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GENERALIZED BERNSTEIN TYPE OPERATORS

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Abstract

In this paper we investigate certain properties of a class of generalized Bernstein type operators.

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1 Preliminaries

Let $B_n : C[0,1] \to C[0,1]$ be the Bernstein operators, defined as follows:

$$(B_n f)(x) = \sum_{k=0}^{n} b_{n,k}(x) \cdot f\left(\frac{k}{n}\right),\ \forall f \in C[0,1],$$

where

$$b_{n,k}(x) = \binom{n}{k} \cdot x^k \cdot (1-x)^{n-k}.$$

In [1] the following Bernstein special operator have been introduced. Denote by V_n the operators defined as:

$$V_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) \cdot f\left(\frac{k}{n}\right)$$

where

$$p_{n,k}(x) = \left(1+\frac{1}{n}\right)^n \cdot \binom{n}{k} \cdot x^k \cdot \left(\frac{n}{n+1} - x\right)^{n-k}$$

and

$$f \in C[0,1],\ x \in \left[0, 1-\frac{1}{n+1}\right],\ n \in \mathbb{N}.$$

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Similarly, we introduce a generalization to Bernstein operator:

$$S_n^t : C[0,1] \to C\Big[0, \frac{n}{n+t}\Big],\ t>0,\ n\in\mathbb{N},$$

$$S_n^t(f)(x) = \sum_{k=0}^{n} s_{n,k}(x)\cdot f\Big(\frac{k}{n+t}\Big),\ \forall f\in C[0,1],\ \ x\in\Big[0,\frac{n}{n+t}\Big], \tag{1}$$

where

$$s_{n,k}(x) = \Big(\frac{n+t}{n}\Big)^n \cdot \binom{n}{k}\cdot x^k \cdot \Big(\frac{n}{n+t} - x\Big)^{n-k}.$$

Operator S_n^t can be defined also on the space $C\Big[0,\frac{n}{n+t}\Big]$, and we denote a function $f\in C[0,1]$ and his restriction to the interval $\Big[0,\frac{n}{n+t}\Big]$ with the same symbol.

2 Auxiliary results

Denote the monomial functions by $e_i(t)=t^i$, $t\in\mathbb{R}$, $i=0,1,2,\ldots$. We use the Pochhammer symbol: $(a)_r = a(a-1)\ldots(a_r+1)$, for $a\in\mathbb{R}$, $r\in\mathbb{N}$.

Proposition 1. *Operator S_n^t satisfies the following relations:*

i) $S_n^t(f)(x)\ge 0$, *if* $f\in C\Big[0,\frac{n}{n+t}\Big]$, $f\ge 0$.

ii) $S_n^t(e_0)(x)=1$;

iii) $S_n^t(e_1)(x)=x$;

iv) $S_n^t(e_2)(x) = x\Big(\frac{n-1}{n}\cdot x + \frac{1}{n+t}\Big)$,

where $x\in\Big[0, 1-\frac{1}{n+t}\Big]$

Proof. i) It is obvious by definition;
ii)

$$\begin{aligned} S_n^t(e_0)(x) &= \Big(\frac{n+t}{n}\Big)^n\cdot\sum_{k=0}^{n}\Big(\frac{n}{n+t}-x\Big)^{n-k}\cdot x^k\cdot\binom{n}{k}\cdot 1\\ &= \Big(\frac{n+t}{n}\Big)^n\cdot\Big(\frac{n}{n+t}\Big)^n = 1; \end{aligned}$$

iii)

$$\begin{aligned} S_n^t(e_1)(x) &= \Big(\frac{n+t}{n}\Big)^n\sum_{k=0}^{n}\Big(\frac{n}{n+t}-x\Big)^{n-k}x^k\binom{n}{k}\cdot\frac{k}{n+t}\\ &= \Big(\frac{n+t}{n}\Big)^n\cdot\frac{x}{n+t}\cdot n\cdot\Big(\frac{n}{n+t}\Big)^{n-1} = x; \end{aligned}$$

iv)

$$\begin{aligned} S_n^t(e_2)(x) &= \left(\frac{n+t}{n}\right)^n \cdot \sum_{k=0}^{n} \left(\frac{n}{n+t} - x\right)^{n-k} \cdot x^k \cdot \binom{n}{k} \cdot \frac{k^2}{(n+t)^2} \\ &= \left(\frac{n+t}{n}\right)^n \cdot \frac{1}{(n+t)^2} \cdot x^2 \cdot n \cdot (n-1) \cdot \left(\frac{n}{n+t}\right)^{n-2} \\ &+ \left(\frac{n+t}{n}\right)^n \cdot \frac{1}{(n+t)^2} \cdot x \cdot n \cdot \left(\frac{n}{n+t}\right)^{n-1} \\ &= \frac{n-1}{n} \cdot x^2 + \frac{1}{n+t} \cdot x = x\left(\frac{n-1}{n} \cdot x + \frac{1}{n+t}\right). \end{aligned}$$

