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RESULT ON UNIQUENESS OF ENTIRE FUNCTIONS RELATED TO DIFFERENTIAL-DIFFERENCE POLYNOMIAL

Sujoy MAJUMDER[*,1] and Jeet SARKAR[2]

Abstract

In the paper, we use the idea of normal family to investigate the uniqueness problems of entire functions when certain types of differential-difference polynomials generated by them sharing a non-zero polynomial. Also we exhibit one example to show that the conditions of our results are the best possible.

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1 Introduction, definitions and results

In the paper by meromorphic functions we shall always mean meromorphic functions in $\mathbb{C}$. We adopt the standard notations of value distribution theory (see [6]). For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set of finite linear measure. A meromorphic function a is called a small function with respect to f, if $T(r,a)=S(r,f)$. The order of f is denoted and defined by

$$\rho=\rho(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}.$$

For $a\in\mathbb{C}\cup\{\infty\}$, we define

$$\delta(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{N(r,a;f)}{T(r,f)}.$$

[1*] *Corresponding author*, Department of Mathematics, Raiganj University, Raiganj, West Bengal-733134, India., e-mail: sm05math@gmail.com, sujoy.katwa@gmail.com, smajumder05@yahoo.in

[2]Department of Mathematics, Raiganj University, Raiganj, West Bengal-733134, India., e-mail: sarkar.jeet05@gmail.com

Let f and g be two non-constant meromorphic functions. Let a be a small function with respect to f and g. We say that f and g share a CM (counting multiplicities) if $f-a$ and $g-a$ have the same zeros with the same multiplicities and we say that f and g share a IM (ignoring multiplicities) if we do not consider the multiplicities.

Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share a with weight k. We write f and g share (a, k) to mean that f and g share a with weight k. Also we note that f and g share a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively.

Let b be a small function of both f and g. We denote by $\overline{N}_E(r, f = b = g)$ the reduced counting function of the common zeros of $f - b$ and $g - b$ with the same multiplicities. We say that f and g share $(b, \infty)_*$ if

$$\overline{N}(r, b; f) - \overline{N}_E(r, f = b = g) = O(\log r) \text{ as } r \to \infty$$

$$\text{and} \qquad \overline{N}(r, b; g) - \overline{N}_E(r, f = b = g) = O(\log r) \text{ as } r \to \infty.$$

Let f be a transcendental meromorphic function and $n \in \mathbb{N}$. Many authors have investigated the value distributions of $f^n f'$. In 1959, W. K. Hayman (see [5], Corollary of Theorem 9) proved the following theorem.

Theorem A. *Let f be a transcendental meromorphic function and $n \in \mathbb{N}$ such that $n \geq 3$. Then $f^n f' = 1$ has infinitely many solutions.*

The case $n = 2$ was settled by Mues [11] in 1979. Bergweiler and Eremenko [1] showed that $ff' - 1$ has infinitely many zeros.

For an analogue of the above result, Laine and Yang [7] investigated the value distribution of difference products of entire functions in the following manner.

Theorem B. *Let f be a transcendental entire function of finite order, $n \in \mathbb{N}$ and $c \in \mathbb{C} \setminus \{0\}$. Then for $n \geq 2$, $f^n(z)f(z+c)$ assumes every non-zero value infinitely often.*

In 2010, X. G. Qi, L. Z. Yang and K. Liu [13] proved the following uniqueness result.

Theorem C. *Let f and g be two transcendental entire functions of finite order, $\eta \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share $(1, \infty)$, then either $fg = t_1$ or $f = t_2 g$ for $t_1, t_2 \in \mathbb{C} \setminus \{0\}$ such that $t_1^{n+1} = t_2^{n+1} = 1$.*

Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \tag{1.1}$$

be a non zero polynomial, where $a_n (\neq 0), a_{n-1}, \ldots, a_0$ are complex constants. We denote Γ_1, Γ_2 by $\Gamma_1 = m_1 + m_2, \Gamma_2 = m_1 + 2m_2$ respectively, where m_1 is the number of simple zeros of $P(z)$ and m_2 is the number of multiple zeros of $P(z)$. Let $d = gcd(\lambda_0, \lambda_1, \ldots, \lambda_n)$, where $\lambda_i = n+1$ if $a_i = 0$, $\lambda_i = i+1$ if $a_i \neq 0$.

In 2011, L. Xudan and W. C. Lin [16] considered the zeros of one certain type of difference polynomial and obtained the following result.

Theorem D. *Let f be a transcendental entire function of finite order and $\eta \in \mathbb{C}\backslash\{0\}$. Then for $n > \Gamma_1$, $P(f(z))f(z+\eta) - \alpha(z) = 0$ has infinitely many solutions, where $\alpha(z)(\not\equiv 0)$ is a small function with respect to f.*

In the same paper the authors also proved the following uniqueness result corresponding to Theorem D.

Theorem E. *Let f and g be two transcendental entire functions of finite order, $\eta \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$ such that $n > 2\Gamma_2 + 1$. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share $(1, \infty)$, then one of the following cases hold:*

(i) $f \equiv tg$, *where* $t^d = 1$;

(ii) $R(f, g) \equiv 0$ *where* $R(w_1, w_2) = P(w_1)w_1(z+\eta) - P(w_2)w_2(z+\eta)$;

(iii) $f = e^{\alpha}$ *and* $g = e^{\beta}$, *where* α, β *are non-constant polynomials and* $\alpha + \beta = c \in \mathbb{C}$ *satisfying* $a_n^2 e^{(n+1)c} = 1$.

We recall the following example due to L. Xudan and W. C. Lin [16].

Example 1. *Let $P(z) = (z-1)^6(z+1)^6 z^{11}$, $f(z) = \sin z$, $g(z) = \cos z$ and $\eta = 2\pi$. It is easily seen that $n > 2\Gamma_2 + 1$ and $P(f(z))f(z+\eta) \equiv P(g(z))g(z+\eta)$. Therefore $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share 1 CM. It is also clear that $R(f, g) \equiv 0$, where $R(w_1, w_2) = P(w_1)w_1(z+\eta) - P(w_2)w_2(z+\eta)$ but $f \not\equiv tg$ for $t \in \mathbb{C} \setminus \{0\}$ satisfying $t^m = 1$, where $m \in \mathbb{Z}^+$.*

From the above example, we see that f and g do not share $(0, \infty)$. Regarding this one may ask the following question.

Question 1. *What can be said about the relationship between f and g, if f and g share $(0, \infty)$ in Theorem E ?*

Keeping the above question in mind, recently W. L. Li and X. M. Li [8] proved the following results.

Theorem F. *Let f and g be two transcendental entire functions of finite order such that f and g share $(0, \infty)$, $\eta \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $n > 2\Gamma_2 + 1$. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share $(1, \infty)$, then one of the following two cases hold:*

(i) $f \equiv tg$, *where* $t^d = 1$;

(ii) $f = e^{\alpha}$ *and* $g = ce^{-\alpha}$, *where α is a non-constant polynomial and* $c \in \mathbb{C} \setminus \{0\}$ *satisfying* $a_n^2 c^{n+1} = 1$.

Theorem G. *Let f and g be two transcendental entire functions of finite order such that f and g share $(0,\infty)$, $\eta\in\mathbb{C}\setminus\{0\}$ and $n\in\mathbb{N}$ such that $n>3\Gamma_1+2\Gamma_2+4$. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share $(1,0)$, then one of the following two cases hold:*

(i) $f\equiv tg$, *where* $t^d=1$;

(ii) $f=e^{\alpha}$ *and* $g=ce^{-\alpha}$, *where* α *is a non-constant polynomial and* $c\in\mathbb{C}\setminus\{0\}$ *satisfying* $a_n^2c^{n+1}=1$.

Regarding Theorems F and G, P. Sahoo and S. Seikh [14] asked the following question.

Question 2. *What happen if one consider the difference polynomials of the form* $(P(f(z))f(z+\eta))^k$, *where* $k\in\mathbb{N}\cup\{0\}$ *?*

Keeping the above question in mind, in 2016, P. Sahoo and S. Seikh [14] proved the following results.

Theorem H. *Let f be a transcendental entire function with finite order and $\alpha(z)(\not\equiv)0$ be a small function with respect to f. Let $\eta\in\mathbb{C}\setminus\{0\}$, $k\in\mathbb{N}\cup\{0\}$ and $n\in\mathbb{N}$. Then for $n>\Gamma_1+km_2$, $(P(f(z))f(z+\eta))^k-\alpha(z)=0$ has infinitely many solutions.*

Theorem I. *Let f and g be two transcendental entire functions of finite order such that f and g share $(0,\infty)$ and $\eta\in\mathbb{C}\setminus\{0\}$. Let $k\in\mathbb{N}\cup\{0\}$ and $n\in\mathbb{N}$ such that $n>2\Gamma_2+2km_2+1$. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share $(1,\infty)$, then one of the following two cases hold:*

(i) $f\equiv tg$, *where* $t^d=1$;

(ii) $f=e^{\alpha}$ *and* $g=ce^{-\alpha}$, *where* α *is a non-constant polynomial and* $c\in\mathbb{C}\setminus\{0\}$ *satisfying* $a_n^2c^{n+1}=1$.

