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ON UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract

In this paper we define an integral operator for analytic functions in the open unit disk and we determine certain univalence criteria for this integral operator.

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1 Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $\mathcal{S}$ the subclass of A consisting of functions $f \in A$, which are univalent in U.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. For $a \in \mathbb{C}$ and $n \in \mathbb{N} - \{0\}$ we note

$$H[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + \cdots\}$$

and

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \cdots\right\},$$

with $\mathcal{A}_1 = A$.

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In this paper we consider the integral operator $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}$ defined by

$$G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z) = \left\{ \gamma \int_0^z u^{\gamma-1} \prod_{j=1}^{n} \left(\frac{u}{g_j(u)} \right)^{\alpha_j} \left(h_j'(u) \right)^{\beta} du \right\}^{\frac{1}{\gamma}}, \qquad (1)$$

for all $z \in U$, $\alpha_j, \gamma, \beta \in \mathbb{C}$, $a = Re\gamma > 0$, $\beta \neq 0$, $g_j, h_j \in \mathcal{A}_n$, $j = \overline{1,n}$ and we obtain sufficient conditions of univalence for this integral operator. Many authors have studied sufficient conditions of univalence for integral operators in papers: [1], [2], [6], [7], [8].

2 Preliminaries

We need the following lemmas.

Lemma 1 (Pescar [5])**.** *Let α be a complex number, $Re\alpha > 0$ and c a complex number, $|c| \leq 1$, $c \neq -1$. If $f(z) = z + a_2 z^2 + \cdots$ is a regular function in U and*

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \qquad (2)$$

for $z \in U$, then function F_α defined by

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{\frac{1}{\alpha}} \qquad (3)$$

is regular and univalent in U.

Lemma 2 (General Schwarz Lemma, [3])**.** *Let f be the function regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If function $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in U_R, \qquad (4)$$

the equality (in the inequality (4) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

Lemma 3 (Mocanu and Şerb, [4])**.** *Let $M_0 = 1,5936\ldots$ be the positive solution of equation*

$$(2 - M)e^M = 2\,. \qquad (5)$$

If $f \in A$ and

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M_0, \quad z \in U, \qquad (6)$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad z \in U. \qquad (7)$$

The edge M_0 is sharp.

3 Main results

Theorem 1. *Let α_j, β, γ be complex numbers, $\beta \neq 0$, $a = Re\gamma > 0$ and functions $g_j, h_j \in \mathcal{A}_n$, L_j, M_j the positive real numbers, $j = \overline{1,n}$.*
If

$$\left|\frac{zg_j'(z)}{g_j(z)} - 1\right| < L_j, \quad z \in U, j = \overline{1,n}\ , \tag{8}$$

$$\left|\frac{zh_j''(z)}{h_j'(z)}\right| < M_j, \quad z \in U, j = \overline{1,n} \tag{9}$$

and

$$|c| + \frac{2n^{\frac{n}{2a}}}{(2a+n)^{\frac{n+2a}{2a}}} \cdot \sum_{j=1}^{n} [|\alpha_j| L_j + |\beta| M_j] \leq 1\ , \tag{10}$$

then function $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ belongs to class S.

Proof. We consider function $f(z)$ defined by:

$$f(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{u}{g_j(u)}\right)^{\alpha_j} \left(h_j'(u)\right)^{\beta} du, \quad z \in U, \tag{11}$$

regular in U and $f(0) = f'(0) - 1 = 0$.

We obtain

$$\frac{zf''(z)}{f'(z)} = \sum_{j=1}^{n} \alpha_j \cdot \left[1 - \frac{zg_j'(z)}{g_j(z)}\right] + \beta \cdot \sum_{j=1}^{n} \frac{zh_j''(z)}{h_j'(z)}, \quad z \in U. \tag{12}$$

We have

$$\left|c|z|^{2\gamma} + (1 - |z|^{2\gamma}) \cdot \frac{zf''(z)}{\gamma f'(z)}\right| =$$

$$= \left|c|z|^{2\gamma} + \frac{1 - |z|^{2\gamma}}{\gamma} \left[\sum_{j=1}^{n} \alpha_j \left(1 - \frac{zg_j'(z)}{g_j(z)}\right) + \beta \sum_{j=1}^{n} \frac{zh_j''(z)}{h_j'(z)}\right]\right| \leq$$

$$\leq |c| \cdot |z|^{2\gamma} + \left|\frac{1 - |z|^{2\gamma}}{\gamma}\right| \cdot \left[\sum_{j=1}^{n} |\alpha_j| \left|1 - \frac{zg_j'(z)}{g_j(z)}\right| + |\beta| \sum_{j=1}^{n} \left|\frac{zh_j''(z)}{h_j'(z)}\right|\right] \tag{13}$$

for all $z \in U$.