□

Proposition 2. *The following recurrence relation holds:*

$$x\left(\frac{n}{n+t} - x\right)s'_{n,k}(x) = n \cdot \left(\frac{k}{n+t} - x\right)s_{n,k}(x).$$

Proof. We have:

$$\begin{aligned} & x\left(\frac{n}{n+t} - x\right)s'_{n,k}(x) \\ = \; & x \cdot \left(\frac{n}{n+t} - x\right) \cdot \left(\frac{n+t}{n}\right)^n \cdot \binom{n}{k} \cdot \left(k \cdot x^{k-1} \cdot \left(\frac{n}{n+t} - x\right)^{n-k} \right. \\ & \left. -x^k \cdot (n-k) \cdot \left(\frac{n}{n+t} - x\right)^{n-k-1}\right) \\ = \; & x \cdot \left(\frac{n}{n+t} - x\right) \cdot \left(\frac{n+t}{n}\right)^n \cdot \binom{n}{k} \cdot x^{k-1} \cdot \left(\frac{n}{n+t} - x\right)^{n-k-1} \\ & \times\left(k \cdot \left(\frac{n}{n+t} - x\right) - x \cdot (n-k)\right) \\ = \; & \left(\frac{n+t}{n}\right)^n \cdot \binom{n}{k} \cdot x^k \cdot \left(\frac{n}{n+t} - x\right)^{n-k} \frac{k \cdot n - n^2 \cdot x - n \cdot t \cdot x}{n+t} \\ = \; & n \cdot \left(\frac{k}{n+t} - x\right)s_{n,k}(x). \end{aligned}$$

□

Theorem 1. *Let be the m-th order moment for the operator (1) be defined by*

$$\mu_{n,m}(x) = \sum_{k=0}^{n} s_{n,k}(x)\left(\frac{k}{n+t} - x\right)^m, m = 0, 1, 2, ...,$$

then we have:

i) $\mu_{n,0}(x) = 1$;

ii) $\mu_{n,1}(x) = 0$;

iii) $n\mu_{n,m+1}(x) = x\Big(\frac{n}{n+t} - x\Big)\Big(\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\Big)$;

iv) $\mu_{n,2}(x) = \frac{x}{n}\Big(\frac{n}{n+t} - x\Big)$;

v) $\mu_{n,3}(x) = \frac{x}{n}\Big(\frac{n}{n+t} - x\Big)\Big(\frac{1}{n+t} - \frac{2x}{n}\Big)$;

vi) $\mu_{n,4}(x) = \frac{x}{n}\Big(\frac{n}{n+t} - x\Big)\Big[\frac{1}{n}\cdot\Big(\frac{n}{n+t} - x\Big)\cdot\Big(\frac{1}{n+t} - \frac{4x}{n} + 3\Big) - \frac{x}{n}\Big(\frac{1}{n+t} - \frac{2x}{n}\Big)\Big]$.

Proof. i) It follows immediately from Proposition 1 - ii)

ii) From Proposition 1 - i), ii) we have

$$\mu_{n,1}(x) = S_n^t(e_1)(x) - xS_n^t(e_0)(x) = 0.$$

iii) First note that

$$\mu_n'(x) = \sum_{k=0}^{n} s'_{n,k}(x)\left(\frac{k}{n+t} - x\right)^m - m\sum_{k=0}^{n} s_{n,k}(x)\left(\frac{k}{n+t} - x\right)^{m-1}.$$