Theorem J. *Let f and g be two transcendental entire functions of finite order such that f and g share $(0,\infty)$ and $\eta\in\mathbb{C}\setminus\{0\}$. Let $k\in\mathbb{N}\cup\{0\}$ and $n\in\mathbb{N}$ such that $n>3\Gamma_1+2\Gamma_2+5km_2+4$. If $P(f(z))f(z+\eta)$ and $P(g(z))g(z+\eta)$ share $(1,0)$, then one of the following two cases hold:*

(i) $f\equiv tg$, *where* $t^d=1$;

(ii) $f=e^{\alpha}$ *and* $g=ce^{-\alpha}$, *where* α *is a non-constant polynomial and* $c\in\mathbb{C}\setminus\{0\}$ *satisfying* $a_n^2c^{n+1}=1$.

In 2017, S. Majumder and R. Mandal [10] executed some errors in the proof of Theorems I and J which were discussed in Section 1 [10]. Also in the same paper S. Majumder and R. Mandal [10] asked the following question.

Question 3. *Can one replace the condition "f and g share $(0,\infty)$" in Theorems I and J by weaker one ?*

Keeping the above question in mind, S. Majumder and R. Mandal [10] obtained the following results which not only rectified Theorems I and J but also improved and generalized Theorems I and J.

Theorem K. *Let f and g be two transcendental entire functions of finite order such that f and g share $(0,\infty)_*$, $c_j \in \mathbb{C}$ $(j = 1,2,\ldots,s)$ be distinct and let $k \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$, $\mu_j \in \mathbb{N}\cup\{0\}$ $(j = 1,2,\ldots,s)$ such that $n > 2\Gamma_2 + 2km_2 + \sigma$, where $\sigma = \sum_{j=1}^{s} \mu_j > 0$. Suppose that P has at least one zeros of multiplicities at least $k+1$ and $\delta(0;f) > 0$ when $k \geq 1$. If $(P(f(z))\prod_{j=1}^{s}(f(z+c_j))^{\mu_j})^{(k)} - p(z)$ and $(P(g(z))\prod_{j=1}^{s}(g(z+c_j))^{\mu_j})^{(k)} - p(z)$ share $(0,2)$, where $p(z)$ is a non-zero polynomial with $\deg(p) \leq n+\sigma-1$, then one of the following cases hold.*

- *(i)* *$f(z) \equiv tg(z)$ for $t \in \mathbb{C}\setminus\{0\}$ such that $t^d = 1$, where d is the GCD of the elements of J, $J = \{p \in I : a_p \neq 0\}$ and $I = \{\sigma, \sigma+1, \ldots, n+\sigma\}$.*
- *(ii)* *If $k = 0$, then $f(z) = e^{\alpha(z)}$ and $g(z) = te^{-\alpha(z)}$ where $\alpha(z)$ is a non-constant polynomial and $t \in \mathbb{C}\setminus\{0\}$ such that $a_n^2 t^{n+\sigma} = c^2$.*
- *(iii)* *If $p \notin \mathbb{C}$, then $f(z) = e^{\alpha(z)}$ and $g(z) = e^{\beta(z)}$, where α and β are two non-constant polynomials such that $n\alpha(z) + \sum_{j=1}^{s}\mu_j\alpha(z+c_j) = c\int_0^z p(z)dz + b_1$, $n\beta(z) + \sum_{j=1}^{s}\mu_j\beta(z+c_j) = -c\int_0^z p(z)dz + b_2$, $b_1, b_2, c(\neq 0) \in \mathbb{C}$ such that $c^2 a_n^2 e^{b_1+b_2} = -1$.*
- *(iv)* *If $p(z) \equiv b \in \mathbb{C}\setminus\{0\}$, then $f(z) = c_1 e^{dz}$ and $g(z) = c_2 e^{-dz}$, where $c_1, c_2, d(\neq 0) \in \mathbb{C}$ such that $(-1)^k a_n^2 (c_1 c_2)^{n+\sigma}(d(n+\sigma))^{2k} = b^2$.*

Theorem L. *Under the same situation in Theorem K if further $n > \frac{1}{2}\Gamma_1 + 2\Gamma_2 + \frac{3}{2}km_2 + \frac{3}{2}\sigma$ and $(P(f(z))\prod_{j=1}^{s}(f(z+c_j))^{\mu_j})^{(k)} - p(z)$ and $(P(g(z))\prod_{j=1}^{s}(g(z+c_j))^{\mu_j})^{(k)} - p(z)$ share $(0,1)$, then conclusions of Theorem K hold.*

Theorem M. *Under the same situation in Theorem K if further $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4\sigma$ and $(P(f(z))\prod_{j=1}^{s}(f(z+c_j))^{\mu_j})^{(k)} - p(z)$ and $(P(g(z))\prod_{j=1}^{s}(g(z+c_j))^{\mu_j})^{(k)} - p(z)$ share $(0,0)$, then conclusions of Theorem K hold.*

Remark 1. *It is easy to see that the conditions "f and g share $(0,\infty)_*$" and "$\delta(0;f) > 0$" in Theorem K are sharp by the following example.*

Example 2. *[16] Let $P(z) = (z-1)^6(z+1)^6 z^{11}$, $f(z) = \sin z$, $g(z) = \cos z$ and $\eta = 2\pi$. Clearly f and g do not share $(0,\infty)_*$ and $\delta(0;f) = 0$. Also it is easily seen that $n > 2\Gamma_2 + 2km_2 + 1$ and $(P(f(z))f(z+\eta))^{(k)} \equiv (P(g(z))g(z+\eta))^{(k)}$. Therefore $(P(f(z))f(z+\eta))^{(k)}$ and $(P(g(z))g(z+\eta))^{(k)}$ share $(1,\infty)$, but conclusions of Theorem K do not hold.*

Theorems K, L and M suggest the following questions as an open problems.

Question 4. *Can one remove the condition "$\deg(p) \leq n+\sigma-1$" in Theorems K-M ?*

Question 5. *Can one deduce generalized results in which Theorems K-M will be included ?*

Throughout the paper we use the following notations:
For two transcendental entire functions f, g and $c_0 \in \mathbb{C}$, we define $f_1(z) = f(z) - c_0$ and $g_1(z) = g(z) - c_0$. For $z_1 = z - c_0$, we define

$$\begin{aligned} P(z) = \sum_{i=0}^{n} a_i(z - c_0 + c_0)^i &= \sum_{i=0}^{n} a_i(z_1 + c_0)^i \\ &= a_{1,n}z_1^n + a_{1,n-1}z_1^{n-1} + \ldots + a_{1,0} = P_1(z_1), \text{ say} \end{aligned}$$

where $a_{1,i} \in \mathbb{C}$ $(i = 0, 1, \ldots, n)$ and $a_{1,n} = a_n$. Also throughout the paper we define $F_1(z) = \prod_{j=1}^{s}(f(z + c_j) - c_0)^{\mu_j} = \prod_{j=1}^{s}(f_1(z + c_j))^{\mu_j}$ and $G_1(z) = \prod_{j=1}^{s}(g(z + c_j) - c_0)^{\mu_j} = \prod_{j=1}^{s}(g_1(z + c_j))^{\mu_j}$, where $c_j \in \mathbb{C} \setminus \{0\}$ are distinct for $j = 1, 2, \ldots, s$ and $\mu_j \in \mathbb{N} \cup \{0\}$ such that $\sigma = \sum_{j=1}^{s} \mu_j > 0$.

2 Main results

Now taking the possible answers of the above Questions 4 and 5 into backdrop we obtain the following results.

Theorem 1. *Let f and g be two transcendental entire functions of finite order such that f and g share $(c_0, \infty)_*$, where $c_0 \in \mathbb{C}$ and let $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ such that $n > 2\Gamma_2 + 2km_2 + \sigma$. Suppose that P_1 has at least one zeros of multiplicities at least $k+1$ and $\delta(c_0; f) > 0$ when $k \geq 1$. If $(P_1(f_1)F_1)^{(k)} - p$ and $(P_1(g_1)G_1)^{(k)} - p$ share $(0, 2)$, where p is a non-zero polynomial, then one of the following cases hold.*

(1) $f - c_0 \equiv t(g - c_0)$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^d = 1$, where $d = \gcd(\sigma + p : p \in \{0, 1, \ldots, n\}$ with $a_{1,p} \neq 0)$.

(2) when $p \notin \mathbb{C}$, then one of the following cases holds.

(2)(i) $f(z) - c_0 = e^{\alpha(z)}$ and $g(z) - c_0 = e^{\beta(z)}$, where α and β are non-constant polynomials such that $n\alpha(z) + \sum_{j=1}^{s} \mu_j \alpha(z + c_j) = c\int_0^z p(z)dz + b_1$, $n\beta(z) + \sum_{j=1}^{s} \mu_j \beta(z + c_j) = -c\int_0^z p(z)dz + b_2$, $b_1, b_2, c(\neq 0) \in \mathbb{C}$ such that $c^2 a_n^2 e^{b_1+b_2} = -1$;

(2)(ii) $f(z) - c_0 = h(z)e^{az}$ and $g(z) - c_0 = th(z)e^{-az}$, where h is a non-constant polynomial and $a, t \in \mathbb{C} \setminus \{0\}$ such that $a_n^2 t^{n+\sigma} h^{2n}(z)(\prod_{j=1}^{s} h(z + c_j))^2 \equiv p^2(z)$.