We get

$$\frac{\left|1 - |z|^{2\gamma}\right|}{|\gamma|} \leq \frac{1 - |z|^{2Re\gamma}}{Re\gamma}, \tag{14}$$

for all $z \in U$.

Aplying Lemma 2 from (8) and (9) we have

$$\left|\frac{zg_j'(z)}{g_j(z)} - 1\right| \le L_j|z|^n, \quad z \in U, j = \overline{1,n}, \tag{15}$$

$$\left|\frac{zh_j''(z)}{h_j'(z)}\right| \le M_j|z|^n, \quad z \in U, j = \overline{1,n}. \tag{16}$$

From (13),(14), (15) and (16) we have

$$\left|c|z|^{2\gamma} + (1-|z|^{2\gamma}) \cdot \frac{zf''(z)}{\gamma f'(z)}\right| \le$$

$$\le |c| + \frac{1-|z|^{2a}}{a} \cdot |z|^n \cdot \sum_{j=1}^{n} \left[|\alpha_j|L_j + |\beta|M_j\right], \tag{17}$$

for all $z \in U$.

We consider function $Q : [0,1] \to \mathbb{R}, Q(x) = \frac{(1-x^{2a})x^n}{a}$, where $x = |z|$, $x \in [0,1]$.

We have

$$\max_{x\in[0,1]} Q(x) = \frac{2n^{\frac{n}{2a}}}{(2a+n)^{\frac{n+2a}{2a}}} \,. \tag{18}$$

By (17), (18) and (10) we obtain:

$$\left|c|z|^{2\gamma} + (1-|z|^{2\gamma}) \cdot \frac{zf''(z)}{\gamma f'(z)}\right| \le 1, \tag{19}$$

for all $z \in U$.

From (19) and Lemma 1 it results that function $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ belongs to class S. □

Theorem 2. *Let α_j, β, γ be complex numbers, $\beta \neq 0$, $a = Re\gamma > 0$, functions $g_j, h_j \in \mathcal{A}_n$, $j = \overline{1,n}$ and M_0 the positive solution of equation $(2-M)e^M = 2$. If*

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \le M_0, \quad z \in U,\ j = \overline{1,n}\,, \tag{20}$$

$$\left|\frac{h_j''(z)}{h_j'(z)}\right| \le M_0, \quad z \in U,\ j = \overline{1,n} \tag{21}$$

and

$$|c| + \frac{1}{a}\sum_{j=1}^{n} |\alpha_j| + \frac{2nM_0|\beta|}{(2a+1)^{\frac{2a+1}{2a}}} \le 1 \tag{22}$$

then function $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ is in class S.

Proof. From (13) we obtain

$$\left|c|z|^{2\gamma}+(1-|z|^{2\gamma})\cdot\frac{zf''(z)}{\gamma f'(z)}\right|\leq$$

$$\leq|c|+\frac{|1-|z|^{2\gamma}|}{|\gamma|}\left[\sum_{j=1}^{n}|\alpha_j|\left|\frac{zg_j'(z)}{g_j(z)}-1\right|+|\beta|\sum_{j=1}^{n}\left|\frac{h_j''(z)}{h_j'(z)}\right|\cdot|z|\right], \tag{23}$$

for all $z\in U$.

From (20), (21), (23) and Lemma 3 we obtain

$$\left|c|z|^{2\gamma}+(1-|z|^{2\gamma})\cdot\frac{zf''(z)}{\gamma f'(z)}\right|\leq$$

$$\leq|c|+\frac{|1-|z|^{2\gamma}|}{|\gamma|}\sum_{j=1}^{n}\left[|\alpha_j|+|\beta|\cdot|z|\cdot M_0\right], \tag{24}$$

for all $z\in U$.

From (14) and (24) we have

$$\left|c|z|^{2\gamma}+(1-|z|^{2\gamma})\cdot\frac{zf''(z)}{\gamma f'(z)}\right|\leq$$

$$\leq|c|+\frac{1}{a}\sum_{j=1}^{n}|\alpha_j|+\frac{|1-|z|^{2a}|}{a}|z|\cdot|\beta|\cdot n\cdot M_0\ , \tag{25}$$

for all $z\in U$.

We consider function $I:[0,1]\to\mathbb{R}$, $I(x)=\frac{(1-x^{2a})x}{a}$, where $x=|z|$, $x\in[0,1]$. We have

$$\max_{x\in[0,1]}I(x)=\frac{2}{(2a+1)^{\frac{2a+1}{2a}}}. \tag{26}$$

By (26), (25) and (22) we obtain

$$\left|c|z|^{2\gamma}+(1-|z|^{2\gamma})\cdot\frac{zf''(z)}{\gamma f'(z)}\right|\leq 1 \tag{27}$$

for all $z\in U$.

From (27) and Lemma 1 it results that function $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ belongs to class S. □

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