Then, using also Proposition 2 we obtain

$$\begin{aligned} & x\Big(\frac{n}{n+t} - x\Big)\Big(\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\Big) \\ = \; & x\Big(\frac{n}{n+t} - x\Big)\sum_{k=0}^{n} s'_{n,k}(x)\left(\frac{k}{n+t} - x\right)^m \\ = \; & n\sum_{k=0}^{n}\Big(\frac{k}{n+t} - x\Big)s_{n,k}(x)\left(\frac{k}{n+t} - x\right)^m \\ = \; & n\mu_{n,m+1}(x). \end{aligned}$$

iv) Using i), ii) and iii), we have:

$$\mu_{n,2}(x) = \frac{x}{n}\Big(\frac{n}{n+t} - x\Big)(\mu'_{n,1}(x) + \mu_{n,0}(x)) = \frac{x}{n}\Big(\frac{n}{n+t} - x\Big).$$

v) Using ii), iii) and iv), we have:

$$\mu_{n,3}(x) = \frac{x}{n}\Big(\frac{n}{n+t} - x\Big)(\mu'_{n,2}(x) + 2\mu_{n,1}(x)) = \frac{x}{n}\Big(\frac{n}{n+t} - x\Big)\Big(\frac{1}{n+t} - \frac{2x}{n}\Big).$$

vi) Using iii), iv) and v), we have:

$$\begin{aligned} \mu_{n,4}(x) &= \frac{x}{n}\Big(\frac{n}{n+t} - x\Big)(\mu'_{n,3}(x) + 3\mu_{n,2}(x)) \\ &= \mu_{n,2}(x)(\mu'_{n,3}(x) + 3\mu_{n,2}(x)) \\ &= \mu_{n,2}(x)\Big[\frac{1}{n}\cdot\Big(\frac{n}{n+t} - x\Big)\cdot\Big(\frac{1}{n+t} - \frac{4x}{n} + 3\Big) - \frac{x}{n}\Big(\frac{1}{n+t} - \frac{2x}{n}\Big)\Big]. \end{aligned}$$

□

3 Convergence properties

Theorem 2. *For all $t > 0$, $f \in C[0,1]$ and $0 < \epsilon < 1$, it is true the following:*

$$\lim_{n\to\infty} S_n^t(f)(x) = f(x)$$

uniformly on $[0, 1-\epsilon]$.

Proof. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\dfrac{n}{n+t} > 1-\epsilon$.

From Proposition 1 we have that $\lim_{n\to\infty} S_n^t(e_i)(x) = x^i, i = 0, 1, 2$ uniformly on $[0, 1-\epsilon]$. Then we can apply Korovkin theorem for the sequence $(S_n^t)_n$ on interval $[0, 1-\epsilon]$. □

Next, we give a Voronovskaja-type theorem.

Theorem 3. *Let be $f \in C[0,1]$ be a bounded function two times derivable at the point $x \in (0,1)$. Then*

$$\lim_{n\to\infty} n\Big[S_n^t(f)(x) - f(x)\Big] = \frac{x(1-x)}{2} f''(x).$$

Proof. Fix a point $x \in (0,1)$. Taylor expansion of f in point x leads to:

$$f(t) = f(x) - (t-x)f'(x) + \frac{(t-x)^2}{2} f''(x) + (t-x)^2 \eta_x(t-x),\ t \in [0,1], \tag{2}$$

where η_x is a bounded function having the property $\lim_{h\to 0} \eta_x(h) = 0$. Denote $\sigma(x,t) = (t-x)^2 \eta_x(t-x)$, $t \in [0,1]$.

We can choose an indice n_0, such that $\frac{n}{n+t} > x$, for $n \geq n_0$. Applying operator S_n^t for $n \geq n_0$ in (2) and taking into account Theorem 1 one obtains

$$\begin{aligned} S_n^t(f)(x) &= f(x)\mu_{n,0}(x) + f'(x)\mu_{n,1}(x) + \frac{f''(x)}{2}\mu_{n,2}(x) + S_n^t(\sigma(x,\cdot))(x) \\ &= f(x) + \Big(-\frac{1}{n}x^2 + \frac{1}{n+t}x\Big)\frac{f''(x)}{2} + S_n^t(\sigma(x,\cdot))(x). \end{aligned}$$

Then

$$\lim_{n\to\infty} n\Big[S_n^t(f)(x) - f(x)\Big] = \frac{x(1-x)}{2} f''(x) + \lim_{n\to\infty} n(S_n^t \sigma(x,\cdot))(x). \tag{3}$$