(3) when $p(z) \equiv b$, then one of the following cases holds.

(3)(i) $f(z) - c_0 = e^{\alpha(z)}$ and $g(z) - c_0 = te^{-\alpha(z)}$ where α is a non-constant polynomial and $t \in \mathbb{C} \setminus \{0\}$ such that $a_n^2 t^{n+\sigma} = b^2$;

(3)(ii) $f(z) - c_0 = c_1 e^{d_1 z}$ and $g(z) - c_0 = c_2 e^{-d_1 z}$, where $c_1, c_2, d_1 \in \mathbb{C} \setminus \{0\}$ such that $(-1)^k a_n^2 (c_1 c_2)^{n+\sigma} (d_1(n + \sigma))^{2k} = b^2$.

Theorem 2. *Under the same situation in Theorem 1 if further* $n > \frac{1}{2}\Gamma_1 + 2\Gamma_2 + \frac{3}{2}km_2 + \frac{3}{2}\sigma$ *and* $(P_1(f_1)F_1)^{(k)} - p$ *and* $(P_1(g_1)G_1)^{(k)} - p(z)$ *share* $(0,1)$*, then conclusions of Theorem 1 hold.*

Theorem 3. *Under the same situation in Theorem 1 if further* $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4\sigma$ *and* $(P_1(f_1)F_1)^{(k)} - p$ *and* $(P_1(g_1)G_1)^{(k)} - p$ *share* $(0,0)$*, then conclusions of Theorem 1 hold.*

Remark 2. *It is easy to see that the conditions "f and g share $(c,\infty)_*$" and "$\delta(c;f) > 0$" in Theorem 1 are sharp by the following example.*

Example 3. *Let* $f(z) = \sin z + c$, $g(z) = \cos z + c$, $P_1(z) = (z-c-1)^6(z-c+1)^6(z-c)^{11}$ *and* $\eta = 2\pi$*. Clearly* f *and* g *do not share* $(c,\infty)_*$*. Note that*

$$f(z) = \sin z + c = \frac{e^{2iz}-1}{2ie^{iz}} + c = \frac{e^{2iz} + 2cie^{iz} - 1}{2ie^{iz}} = \frac{(e^{iz}-\alpha)(e^{iz}-\beta)}{2ie^{iz}}, \text{ say.}$$

Clearly $\alpha, \beta \neq 0$*. Also we have* $T(r,f) = 2\,T(r,e^{iz}) + S(r,e^{iz})$*. Since* $e^{iz} \neq 0, \infty$*, it follows that* $N(r,\alpha;e^{iz}) \sim T(r,e^{iz})$ *and* $N(r,\beta;e^{iz}) \sim T(r,e^{iz})$*. Therefore* $N(r,c;f) = N(r,\alpha;e^{iz}) + N(r,\beta;e^{iz}) \sim 2\,T(r,e^{iz})$*. Consequently*

$$\begin{aligned}\delta(c;f) = 1 - \limsup_{r\longrightarrow\infty} \frac{N(r,c;f)}{T(r,f)} &= 1 - \limsup_{r\longrightarrow\infty} \frac{N(r,\alpha;e^{iz}) + N(r,\beta;e^{iz})}{2T(r,e^{iz}) + S(r,e^{iz})} \\ &= 1 - \limsup_{r\longrightarrow\infty} \frac{2T(r,e^{iz})}{2T(r,e^{iz}) + S(r,e^{iz})} = 0.\end{aligned}$$

Also we see that $n > 2\Gamma_{62} + 2km_{62} + 1$ *and* $\big(P_1(f(z)-c)(f(z+\eta)-c)\big)^{(k)} \equiv \big(P_1(g(z)-c)(g(z+\eta)-c)\big)^{(k)}$*. Therefore* $\big(P_1(f(z)-c)(f(z+\eta)-c)\big)^{(k)}$ *and* $\big(P_1(g(z)-c)(g(z+\eta)-c)\big)^{(k)}$ *share* $(1,\infty)$*, but the conclusions of Theorem 1 do not hold.*

3 Lemmas

Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M\ \forall\ z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1+|h(z)|^2}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [15]).

For two non-constant entire functions F and G we define the auxiliary function H as follows

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{3.1}$$

Lemma 1. *[17] Let f be a non-constant meromorphic function and let $a_n(\not\equiv 0)$, a_{n-1}, ... , a_0 be meromorphic functions such that $T(r, a_i) = S(r, f)$ for $i = 0, 1, 2, ..., n$. Then $T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f)$.*

Lemma 2. *[3] Let f be a meromorphic function of finite order ρ and let $c \in \mathbb{C}\backslash\{0\}$ be fixed. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\left(r^{\rho-1+\varepsilon}\right).$$

Lemma 3. *[4] Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$N(r, 0; f(z+c)) \leq N(r, 0; f(z)) + S(r, f)$$
$$\text{and} \quad \overline{N}(r, 0; f(z+c)) \leq \overline{N}(r, 0; f(z)) + S(r, f).$$

Lemma 4. *Let f be a transcendental entire function of finite order and $n \in \mathbb{N}$. Then for each $\varepsilon > 0$, we have $T(r, P_1(f_1)F_1) = (n + \sigma)\, T(r, f_1) + O(r^{\rho-1+\varepsilon})$.*

Proof. Proof follows directly from Lemma 2.6 [10]. □

Lemma 5. *[9] Let h be a non-constant meromorphic function such that $\overline{N}(r, 0; h) + \overline{N}(r, \infty; h) = S(r, h)$. Let $f = a_0 h^p + a_1 h^{p-1} + \ldots + a_p$ and $g = b_0 h^q + b_1 h^{q-1} + \ldots + b_q$ be polynomials in h with co-efficients $a_0, a_1, \ldots, a_p, b_0, b_1, \ldots, b_q$ being small functions of h and $a_0 b_0 a_p \neq 0$. If $q \leq p$, then $m\big(r, \frac{g}{f}\big) = S(r, h)$.*

Lemma 6. *Let f and g be two transcendental entire functions of finite order, $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ such that $n > 2\Gamma_1 + 2km_2 + \sigma$. Let $F = \frac{(P_1(f_1)F_1)^{(k)}}{\alpha}$ and $G = \frac{(P_1(g_1)G_1)^{(k)}}{\alpha}$, where α is a small function of f and g. If $H \equiv 0$, then one of the following two cases holds.*

(i) $(P_1(f_1)F_1)^{\mu_j})^{(k)} \equiv (P_1(g_1)G_1)^{(k)}$,

(ii) $(P_1(f_1)F_1)^{(k)}(P_1(g_1)G_1)^{(k)} \equiv \alpha^2$,
where $(P_1(f_1)F_1)^{(k)} - \alpha$ *and* $(P_1(g_1)G_1)^{(k)} - \alpha$ *share* $(0, \infty)$.

Proof. Proof follows directly from Lemma 2.8 [10]. □

Lemma 7. *Let f and g be two transcendental entire functions of finite order such that f and g share $(c_0, \infty)_*$. Let $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ such that $n > 2m_1 + 2km_2 + \sigma$. If $(P_1(f_1)F_1)^{(k)} \equiv (P_1(g_1)G_1)^{(k)}$, then $f - c_0 \equiv t(g - c_0)$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^d = 1$, where $d = \gcd(\sigma + p : p \in \{0, 1, \ldots, n\}$ with $a_{1,p} \neq 0)$.*

Proof. Suppose $(P_1(f_1)F_1)^{(k)} \equiv (P_1(g_1)G_1)^{(k)}$. Using Lemma 2.9 [10], one can easily obtain

$$P_1(f_1(z)) \prod_{j=1}^{s} (f_1(z + c_j))^{\mu_j} \equiv P_1(g_1(z)) \prod_{j=1}^{s} (g_1(z + c_j))^{\mu_j}. \tag{3.2}$$

Let $h=\frac{f_1}{g_1}$. Now by putting $f_1=hg_1$ into (3.2), we get

$$\begin{aligned}
& a_{1,n}g_1^n(z)\left(h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\right) \qquad (3.3)\\
& +a_{1,n-1}g_1^{n-1}(z)\left(h^{n-1}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\right)+\ldots\\
& +a_{1,1}g_1(z)\left(h(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\right)+a_{1,0}\left(\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\right)\equiv 0.
\end{aligned}$$

