Tx = \begin{cases} 1, & x = 0 \\ \frac{1}{6}, & x \in (0,1] \end{cases}$ *and* $F(x) = \ln x$, $d(x,y) = |x-y|$. *Then* $(X,d)$ *is a complet metric space.*

*We choosing* $F(\alpha) = \ln \alpha$, $\alpha \in (0,\infty)$ *and. Since* $T$ *is not continuous,* $T$ *is not a contraction.*

*From definition of mapping* $T$*, we observe that* $d(Tx, Ty) > 0$*, only for two situations:*

1. $x \in (0,1]$ *and* $y = 0$ *and*

2. $x = 0$, *and* $y \in (0,1]$*, which can be reduced to only one, from symmety of* $d(x,y)$.

*First we show that $T$ does not satisfy the conditions of Theorem 1. Indeed, let $\tau > 0$ arbitrary. We can write $\tau = \ln a$, with $a > 1$. If we take $x = \frac{1}{2}$ and $y = 0$, then inequality*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

*is equivalent with*

$$\ln a \cdot \frac{5}{6} \leq \ln \frac{1}{2} \text{ (false).}$$

*So that $T$ is not a F-weak contraction,*

*On the other hand, if we take $\tau = \ln \frac{11}{10}$, the conditions of Theorem 3 are satisfied. Indeed, first we obtain:*

$$\begin{aligned} M^*(x,0) &= \max\{d(x,0) + |d(x,Tx) - d(0,T0)|, \\ &\quad d(x,Tx) + |d(0,T0) - d(x,0)|, \\ &\quad d(0,T0) + |d(x,0) - d(x,Tx)|, \\ &\quad \frac{1}{2}[d(0,Tx) + d(x,T0) + \\ &\quad + |d(x,Tx) - d(0,T0)|]\} \\ &= \max\left\{x + \left|\left|x - \frac{1}{6}\right| - 1\right|, \right. \\ &\quad \left(\left|x - \frac{1}{6}\right| + |x-1|\right), \left(1 + \left|\left|x - \frac{1}{6}\right| - x\right|\right), \\ &\quad \left.\frac{1}{2}\left(\frac{1}{6} + 1 - x + \left|\left|x - \frac{1}{6}\right| - 1\right|\right)\right\}. \end{aligned}$$

*We have the following cases:*

*1)* $x \in \left(0, \frac{1}{12}\right]$

$$M^*(x,0) = \max\left\{x + \left|\left|x - \frac{1}{6}\right| - 1\right|, \left|x - \frac{1}{6}\right| + |x-1|, 1 + \left|\left|x - \frac{1}{6}\right| - x\right|, \right.$$
$$\left.\frac{1}{2}\left(\frac{1}{6} + 1 - x + \left|\left|x - \frac{1}{6}\right| - 1\right|\right)\right\} =$$
$$= \max\left(2x + \frac{5}{6}; \frac{7}{6} - 2x; \frac{7}{6} - 2x; 1\right) = \frac{7}{6} - 2x$$

*So, relation (1) from Theorem 3 became* $\ln \frac{11}{10} \cdot \frac{5}{6} < \ln\left(\frac{7}{6} - 2x\right)$, *which is true for every* $x \in \left(0, \frac{1}{12}\right]$.

*2)* $x \in \left(\frac{1}{12}; \frac{1}{6}\right]$, $M^*(x,0) = \max\left\{2x + \frac{5}{6}; \frac{7}{6} - 2x; 2x + \frac{5}{6}; 1\right\}$

$M^*(x,0) = 2x + \frac{5}{6}$, *so relation (1) from Theorem 3 became*

$$\ln \frac{11}{10} \cdot \frac{5}{6} < \ln\left(2x + \frac{5}{6}\right),$$

*which is true for every* $x \in \left(\frac{1}{12}; \frac{1}{6}\right]$

*3)* $\frac{1}{6} < x \leq 1$

$$M^*(x,0) = \max\left\{\frac{7}{6}, \frac{5}{6}, \frac{7}{6}, \frac{7}{6} - x\right\} = \frac{7}{6},$$

*and relation (1) from Theorem 3 is true*

$$\ln\frac{11}{10}\cdot\frac{5}{6} < \ln\frac{7}{6}$$

*The second condition of Theorem 3 is satisfied by $F(\alpha)$, which is continuous,* $\forall\alpha > 0$.

*Since the conditions of Theorem 3 are satisfied, then $T$ has a unique fixed point*

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