Using Hölder inequality it follows

$$\begin{aligned} \big|nS_n^t(\sigma(x,\cdot))(x)\big| &= n\left|\sum_{k=0}^{n}\Big(\frac{k}{n+t} - x\Big)^2 \eta_x\Big(\frac{k}{n+t} - x\Big) s_{n,k}(x)\right| \\ &\leq n\sqrt{\sum_{k=0}^{n} s_{n,k}(x)\Big(\frac{k}{n+t} - x\Big)^4 \cdot \sum_{k=0}^{n} s_{n,k}(x)\eta_x^2\Big(\frac{k}{n+t} - x\Big)} \\ &= n\sqrt{\mu_{n,4}(x)}\sqrt{S_n^t((\eta_x)^2)(x)} \end{aligned}$$

From Theorem 1-vi) we obtain $\mu_{n,4}(x) = \mathrm{O}\left(\frac{1}{n^2}\right)$. Then $n\sqrt{\mu_{n,4}(x)} = \mathrm{O}(1)$.

On the other hand from Theorem 2 we have

$$\lim_{n\to\infty} S_n^t((\eta_x)^2)(x) = \eta_x(x) = 0.$$

We deduce

$$\lim_{n\to\infty} n(S_n^t\sigma(x,\cdot))(x) = 0.$$

This finishes the proof.

□

4 Simultaneous approximations

We adopt this known definitions

Definition 1. *A function $g : I \to \mathbb{R}$, I interval, is named convex of order $r \geq -1$ if all the divided differences on $r+2$ points in I are positive.*

Hence o positive function is a convex function of order -1, an increasing function is convex of order 0 and so on. In other words, for $r \geq 0$ if $f \in C^{r+1}(I)$, then f is convex of order r iff $f^{(r+1)} \geq 0$.

Definition 2. *A linear operator is named convex operator of order r, $r \geq -1$ if it transforms any r convex function into a r-convex function.*

The following property is essential for proving the existence of the simultaneous approximation.

Theorem 4. *Operators S_n^t is convex of order $r-1$, $\forall r \in [0,n]$.*

Proof. It suffices to prove that we have $S_n^t(f)^{(r)} \geq 0, \forall f \in C^r\left[0, \dfrac{n}{n+t}\right]$, such that $f^{(r)} \geq 0$, because if f is convex of order $r-1$ on I, there is $g \in C^{(r)}\left[0, \dfrac{n}{n+t}\right]$, such that g coincides with f on the knots $\frac{k}{n+1}$, $0 \leq k \leq n$ and we ca take g instead of f.

For $r = 0$, the affirmation is true from Proposition 1 - i).

Let be $r \geq 1$ and a function $f \in C^r\left[0, \dfrac{n}{n+t}\right]$ such that $f^{(r)} \geq 0$.

We use formula:

$$s'_{n,k}(x) \tag{4}$$
$$= (n+t)\Big[s_{n-1,k-1}\Big(\frac{(n-1)(n+t)}{n(n+t-1)}x\Big) - s_{n-1,k}\Big(\frac{(n-1)(n+t)}{n(n+t-1)}x\Big)\Big],\ 0 \leq k \leq n.$$

Here we made the convention $s_{n,-1}(t) = 0$ and $s_{n,n+1}(t) = 0$, for any t and $n \geq 1$.

We denote by $\Delta_h f(x) := f(x+h) - f(h)$ and by Δ_h^r, the r-th iterate of Δ_h.

From formula (4). We have:

$$
\begin{aligned}
&(S_n^t(f))'(x)\\
= &\sum_{k=0}^{n-1}(n+t)\Big[s_{n-1,k-1}\Big(\frac{(n-1)(n+t)}{n(n+t-1)}x\Big)-s_{n-1,k}\Big(\frac{(n-1)(n+t)}{n(n+t-1)}x\Big)\Big]f\Big(\frac{k}{n+t}\Big)\\
= &(n+t)\sum_{k=0}^{n-1}s_{n-1,k}\Big(\frac{(n-1)(n+t)}{n(n+t-1)}x\Big)\Big[f\Big(\frac{k+1}{n+t}\Big)-f\Big(\frac{k}{n+t}\Big)\Big]\\
= &(n+t)\sum_{k=0}^{n-1}s_{n-1,k}\Big(\frac{(n-1)(n+t)}{n(n+t-1)}x\Big)\Delta_{\frac{1}{n+t}}f\Big(\frac{k}{n+t}\Big). \qquad (5)
\end{aligned}
$$

Now we apply induction. Suppose that the following is true:

$$
(S_n^t(f))^{(r)}(x)=(n+t)_r\sum_{k=0}^{n-r}s_{n-r,k}\Big(\frac{(n-1)_r(n+t)_r}{(n)_r(n+t-1)_r}x\Big)\Delta^r_{\frac{1}{n+t}}f\Big(\frac{k}{n+t}\Big), \qquad (6)
$$

where $(n+t)_r$ is the Pochhammer symbol.