First we suppose $h\in\mathbb{C}\setminus\{0\}$. Now from (3.3), we get

$$\begin{aligned}
a_{1,n}g_1^n\left(h^{n+\sigma}-1\right)+a_{1,n-1}g_1^{n-1}\left(h^{n+\sigma-1}(z)-1\right)+\\
\ldots+a_{1,1}g_1\left(h^{\sigma+1}-1\right)+a_{1,0}\left(h^{\sigma}-1\right)\equiv 0,
\end{aligned}$$

which implies that $h^d=1$, where $d=\gcd(\sigma+p:p\in\{0,1,\ldots,n\}$ with $a_{1,p}\neq 0)$. Thus $f-c_0\equiv t(g-c_0)$ for a constant t such that $t^d=1$, where $d=\gcd(\sigma+p:$ $p\in\{0,1,\ldots,n\}$ with $a_{1,p}\neq 0)$.
Next we suppose $h\notin\mathbb{C}$. Since f_1 and g_1 share $(0,\infty)_*$, it follows that h is a non-constant meromorphic function such that $N(r,0;h)+N(r,\infty;h)=O(\log r)$ as $r\to\infty$. Also we note that $\rho(h)\leq\max\{\rho(f),\rho(g)\}<\infty$, i.e., h is of finite order.
Suppose that h is a rational function. Let $P_1(f_1)=a_{1,n}f_1^n$. Then from (3.3), we get

$$h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}\equiv 1,\ \text{i.e.,}\ h^n(z)=\frac{1}{\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}}. \qquad (3.4)$$

Let

$$h=\frac{h_1}{h_2}, \qquad (3.5)$$

where h_1 and h_2 are two nonzero relatively prime polynomials. From (3.5), we have

$$T(r,h)=\max\{\deg(h_1),\deg(h_2)\}\log r+O(1). \qquad (3.6)$$

Now from (3.4), (3.5) and (3.6), we have

$$\begin{aligned}
& n\ \max\{\deg(h_1),\deg(h_2)\}\ \log r \qquad (3.7)\\
= \ & T(r,h^n)+O(1)\\
\leq \ & T(r,\prod_{j=1}^{s}(h(z+c_j))^{\mu_j})+O(1)\\
\leq \ & \sigma\ \max\{\deg(h_1),\deg(h_2)\}\ \log r+O(1).
\end{aligned}$$

We see that $\max\{\deg(h_2), \deg(h_3)\} \geq 1$. Since $n > \sigma$, we arrive at a contradiction from (3.7).
Let $P_1(f_1) \not\equiv a_{1,n}f_1^n$. Suppose $a_{1,p}$ is the last non-vanishing term of $P_1(z_1)$, where $p \in \{0, 1, \ldots, n-1\}$. Then from (3.3), we have

$$\begin{aligned}
& a_{1,n}g_1^{n-p}(z)\left(h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\right) \\
& +a_{1,n-1}g_1^{n-p-1}(z)\left(h^{n-1}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\right)+\ldots \\
& +a_{1,p+1}g_1(z)\left(h^{p+1}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\right) \\
\equiv\ & -a_{1,p}\left(h^{p}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1\right).
\end{aligned} \tag{3.8}$$

Now from Lemma 1 and (3.8), we get $(n-p)\ T(r, g_1) = S(r, g_1)$, which is a contradiction.
Next we suppose that h is a transcendental meromorphic function. We claim that

$$h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j} \not\equiv 1.$$

If not, suppose

$$h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j} \equiv 1, \text{ i.e., } h^n(z) = \frac{1}{\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}}. \tag{3.9}$$

Now by Lemmas 1, 2 and 3, we get

$$\begin{aligned}
n\,T(r,h) &= T(r,h^n)+S(r,h) \\
&= T\left(r,\frac{1}{\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}}\right)+S(r,h) \\
&\leq \sum_{j=1}^{s}\mu_j N(r,0;h(z+c_j))+\sum_{j=1}^{s}\mu_j\ m\left(r,\frac{1}{h(z+c_j)}\right)+S(r,h) \\
&\leq \sum_{j=1}^{s}\mu_j\ N(r,0;h(z))+\sum_{j=1}^{s}\mu_j\ m\left(r,\frac{1}{h(z)}\right)+S(r,h) \\
&\leq \sigma\,T(r,h)+S(r,h),
\end{aligned}$$

which is a contradiction.
Let $P_1(f_1) = a_{1,n}f_1^n$. Then from (3.3), we get $h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j} \equiv 1$, which

is a contradiction.
Let $P_1(f_1) \not\equiv a_{1,n}f_1^n$. Suppose $a_{1,p}$ is the last non-vanishing term of $P_1(z_1)$, where $p \in \{0, 1, \ldots, n-1\}$. Then from (3.8), we have

$$a_{1,n-1}g_1^{n-p-1}(z)\frac{h^{n-1}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1}{h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1}+\ldots \tag{3.10}$$
$$+a_{1,p+1}g_1(z)\frac{h^{p+1}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1}{h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1}$$
$$+a_{1,p}\frac{h^{p}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1}{h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1}\equiv -a_{1,n}g_1^{n-p},$$

where $p \in \{0, 1, \ldots, n-1\}$. Let

$$H_i(z)=\frac{h^{i}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1}{h^n(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}-1},$$

where $i = p, p+1, \ldots, n-1$. Then we have

$$H_i(z)=\frac{h^{\sigma+i}(z)\frac{\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}}{h^{\sigma}(z)}-1}{h^{n+\sigma}(z)\frac{\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}}{h^{\sigma}(z)}-1}.$$

Since h is a transcendental meromorphic function, we have $S(r,h)+O(\log r)=S(r,h)$. Now using Lemma 2, we get

$$\begin{aligned}
&T\left(r,\prod_{j=1}^{s}\left(\frac{h(z+c_j)}{h(z)}\right)^{\mu_j}\right)\\
\leq\ &\sum_{j=1}^{s}\mu_j\,T\left(r,\frac{h(z+c_j)}{h(z)}\right)\\
\leq\ &\sum_{j=1}^{s}\mu_j\left(m\left(r,\frac{h(z+c_j)}{h(z)}\right)+N\left(r,\infty;\frac{h(z+c_j)}{h(z)}\right)\right)\\
\leq\ &\sum_{j=1}^{s}\mu_j\left(S(r,h)+O(\log r)\right)=S(r,h).
\end{aligned}$$

This implies that $\prod_{j=1}^{s}\left(\frac{h(z+c_j)}{h(z)}\right)^{\mu_j}\in S(h)$. Since $n+\sigma > i+\sigma$, using Lemma 5, we get $m(r,H_i)=S(r,h)$, where $i = p, p+1, \ldots, n-1$.
Also Lemma 4 and (3.2) yield $T(r,f_1)+S(r,f_1)=T(r,g_1)+S(r,g_1)$. Since $h=\frac{f_1}{g_1}$, it follows that $T(r,h)\leq 2T(r,g_1)+S(r,g_1)$ and $S(r,h)$ can be replaced by $S(r,g_1)$. Therefore $m(r,H_i)=S(r,g_1)$, where $i=p,p+1,\ldots,n-1$. For the sake of simplicity we assume that $a_{1,n-1}\neq 0$. Then from (3.10), we have

$$-a_{1,n}g_1^{n-p} \equiv a_{1,n-1}g_1^{n-p-1}H_{n-1}+\ldots+a_{1,p+1}g_1H_{p+1}+a_{1,p}H_p, \tag{3.11}$$

where $p \in \{0, 1, \ldots, n-1\}$. Now from (3.11), we obtain

$$\begin{aligned}
& (n-p)m(r, g_1) \\
\leq \;& m\left(r, -a_{1,n}g_1^{n-p}\right) + O(1) \\
= \;& m\left(r, a_{1,n-1}g_1^{n-p-1}H_{n-1} + \ldots + a_{1,p+1}g_1H_{p+1} + a_{1,p}H_p\right) + O(1) \\
\leq \;& m\left(r, a_{1,n-1}g_1^{n-p-1}H_{n-1} + \ldots + a_{1,p+1}g_1H_{p+1}\right) + S(r, g_1) \\
\leq \;& m(r, g_1) + m\left(r, a_{1,n-1}g_1^{n-p-2}H_{n-1} + \ldots + a_{1,p+1}H_{p+1}\right) + S(r, g_1) \\
\leq \;& m(r, g_1) + m\left(r, a_{1,n-1}g_1^{n-p-2}H_{n-1} + \ldots + a_{1,p+2}g_1H_{p+2}\right) + S(r, g_1) \\
\leq \;& 2m(r, g_1) + m\left(r, a_{1,n-1}g_1^{n-p-3}H_{n-1} + \ldots + a_{1,p+2}H_{p+2}\right) + S(r, g_1) \\
\leq \;& \ldots\ldots\ldots\ldots\ldots \\
\leq \;& (n-p-1)m(r, g_1) + S(r, g_1).
\end{aligned}$$

This intimates that $m(r, g_1) = S(r, g_1)$. Since g_1 is a transcendental entire function, $N(r, \infty; g_1) = 0$ and so $T(r, g_1) = m(r, g_1) = S(r, g_1)$, which is a contradiction. This completes the proof. □

Lemma 8. *[21] Let F be a family of meromorphic functions in the unit disc Δ such that all zeros of functions in F have multiplicity greater than or equal to l and all poles of functions in F have multiplicity greater than or equal to j and α be a real number satisfying $-l < \alpha < j$. Then F is not normal in any neighborhood of $z_0 \in \Delta$, if and only if there exist*

(i) points $z_n \in \Delta$, $z_n \to z_0$,

(ii) positive numbers ρ_n, $\rho_n \to 0^+$ and

(iii) functions $f_n \in F$,

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where g is a non-constant meromorphic function. The function g may be taken to satisfy the normalisation $g^{\#}(\zeta) \leq g^{\#}(0) = 1(\zeta \in \mathbb{C})$.