Taking the derivative in (6) an using formula (5) we obtain:

$$
\begin{aligned}
&\Big((S_n^t(f))^{(r)}(x)\Big)'\\
= &(n+t)_r\sum_{k=0}^{n-r-1}(n-r+t)s_{n-r-1,k}\Big(\frac{(n-1)_r(n+t)_r}{(n)_r(n+t-1)_r}\frac{(n-r-1)(n-r+t)}{(n-r)(n-r+t-1)}x\Big)\\
&\qquad\times\Delta_{\frac{1}{n+t}}\Big(\Delta^r_{\frac{1}{n+t}}f\Big(\frac{k}{n+t}\Big)\Big)\\
= &(n+t)_{r+1}\sum_{k=0}^{n-r-1}s_{n-r-1,k}\Big(\frac{(n-1)_{r+1}(n+t)_{r+1}}{(n)_{r+1}(n+t-1)_{r+1}}x\Big)x\Big)\Delta^{r+1}_{\frac{1}{n+t}}f\Big(\frac{k}{n+t}\Big).
\end{aligned}
$$

So, relation (6) was proved.

Now if f is convex of order $r-1$ then all the finite differences $\Delta^r_{\frac{1}{n+t}}f\Big(\frac{k}{n+t}\Big)$ are positive. Then from formula (6) one obtains that $(S_n^t(f))'\geq 0$. This means that S_n^t is convex of order r. □

The study of simultaneous approximation is based on the use of Kantorovich operators of higher order. First we prove the following additional theorems.

Theorem 5. *Writing $T_{n,r}(x)=S_n^t(e_r)(x)$, we have:*

$$
T_{n,r+1}(x)=x\cdot T_{n,r}(x)+\frac{x}{n}\Big(\frac{n}{n+t}-x\Big)T'_{n,r}(x).
$$

Proof. Using Proposition 2, we obtain

$$\begin{aligned}
x\Big(\frac{n}{n+t}-x\Big)T'_{n,r}(x) &= x\Big(\frac{n}{n+t}-x\Big)\sum_{k=0}^{n} s'_{n,k}(x)\Big(\frac{k}{n+t}\Big)^r\\
&= n\sum_{k=0}^{n}\Big(\frac{k}{n+t}-x\Big)s_{n,k}(x)\Big(\frac{k}{n+t}\Big)^r\\
&= n\Big(\sum_{k=0}^{n} s_{n,k}(x)\Big(\frac{k}{n+t}\Big)^{r+1}-x\sum_{k=0}^{n} s_{n,k}(x)\Big(\frac{k}{n+t}\Big)^r\Big)\\
&= nT_{n,r+1}(x)-nxT_{n,r}(x).
\end{aligned}$$

From this it results

$$T_{n,r+1}(x)=\frac{1}{n}\Big(\frac{n}{n+t}-x\Big)T'_{n,r}(x)+xT_{n,r}(x).$$

□

Theorem 6. *For $n \geq 1, r \geq 0, x \in [0,1]$, we have:*

$$T_{n,r}(x)=A_{n,r}x^r+B_{n,r}x^{r-1}+C_{n,r}x^{r-2}+R_{n,r}(x)$$

where

$$\begin{aligned}
A_{n,r} &= \frac{(n-1)_{r-1}}{n^{r-1}},\\
B_{n,r} &= \frac{r(r-1)}{2}\cdot\frac{(n-1)_{r-2}}{n^{r-2}(n+t)},\\
C_{n,r} &= \frac{r(r-1)(r-2)(3r-5)}{24}\cdot\frac{(n-1)_{r-3}}{n^{r-3}(n+t)^2}
\end{aligned}$$

and $R_{n,r}$ is a polynomial of degree $r-3$.

Proof. From relation

$$\begin{aligned}
T_{n,r+1}(x) &= x\Big(A_{n,r}x^r+B_{n,r}x^{r-1}+C_{n,r}x^{r-2}+R_{n,r}(x)\Big)\\
&+ \frac{x}{n}\Big(\frac{n}{n+t}-x\Big)\Big(rA_{n,r}x^{r-1}+(r-1)B_{n,r}x^{r-2}\\
&\quad +(r-2)C_{n,r}x^{r-3}+R_{n,r-1}(x)\Big),
\end{aligned}$$

by identifying the coefficients, we obtain:

$$A_{n,r+1} = \frac{n-r}{n}\cdot A_{n,r};$$

$$B_{n,r+1} = \frac{n-r+1}{n}\cdot B_{n,r}+\frac{r}{n+t}\cdot A_{n,r};$$

$$C_{n,r+1} = \frac{n-r+2}{n}\cdot C_{n,r} + \frac{r-1}{n+t}\cdot B_{n,r}.$$

Using Proposition 1 and Theorem 5 one obtains

$$\begin{aligned} T_{n,1}(x) &= x \\ T_{n,2}(x) &= \frac{n-1}{n}\cdot x^2 + \frac{1}{n+t}\cdot x \end{aligned}$$

and then

$$\begin{aligned} T_{n,3}(x) &= x\cdot T_{n,2}(x) + \frac{x}{n}\Big(\frac{n}{n+t} - x\Big)T'_{n,2}(x) \\ &= \frac{(n-1)(n-2)}{n^2}x^3 + \frac{3(n-1)}{n(n+t)}x^2 + \frac{x}{(n+t)^2}. \end{aligned}$$

Then
$$A_{n,3} = \frac{(n-1)(n-2)}{n^2}, \quad B_{n,3} = \frac{3(n-1)}{n(n+t)}, \quad C_{n,3} = \frac{1}{n(n+t)}.$$

So that Theorem is true for Theorem is true for $r = 1, 2, 3$.

Further suppose by induction that Theorem is true for r. Then applying the relations of recurrence one obtains:

$$\begin{aligned} A_{n,r+1} &= \frac{n-r}{n}\cdot A_{n,r} = \frac{n-r}{n}\cdot\frac{(n-1)_{r-1}}{n^{r-1}} \\ &= \frac{(n-1)_r}{n^r}. \\ B_{n,r+1} &= \frac{n-r+1}{n}\cdot B_{n,r} + \frac{r}{n+t}\cdot A_{n,r} \\ &= \frac{n-r+1}{n}\cdot\frac{r(r-1)}{2}\cdot\frac{(n-1)_{r-2}}{n^{r-2}(n+t)} + \frac{r}{n+t}\cdot\frac{(n-1)_{r-1}}{n^{r-1}} \\ &= \frac{(n-1)_{r-1}}{n^{r-1}(n+t)}\Big(\frac{r(r-1)}{2} + r\Big) \\ &= \frac{r(r+1)}{2}\cdot\frac{(n-1)_{r-1}}{n^{r-1}(n+t)} \\ C_{n,r+1} &= \frac{n-r+2}{n}\cdot C_{n,r} + \frac{r-1}{n+t}\cdot B_{n,r} \\ &= \frac{n-r+2}{n}\cdot\frac{r(r-1)(r-2)(3r-5)}{24}\cdot\frac{(n-1)_{r-3}}{n^{r-3}(n+t)^2} \\ &\quad + \frac{r-1}{n+t}\cdot\frac{r(r-1)}{2}\cdot\frac{(n-1)_{r-2}}{n^{r-2}(n+t)} \\ &= \frac{(n-1)_{r-2}}{n^{r-2}(n+t)^2}\cdot\Big(\frac{r(r-1)(r-2)(3r-5)}{24} + \frac{r(r-1)^2}{2}\Big) \\ &= \frac{(r+1)r(r-1)(3r-2)}{24}\cdot\frac{(n-1)_{r-2}}{n^{r-2}(n+t)^2} \end{aligned}$$

□

The main result is the following

Theorem 7. *For any function $f \in C^r[0,1]$, $r \geq 1$ any $t > 0$ and $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} (S_n^t(f)(x))^{(r)} = f^{(r)}(x), \text{ uniformly for } x \in [0, 1-\varepsilon]. \tag{7}$$

Proof. We take $n \in \mathbb{N}$, such that $\frac{n}{n+t} \leq 1-\varepsilon$ and $n > r$.