Lemma 9. *[2] Let f be a meromorphic function on $\mathbb{C}$ with finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, then f is of order at most 1.*

Lemma 10. *[6] If f is an integral function of finite order, then*

$$\sum_{a \neq \infty} \delta(a, f) \leq \delta(0, f').$$

Lemma 11. *[[6], Lemma 3.5] Suppose that F is meromorphic in a domain D and set $f = \frac{F'}{F}$. Then for $n \in \mathbb{N}$,*

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4} (f')^2 + P_{n-3}(f),$$

where $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ *and* $P_{n-3}(f)$ *is a differential polynomial with constant coefficients, which vanishes identically for* $n \leq 3$ *and has degree* $n-3$ *when* $n > 3$.

Lemma 12. *Let* f *and* g *be two transcendental entire functions of finite order such that* f *and* g *share* $(c_0, \infty)_*$ *and* $\delta(c_0, f) > 0$. *Let* $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $\mu_j \in \mathbb{N} \cup \{0\}$ $(j = 1, 2, \ldots, s)$ *and* p *be a non-zero polynomial. Suppose* $\big(P_1(f_1)F_1\big)^{(k)}\big(P_1(g_1)G_1\big)^{(k)} \equiv p^2$, *where* $\big(P_1(f_1)F_1\big)^{(k)} - p$ *and* $\big(P_1(g_1)G_1\big)^{(k)} - p$ *share* $(0, \infty)$. *Now*

(1) when $p \notin \mathbb{C}$, *then one of the following cases holds.*

(1)(i) $f(z) - c_0 = e^{\alpha(z)}$ *and* $g(z) - c_0 = e^{\beta(z)}$, *where* α *and* β *are non-constant polynomials such that* $n\alpha(z) + \sum_{j=1}^{s} \mu_j \alpha(z + c_j) = c\int_0^z p(z)dz + b_1$, $n\beta(z) + \sum_{j=1}^{s} \mu_j \beta(z + c_j) = -c\int_0^z p(z)dz + b_2$, $b_1, b_2, c(\neq 0) \in \mathbb{C}$ *such that* $c^2 a_n^2 e^{b_1+b_2} = -1$;

(1)(ii) $f(z) - c_0 = h(z)e^{az}$ *and* $g(z) - c_0 = th(z)e^{-az}$, *where* h *is a non-constant polynomial and* $a, t \in \mathbb{C}\backslash\{0\}$ *such that* $a_n^2 t^{n+\sigma} h^{2n}(z)(\prod_{j=1}^{s} h(z+c_j))^2 \equiv p^2(z)$.

(2) when $p(z) \equiv b$, *then one of the following cases holds.*

(2)(i) $f(z) - c_0 = e^{\alpha(z)}$ *and* $g(z) - c_0 = te^{-\alpha(z)}$ *where* α *is a non-constant polynomial and* $t \in \mathbb{C} \setminus \{0\}$ *such that* $a_n^2 t^{n+\sigma} = b^2$;

(2)(ii) $f(z) - c_0 = c_1 e^{d_1 z}$ *and* $g(z) - c_0 = c_2 e^{-d_1 z}$, *where* $c_1, c_2, d_1 \in \mathbb{C} \setminus \{0\}$ *such that* $(-1)^k a_n^2 (c_1 c_2)^{n+\sigma}(d_1(n+\sigma))^{2k} = b^2$.

Proof. Suppose
$$\left(P_1(f_1)F_1)^{\mu_j}\right)^{(k)}\left(P_1(g_1)G_1\right)^{(k)} \equiv p^2. \tag{3.12}$$
Using Lemma 2.12 [10], one can easily prove that $(P_1(f_1)F_1)^{(k)}$ and $(P_1(g_1)G_1)^{(k)}$ share $(0, \infty)$. Now we want to show that $P_1(z_1) = a_{1,n}z_1^n$.
First we suppose $k = 0$. Then from (3.12) we get

$$P_1(f_1)F_1P_1(g_1)G_1 \equiv p^2. \tag{3.13}$$

From (3.13), we have $N(r, 0; P_1(f_1)) = O(\log r)$. Clearly $P_1(z_1)$ can not have more than one distinct zeros otherwise we get a contradiction from the second fundamental theorem. Hence we conclude that $P_1(z_1)$ has only one zero and so we may write $P_1(f_1) = a_{1,n}(f_1 - a)^n$, where $a \in \mathbb{C}$. Since f_1 and g_1 are transcendental entire functions of finite order, from (3.13) we obtain that

$$f_1(z) = \alpha_1(z)e^{\beta_1(z)} + a, \quad g_1(z) = \alpha_2(z)e^{\beta_2(z)} + a \tag{3.14}$$

$$F_1(z) = \alpha_3(z)e^{\beta_3(z)} \text{ and } G_1(z) = \alpha_4(z)e^{\beta_4(z)}, \tag{3.15}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are non-zero polynomials and $\beta_1, \beta_2, \beta_3, \beta_4$ are non-constant polynomials. Now from (3.14) and (3.15), we have

$$\prod_{j=1}^{s} \left(\alpha_1(z+c_j)e^{\beta_1(z+c_j)} + a\right)^{\mu_j} = \alpha_3(z)e^{\beta_3(z)}$$

and so we have $\overline{N}(r, -a; \alpha_1(z+c_1)e^{\beta_1(z+c_1)}) = O(\log r)$. Now using Lemma 1, we get from the second fundamental theorem that

$$\begin{aligned} & T\left(r, e^{\beta_1(z+c_1)}\right) \\ = \;& T\left(r, \alpha_1(z+c_1)e^{\beta_1(z+c_1)}\right) + S\left(r, \alpha_1(z+c_1)e^{\beta_1(z+c_1)}\right) \\ \leq \;& \overline{N}\left(r, \infty; \alpha_1(z+c_1)e^{\beta_1(z+c_1)}\right) + \overline{N}\left(r, 0; \alpha_1(z+c_1)e^{\beta_1(z+c_1)}\right) \\ & + \overline{N}\left(r, -a; \alpha_1(z+c_1)e^{\beta_1(z+c_1)}\right) + S\left(r, \alpha_1(z+c_1)e^{\beta_1(z+c_1)}\right) \\ = \;& O(\log r) + S\left(r, \alpha_1(z+c_1)e^{\beta_1(z+c_1)}\right) = S\left(r, \alpha_1(z+c_1)e^{\beta_1(z+c_1)}\right), \end{aligned}$$

which is impossible. Hence $a = 0$ and so $P_1(z_1) = a_{1,n}z_1^n$. Therefore

$$(a_n f_1^n F_1)(a_n g_1^n G_1) \equiv p.$$

Next we suppose $k \in \mathbb{N}$. Here it is given that $P_1(z_1) = a_{1,n}z_1^n + a_{1,n-1}z_1^{n-1} + \ldots + a_{1,1}z_1 + a_{1,0}$. Suppose that

$$P_1(z_1) = (z_1 - a)^m (b_{l_1} z_1^{l_1} + b_{l_1-1} z_1^{l_1-1} + \ldots + b_1 z_1 + b_0), \tag{3.16}$$

where $m + l_1 = n$ and $a_{1,n} = b_{l_1}$. Let $z_2 = z_1 - a$. Then (3.16) becomes

$$\left.\begin{aligned} P_1(z_1) &= z_2^m (d_{l_1} z_2^{l_1} + d_{l_1-1} z_2^{l_1-1} + \ldots + d_1 z_2 + d_0), \\ \text{i.e.,}\quad P_1(z_1) &= z_2^m P_2(z_2), \end{aligned}\right\} \tag{3.17}$$

where $P_2(z_2) = d_{l_1} z_2^{l_1} + d_{l_1-1} z_2^{l_1-1} + \ldots + d_1 z_2 + d_0$. Clearly

$$P_1(f_1) = f_2^m P_2(f_2). \tag{3.18}$$

By the given condition, since P_1 has at least one zero of multiplicity at least $k+1$ when $k \in \mathbb{N}$, for the sake of simplicity we may assume that $m > k$. Since $P_1(f_1) = f_2^m P_2(f_2)$ and $m > k$, from (3.12) we conclude that the zeros of both f_2 and g_2 are the zeros of p. As the the number of zeros of p is finite, it follows that f_2 as well as g_2 have finitely many zeros. Therefore f_2 takes the form $f_2 = h_0 e^{\alpha}$, where h_0 is a non-zero polynomial and α is a non-constant polynomial.
Note that $f_2' = f_1' = (h_0' + h_0\alpha')e^{\alpha}$. Therefore $\delta(0, f_1') = 1$ and $\delta(a, f_1) = 1$. Since $\delta(0, f_1) > 0$, then by Lemma 10 we conclude that $a = 0$ and so $f_1 = h_0 e^{\alpha}$. Also in that case we have $P_1(f_1) = f_1^m (b_{l_1} f_1^{l_1} + b_{l_1-1} f_1^{l_1-1} + \ldots + b_1 f_1 + b_0)$.

Now we claim that $b_i = 0$ for $i = 0, 1, \ldots, l_1 - 1$. If not, for the sake of simplicity we may assume that $b_{l_1}, b_0 \neq 0$. Let

$$\mathcal{H}_i(z) = h^{m+i}(z) \prod_{j=1}^{s} (h(z+c_j))^{\mu_j}$$

$$\text{and} \qquad \xi_i(z) = (m+i)\alpha(z) + \sum_{j=1}^{s} \mu_j \alpha(z+c_j),$$

where $i = 0, 1, \ldots, l_1$. Clearly $f_1^{m+i}(z)F_1(z) = \mathcal{H}_i(z)\, e^{\xi_i(z)}$, where $i = 0, 1, \ldots, l_1$. Then by induction we have

$$\left(b_i f_1^{m+i} F_1\right)^{(k)} = \eta_i \left(\xi_i', \xi_i'', \ldots, \xi_i^{(k)}, \mathcal{H}_i, \mathcal{H}_i', \ldots, \mathcal{H}_i^{(k)}\right) e^{\xi_i}, \tag{3.19}$$

where $\eta_i \left(\xi_i', \xi_i'', \ldots, \xi_i^{(k)}, \mathcal{H}_i, \mathcal{H}_i', \ldots, \mathcal{H}_i^{(k)}\right)$, $i = 0, 1, \ldots, l_1$ are differential polynomials in $\xi_i', \xi_i'', \ldots, \xi_i^{(k)}, \mathcal{H}_i, \mathcal{H}_i', \ldots, \mathcal{H}_i^{(k)}$. If possible suppose

$$\eta_i \left(\xi_i', \xi_i'', \ldots, \xi_i^{(k)}, \mathcal{H}_i, \mathcal{H}_i', \ldots, \mathcal{H}_i^{(k)}\right) \equiv 0, \; i = 0, 1, \ldots, l_1.$$

Then from (3.19), we have $f_1^{m+i}F_1 \equiv p_1$, where p_1 is a polynomial such that $\deg(p_1) \leq k-1$. Therefore $T\left(r, f_1^{m+i}F_1\right) = O(\log r)$ and so by Lemma 4, we get $T(r, f_1) = O(\log r) + O\left(r^{\rho-1+\varepsilon}\right)$ for all $\varepsilon > 0$, which contradicts the fact that f_1 is a transcendental entire function. Hence $\eta_i \left(\xi_i', \xi_i'', \ldots, \xi_i^{(k)}, \mathcal{H}_i, \mathcal{H}_i', \ldots, \mathcal{H}_i^{(k)}\right) \not\equiv 0$, $i = 0, 1, \ldots, l_1$. Therefore

$$\begin{aligned} (P_1(f_1)F_1)^{(k)} &= \left(f_1^m (b_{l_1} f_1^{l_1} + b_{l_1-1} f_1^{l_1-1} + \ldots + b_1 f_1 + b_0) F_1\right)^{(k)} \qquad (3.20) \\ &= \sum_{i=0}^{l_1} \left(b_i f_1^{m+i} F_1\right)^{(k)} \\ &= \sum_{i=0}^{l_1} \eta_i e^{\xi_i} \\ &= \exp\left\{ m\alpha(z) + \sum_{j=1}^{s} \alpha(z+c_j) \right\} \times \sum_{i=0}^{l_1} \eta_i(z) e^{i\alpha(z)}. \end{aligned}$$

Note that H_i and ξ_i are polynomials for $i = 0, 1, \ldots, l_1$ and so η_i are also polynomials for $i = 0, 1, \ldots, l_1$. Since f_1 is a transcendental entire function, it follows that $T(r, \eta_i) = S(r, f)$ for $i = 0, 1, \ldots, l_1$. Also from (3.12), we see that $\overline{N}(r, 0; (P_1(f_1)F_1)^{(k)}) = O(\log r)$ and so from (3.20), we have

$$\overline{N}(r, 0; \eta_{l_1} e^{l_1 \alpha} + \ldots + \eta_1 e^{\alpha} + \eta_0) \leq S(r, f_1). \tag{3.21}$$

Since $\eta_{l_1} e^{l_1 \alpha} + \ldots + \eta_1 e^{\alpha}$ is a transcendental entire function and η_0 is a polynomial, it follows that η_0 is a small function of $\eta_{l_1} e^{l_1 \alpha} + \ldots + \eta_1 e^{\alpha}$. Now in view of Lemma

1, (3.21) and using second fundamental theorem for small functions (see [19]), we obtain

$$\begin{aligned} l_1\, T(r,f_1)=l_1 T(r,e^{\alpha}) &= T\left(r,\eta_{l_1}e^{l_1\alpha}+\ldots+\eta_1 e^{\alpha}\right)+S(r,f_1)\\ &\leq \overline{N}\left(r,0;\eta_{l_1}e^{l_1\alpha}+\ldots+\eta_1 e^{\alpha}\right)\\ &\quad +\overline{N}\left(r,0;\eta_{l_1}e^{l_1\alpha}+\ldots+\eta_1 e^{\alpha}+\eta_0\right)+S(r,f_1)\\ &\leq \overline{N}\left(r,0;\eta_{l_1}e^{(l_1-1)\alpha}+\ldots+\eta_1\right)+S(r,f_1)\\ &\leq (l_1-1)T(r,f)+S(r,f_1), \end{aligned}$$

which is a contradiction. Hence $b_i=0$ for $i=0,1,\ldots,l_1-1$ and so $P_1(f_1)=a_{1,n}f_1^n$. By the given condition, since P_1 has at least one zero of multiplicity at least $k+1$ when $k\in\mathbb{N}$, it follows that $n\geq k+1$. Therefore (3.12) yields $(a_n f_1^n F_1)^{(k)}(a_n g_1^n G_1)^{(k)}\equiv p^2$.
Thus in either cases we have

$$(a_n f_1^n F_1)^{(k)}(a_n g_1^n G_1)^{(k)}\equiv p^2, \tag{3.22}$$

where $(a_n f_1^n F_1)^{(k)}$ and $(a_n g_1^n G_1)^{(k)}$ share $(0,\infty)$. Let z_q be a zero of f_1 of multiplicity q and z_r be a zero of g_1 of multiplicity r. Clearly z_q will be a zero of f_1^n of multiplicity nq and z_r will be a zero of g_1^n of multiplicity nr. Since f_1 and g_1 are transcendental entire functions, it follows that z_q and z_r must be the zeros of $(a_n f_1^n F_1)^{(k)}$ and $(a_n g_1^n G_1)^{(k)}$ of multiplicities at least $q_1-k(\geq nq-k\geq 1)$ and $r_1-k(\geq nr-k\geq 1)$ respectively. Since $(a_n f_1^n F_1)^{(k)}$ and $(a_n g_1^n G_1)^{(k)}$ share $(0,\infty)$, it follows that $z_p=z_q$. Hence f_1 and g_1 share $(0,\infty)$. Consequently F_1 and G_1 share $(0,\infty)$ and so $a_n f_1^n F_1$ and $a_n g_1^n G_1$ share $(0,\infty)$.
We consider the following cases.
Case 1. Suppose 0 is a Picard exceptional value of both f_1 and g_1. Since $f_1\neq 0$ and $g_1\neq 0$, so we can take

$$f_1(z)=e^{\alpha(z)} \text{ and } g_1(z)=e^{\beta(z)}, \tag{3.23}$$

where α and β are two non-constant entire functions. Since f_1 and g_1 are of finite order, both α and β are non-constant polynomials. Let

$$\alpha_1(z)=n\,\alpha(z)+\sum_{j=1}^{s}\mu_j\alpha(z+c_j) \text{ and } \beta_1(z)=n\,\beta(z)+\sum_{j=1}^{s}\mu_j\beta(z+c_j). \tag{3.24}$$

We now consider the following sub-cases.
Sub-case 1.1. Let $\deg(p)=l\in\mathbb{N}$. Following sub-cases are immediately.
Sub-case 1.1.1. Let $k=0$. Note that $a_n f_1^n F_1\neq 0$ and $a_n g_1^n G_1\neq 0$. Since $\deg(p)\geq 1$, from (3.22) we arrive at a contradiction.
Sub-case 1.1.2. Let $k=1$. Then from (3.22), we get

$$a_n^2\alpha_1'\beta_1' e^{\alpha_1+\beta_1}\equiv p^2. \tag{3.25}$$

Also from (3.25), we can conclude that $\alpha_1 + \beta_1 \equiv c_1 \in \mathbb{C}$ and so $\alpha_1' + \beta_1' \equiv 0$. Thus from (3.25), we get $a_n^2 e^{c_1}\alpha_1'\beta_1' \equiv p^2$. By computation we get

$$\alpha_1'(z) = cp(z) \text{ and } \beta_1'(z) = -cp(z), \text{ where } c \in \mathbb{C}\setminus\{0\}. \tag{3.26}$$

Hence

$$\alpha_1(z) = cQ(z) + b_1 \text{ and } \beta_1(z) = -cQ(z) + b_2, \tag{3.27}$$

where $Q(z) = \int_0^z p(z)dz$ and $b_1, b_2 \in \mathbb{C}$. Finally we take f and g as

$$f(z) - c_0 = e^{\alpha(z)} \text{ and } g(z) - c_0 = e^{\beta(z)}$$

such that $n\alpha(z)+\sum_{j=1}^s \mu_j\alpha(z+c_j) = c\int_0^z p(z)dz+b_1$, $n\beta(z)+\sum_{j=1}^s \mu_j\beta(z+c_j) = -c\int_0^z p(z)dz + b_2$, where $b_1, b_2 \in \mathbb{C}$ and $c \in \mathbb{C}\setminus\{0\}$ such that $a_n^2c^2e^{b_1+b_2} = -1$.