For $r \in \mathbb{N}$, we denote the derivative operator of order r, by

$$D^r(f)(x) = f^{(r)}(x),\ f \in C^r[0,1],\ x \in [0,1]. \tag{8}$$

The antiderivative operator of degree r, is defined by

$$J^r(f)(x) = \int_0^x \frac{(x-u)^{r-1}}{(r-1)!} f(u)du,\ f \in C[0,1]. \tag{9}$$

Consider be the Kantorovich operator of order r attached to S_n^t, namely

$$K_{n,r}(f)(x) = \Big(D^r \circ S_n^t \circ J^r\Big)(f)(x). \tag{10}$$

Let show that operator $K_{n,r}$ is positive. If $g \in C\Big[0, \frac{n}{n+t}\Big]$ is positive then $J^r(g)$ has the derivative of order r positive and hence all the divided differences on $r+1$ points are positive. In particular his finite differences on $r+1$ point are positive. If in formula (6) we replace function f by function $J^r(g)$ one obtains

$$(S_n^t(J^r(g)))^{(r)}(x) = (n+t)_r \sum_{k=0}^{n-r} s_{n-r,k}\Big(\frac{(n-1)_r(n+t)_r}{(n)_r(n+t-1)_r}x\Big)\Delta^r_{\frac{1}{n+t}} J^r(g)\Big(\frac{k}{n+t}\Big) \geq 0.$$

But if this is equivalent with the condition that $K_{n,r}$ is positive operator.

Using (8),(9),(10), we deduce

$$D^r(S_n^t)(f) = K_{n,r}(f^{(r)}),\ \forall f \in C\Big[0, \frac{n}{n+t}\Big]. \tag{11}$$

So that, in order to prove the theorem it suffices to show that the sequence of operators $(K_{n,r})_n$ satisfies the conditions of the theorem of Korovkin.

We begin with the following relation, which are true for any $x \in [0,1]$:

$$\begin{aligned} J_r(e_0)(x) &= \frac{x^r}{r!} \\ J_r(e_1)(x) &= \frac{x^{r+1}}{(r+1)!}. \\ J_r(e_2)(x) &= 2 \cdot \frac{x^{r+2}}{(r+2)!}. \end{aligned}$$

Consequently,

$$\begin{aligned}
K_{n,r}(e_0)(x) &= \Big(S_n^t\Big(\frac{e^r}{r!}\Big)(x)\Big)^{(r)} \\
&= \frac{1}{r!}\Big(A_{n,r}x^r + B_{n,r}x^{r-1} + C_{n,r}x^{r-2} + R_{n,r}(x)\Big)^{(r)} \\
&= A_{n,r}, \\
K_{n,r}(e_1)(x) &= \Big(S_n^t\Big(\frac{e^{r+1}}{(r+1)!}\Big)(x)\Big)^{(r)} \\
&= \frac{1}{(r+1)!}\Big(A_{n,r+1}x^{r+1} + B_{n,r+1}x^r + C_{n,r+1}x^{r-1} + R_{n,r+1}(x)\Big)^{(r)} \\
&= A_{n,r+1}\cdot x + \frac{1}{r+1}\cdot B_{n,r+1}, \\
K_{n,r}(e_2)(x) &= \Big(S_n^t\Big(\frac{2\cdot e^{r+2}}{(r+2)!}\Big)(x)\Big)^{(r)} \\
&= \frac{2}{(r+2)!}\Big(A_{n,r+2}x^{r+2} + B_{n,r+2}x^{r+1} + C_{n,r+2}x^r + R_{n,r+2}(x)\Big)^{(r)} \\
&= A_{n,r+2}\cdot x^2 + \frac{2}{r+2}\cdot B_{n,r+2}\cdot x + \frac{2r!}{(r+2)!} + C_{n,r+2}.
\end{aligned}$$

Since

$$\begin{aligned}
\lim_{n\to\infty} A_{n,s} &= 1,\ \forall s \geq 1 \\
\lim_{n\to\infty} B_{n,s} &= 0,\ \forall s \geq 2 \\
\lim_{n\to\infty} C_{n,s} &= 0,\ \forall s \geq 3
\end{aligned}$$

it results

$$\lim_{n\to\infty} K_{n,r}(e_j)(x) = e_j(x), \text{uniformly on } [0, 1-\varepsilon],\ \text{for } j = 0,1,2. \qquad (12)$$

□

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