Sub-case 1.1.3. Let $k \in \mathbb{N}\setminus\{1\}$. Then from (3.22), we see that $\alpha_1+\beta_1 \in \mathbb{C}$, i.e., $\alpha_1' \equiv -\beta_1'$. Therefore $\deg(\alpha_1) = \deg(\beta_1)$. If possible suppose $\deg(\alpha_1) = \deg(\beta_1) = 1$. Then clearly $(a_nf_1^nF_1)^{(k)} \neq 0$ and $(a_ng_1^nG_1)^{(k)} \neq 0$. Since $\deg(p) \geq 1$, we get a contradiction from (3.22). Hence $\deg(\alpha_1) = \deg(\beta_1) \geq 2$. Now from (3.23) and Lemma 11, we see that

$$(a_nf_1^nF_1)^{(k)} = \left(n^k(\alpha_1')^k + \frac{k(k-1)}{2}n^{k-1}(\alpha_1')^{k-2}\alpha_1'' + P_{k-2}(\alpha_1')\right)e^{\alpha_1}.$$

Similarly we have

$$\begin{aligned}&(a_ng_1^nG_1)^{(k)}\\ =\ &\left(n^k(\beta_1')^k + \frac{k(k-1)}{2}n^{k-1}(\beta_1')^{k-2}\beta_1'' + P_{k-2}(\beta_1')\right)e^{\beta_1}\\ =\ &\left((-1)^kn^k(\alpha_1')^k - \frac{k(k-1)}{2}n^{k-1}(-1)^{k-2}(\alpha_1')^{k-2}\alpha_1'' + P_{k-2}(-\alpha_1')\right)e^{\beta_1}.\end{aligned}$$

Since $\deg(\alpha_1) \geq 2$, we observe that $\deg((\alpha_1')^k) \geq k\ \deg(\alpha_1')$ and so $(\alpha_1')^{k-2}\alpha_1''$ is either a non-zero constant or $\deg((\alpha_1')^{k-2}\alpha_1'') \geq (k-1)\ \deg(\alpha_1') - 1$. Also we see that

$$\deg\left((\alpha_1')^k\right) > \deg\left((\alpha_1')^{k-2}\alpha_1''\right) > \deg\left(P_{k-2}(\alpha_1')\right) (\text{or } \deg\left(P_{k-2}(-\alpha_1')\right)).$$

Let

$$\alpha_1'(z) = e_tz^t + e_{t-1}z^{t-1} + \ldots + e_0,$$

where $e_0, e_1, \ldots, e_t(\neq 0) \in \mathbb{C}$. Then we have

$$(\alpha_1'(z))^i = e_t^iz^{it} + ie_t^{i-1}e_{t-1}z^{it-1} + \ldots...,$$

where $i \in \mathbb{N}$. Therefore we have

$$(a_nf_1^nF_1)^{(k)} = \left(n^ke_t^kz^{kt} + kn^ke_t^{k-1}e_{t-1}z^{kt-1} + \ldots + (D_1+D_2)z^{kt-t-1} + \ldots\right)e^{\alpha_1}$$

and

$$\begin{aligned}(a_n g_1^n G_1)^{(k)} &= \Big((-1)^k n^k e_t^k z^{kt} + (-1)^k k n^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots \\ &\quad + \Big((-1)^k D_1 + (-1)^{k-1} D_2\Big) z^{kt-t-1} + \ldots\Big) e^{\beta_1},\end{aligned}$$

where $D_1, D_2 \in \mathbb{C}$ such that $D_2 = \frac{k(k-1)}{2} t n^{k-1} e_t^{k-1}$. Since $(a_n f_1^n F_1)^{(k)}$ and $(a_n g_1^n G_1)^{(k)}$ share $(0, \infty)$, we have

$$\begin{aligned}& n^k e_t^k z^{kt} + k n^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots + (D_1 + D_2) z^{kt-t-1} + \ldots \\ = \;& d_1^* \Big((-1)^k n^k e_t^k z^{kt} + (-1)^k k n^k e_t^{k-1} e_{t-1} z^{kt-1} + \ldots \\ & + \big((-1)^k D_1 + (-1)^{k-1} D_2\big) z^{kt-t-1} + \ldots\Big)\end{aligned} \tag{3.28}$$

where $d_1^* \in \mathbb{C} \setminus \{0\}$. From (3.28), we get $D_2 = 0$, i.e., $\frac{k(k-1)}{2} t n^{k-1} e_t^{k-1} = 0$, which is impossible for $k \geq 2$.

Sub-case 1.2. Let $p(z) = b \in \mathbb{C} \setminus \{0\}$. Since $n > k$, we have $f_1 \neq 0$ and $g_1 \neq 0$. In this case also we have $f_1(z) = e^{\alpha(z)}$ and $g_1(z) = e^{\beta(z)}$, where α and β are non-constant polynomials. We now consider the following two sub-cases.

Sub-case 1.2.1. Let $k = 0$. Now from (3.22) and (3.23), we have

$$a_n^2 \exp\{n(\alpha(z) + \beta(z)) + \sum_{i=1}^{s} \mu_j(\alpha(z + c_j) + \beta(z + c_j))\} \equiv b.$$

Therefore we must have $n(\alpha(z)+\beta(z)) + \sum_{i=1}^{s} \mu_j(\alpha(z+c_j)+\beta(z+c_j)) \in \mathbb{C}$ and so $\alpha(z) + \beta(z) \in \mathbb{C}$. Finally we can take $f_1(z)$ and $g_1(z)$ as follows $f(z) - c_0 = e^{\alpha(z)}$ and $g(z) - c_0 = te^{-\alpha(z)}$, where α is a non-constant polynomial and $t \in \mathbb{C} \setminus \{0\}$ such that $a_n^2 t^{n+\sigma} = b^2$.

Sub-case 1.2.2. Let $k = 1$. Considering **Sub-case 1.1.2** one can easily conclude that $\deg(\alpha_1) = \deg(\beta_1) = 1$, i.e., $\deg(\alpha) = \deg(\beta) = 1$. Finally observing (3.22), we can take $f(z) - c_0 = c_1 e^{d_1 z}$ and $g(z) - c_0 = c_2 e^{-d_1 z}$, where $c_1, c_2, d_1 \in \mathbb{C}$ such that $(-1)^k a_n^2 (c_1 c_2)^{n+\sigma} (d_1(n + \sigma))^{2k} = b^2$.

Sub-case 1.2.3. Let $k \in \mathbb{N} \setminus \{1\}$. Then from (3.22), we see that

$$(a_n f_1^n F_1)^{(k)} \neq 0 \text{ and } (a_n g_1^n G_1)^{(k)} \neq 0. \tag{3.29}$$

Again from (3.22), we see that $\alpha_1 + \beta_1 \in \mathbb{C}$, i.e., $\alpha_1' \equiv -\beta_1'$. Therefore $\deg(\alpha_1) = \deg(\beta_1)$. Suppose $\deg(\alpha_1) = \deg(\beta_1) \geq 2$. Considering **Sub-case 1.1.3** one can easily get

$$(a_n f_1^n F_1)^{(k)} = \Big(n^k (\alpha_1')^k + \frac{k(k-1)}{2} n^{k-1} (\alpha_1')^{k-2} \alpha_1'' + P_{k-2}(\alpha_1')\Big) e^{\alpha_1}$$

and

$$\begin{aligned}&(a_n g_1^n G_1)^{(k)} \\ = \;& \left((-1)^k n^k (\alpha_1')^k - \frac{k(k-1)}{2} n^{k-1} (-1)^{k-2} (\alpha_1')^{k-2} \alpha_1'' + P_{k-2}(-\alpha_1')\right) e^{\beta_1},\end{aligned}$$

where

$$n^k(\alpha_1')^k + \frac{k(k-1)}{2}n^{k-1}(\alpha_1')^{k-2}\alpha_1'' + P_{k-2}(\alpha_1')$$
$$\text{and} \qquad (-1)^k n^k(\alpha_1')^k - \frac{k(k-1)}{2}n^{k-1}(-1)^{k-2}(\alpha_1')^{k-2}\alpha_1'' + P_{k-2}(-\alpha_1')$$

are non-constant polynomials. Then from (3.29), we arrive at a contradiction. Hence $\deg(\alpha_1) = \deg(\beta_1) = 1$ and so $\deg(\alpha) = \deg(\beta) = 1$. Finally observing (3.22), we can take $f(z)-c_0 = c_1e^{d_1z}$ and $g(z)-c_0 = c_2e^{-d_1z}$, where $c_1, c_2, d_1 \in \mathbb{C}$ such that $(-1)^k a_n^2(c_1c_2)^{n+\sigma}(d_1(n+\sigma))^{2k} = b^2$.

Case 2. Suppose 0 is not a Picard exceptional value of f_1 and g_1. Since $n > k$, from (3.22) we see that zeros of both f_1 and g_1 are the zeros of p and so f_1 and g_1 have finitely many zeros. Consequently both $f_1^nF_1$ and $g_1^nG_1$ have finitely many zeros.

Let $H = f_1^nF_1$, $\hat{H} = g_1^nG_1$, $F = \frac{H}{p}$ and $G = \frac{\hat{H}}{p}$. Clearly F and G have finitely many poles. Let $\mathcal{F} = \{F_\omega\}$ and $\mathcal{G} = \{G_\omega\}$, where $F_\omega(z) = F(z+\omega) = \frac{H(z+\omega)}{p(z+\omega)}$ and $G_\omega(z) = G(z+\omega) = \frac{\hat{H}(z+\omega)}{p(z+\omega)}$, $z \in \mathbb{C}$. Clearly $\mathcal{F}$ and $\mathcal{G}$ are two families of meromorphic functions defined on $\mathbb{C}$. We now consider following two sub-cases.

Sub-case 2.1. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is normal on $\mathbb{C}$. Then by Marty's theorem $F^{\#}(\omega) = F_\omega^{\#}(0) \le M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence by Lemma 9, we have $F\left(= \frac{f_1^nF_1}{p}\right)$ is of order at most 1. From Lemma 4, we have $\rho(f_1^nF_1) = \rho(f_1)$. Now from (3.22), we have

$$\begin{aligned}\rho(f_1) = \rho\left(\frac{f_1^nF_1}{p}\right) = \rho(f_1^nF_1) &= \rho\left((f_1^nF_1)^{(k)}\right) \\ &= \rho\left((g_1^nG_1)^{(k)}\right) = \rho(g_1^nG_1) = \rho\left(\frac{g_1^nG_1}{p}\right) \\ &= \rho(g_1) \le 1.\end{aligned}$$

Since f_1 and g_1 are transcendental entire functions having finitely many zeros and are of order at most 1, we have

$$f_1 = h_1e^{\alpha} \text{ and } g_1 = h_1e^{\beta}, \tag{3.30}$$

where h_1 is a non-constant polynomial and α, β polynomials of degree 1. Now from (3.22) and (3.24), we see that $\alpha_1 + \beta_1 \in \mathbb{C}$ and so $\alpha_1' + \beta_1' \equiv 0$. Since α and β are polynomials of degree 1, without loss of generality we may assume that $\alpha(z) = a_1z+b_1$ and $\beta(z) = a_2z+b_2$, where $a_1(\neq 0), b_1, a_2(\neq 0), b_2 \in \mathbb{C}$. Then from (3.24), we have $\alpha_1'(z) = (n+\sigma)a_1$ and $\beta_1'(z) = (n+\sigma)a_2$. Since $\beta_1'(z) \equiv -\alpha_1'(z)$, it follows that $a_2 = -a_1$ and so $\alpha'(z) \equiv -\beta'(z)$. Now we consider the following two sub-cases.

Sub-case 2.1.1. Let $k = 0$. Now observing (3.22), we can take $f(z)-c_0 = h(z)e^{az}$ and $g(z)-c_0 = th(z)e^{-az}$, where h is a non-constant polynomial and $a, t \in \mathbb{C}\setminus\{0\}$ such that $a_n^2t^{n+\sigma}h^{2n}(z)(\prod_{j=1}^{s} h(z+c_j))^2 \equiv p^2(z)$.

Sub-case 2.1.2. Let $k \in \mathbb{N}$. Suppose $\alpha'(z) = a_1$. Therefore $\beta'(z) \equiv -\alpha'(z) \equiv$

$-a_1$. Now from (3.24), we have $\alpha_1'(z) = (n+\sigma)a_1$ and $\beta_1'(z) = -(n+\sigma)a_1$. Now from (3.22) and (3.30), we have respectively

$$(f_1^n F_1)^{(k)} = e^{\alpha_1} \sum_{i=0}^{k} {}^kC_i (n\alpha_1')^{k-i} (h_1^n)^{(i)} = e^{\alpha_1} \sum_{i=0}^{k} {}^kC_i \big((n+\sigma)a_1\big)^{k-i} (h_1^n)^{(i)}$$

and

$$(g_1^n G_1)^{(k)} = e^{\beta_1} \sum_{i=0}^{k} {}^kC_i (\beta_1')^{k-i} (h_1^n)^{(i)} = e^{\beta_1} \sum_{i=0}^{k} {}^kC_i \big(-(n+\sigma)a_1\big)^{k-i} (h_1^n)^{(i)},$$

where we define $(h_1^n)^{(0)} = h_1^n$. Since $(f_1^n F_1)^{(k)}$ and $(g_1^n G_1)^{(k)}$ share $(0,\infty)$, it follows that

$$\sum_{i=0}^{k} {}^kC_i \big((n+\sigma)a_1\big)^{k-i} (h_1^n)^{(i)} \equiv d_2^* \sum_{i=0}^{k} {}^kC_i (-1)^{k-i} \big((n+\sigma)a_1\big)^{k-i} (h_1^n)^{(i)}, \quad (3.31)$$

where $d_2^* \in \mathbb{C} \setminus \{0\}$. But from (3.31), we arrive at a contradiction.

Sub-case 2.2. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$ is not normal on $\mathbb{C}$. Then there exists at least one point $z_0 \in \Delta$ such that $\mathcal{F}$ is not normal at z_0. Without loss of generality we may assume that $z_0 = 0$. Now by Marty's theorem there exists a sequence of meromorphic functions $\{F(z+\omega_j)\} \subset \mathcal{F}$, where $z \in \Delta$ and $\{\omega_j\} \subset \mathbb{C}$ is some sequence of complex numbers such that $F^{\#}(\omega_j) \to \infty$ as $|\omega_j| \to \infty$. Since p has only finitely many zeros, so there exists a $r_1 > 0$ such that $p(z) \neq 0$ in $\{z : |z| \geq r\}$. Again since $f_1^n F_1$ has finitely many zeros, so there exists a $r_2 > 0$ such that $f_1^n(z)F_1(z) \neq 0$ in $\{z : |z| \geq r_2\}$. Let $r = \max\{r_1, r_2\}$ and $D = \{z : |z| \geq r\}$. Also since $w_j \to \infty$ as $j \to \infty$, without loss of generality we may assume that $|w_j| \geq r+1$ for all j. Let

$$F(w_j + z) = \frac{H(w_j + z)}{p(w_j + z)}.$$

Since $|w_j + z| \geq |w_j| - |z|$, it follows that $w_j + z \in D$ for all $z \in \Delta$. Also since $f_1^n(z)F_1(z) \neq 0$ and $p(z) \neq 0$ in D, it follows that $f_1^n(\omega_j + z)F_1(\omega_j + z) \neq 0$ and $p(\omega_j + z) \neq 0$ in Δ for all j. Observing that $F(z)$ is analytic in D, so $F(\omega_j + z)$ is analytic in Δ. Therefore all $F(\omega_j + z)$ are analytic in Δ. Thus we have structured a family $\{F(\omega_j + z)\}$ of holomorphic functions such that $F(\omega_j + z) \neq 0$ in Δ for all j. Then by Lemma 8, there exist

(i) points $z_j \in \Delta$ such that $|z_j| < 1$,

(ii) positive numbers ρ_j, $\rho_j \to 0^+$,

(iii) a subsequence $\{F(\omega_j + z_j + \rho_j\zeta)\}$ of $\{F(\omega_j + z)\}$

such that

$$h_j(\zeta) = \rho_j^{-k} F(\omega_j + z_j + \rho_j\zeta) \to h(\zeta),$$

$$\text{i.e.,} \qquad h_j(\zeta) = \rho_j^{-k} \frac{H(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h(\zeta) \qquad (3.32)$$

spherically locally uniformly in $\mathbb{C}$, where $h(\zeta)$ is some non-constant holomorphic function such that $h^{\#}(\zeta) \leq h^{\#}(0) = 1$. Now from Lemma 9, we see that $\rho(h) \leq 1$. In the proof of Zalcman's lemma (see [12, 20]) we see that

$$\rho_j = \frac{1}{F^{\#}(b_j)}, \tag{3.33}$$

where $b_j = \omega_j + z_j$. By Hurwitz's theorem we see that $h(\zeta) \neq 0$. Note that

$$\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to 0, \tag{3.34}$$

as $j \to \infty$. We now prove that

$$(h_j(\zeta))^{(k)} = \frac{H^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(k)}(\zeta), \text{ where } k \in \mathbb{N} \cup \{0\}. \tag{3.35}$$

Clearly (3.35) is true for $k = 0$. Therefore we have to show that (3.35) is true for $k \in \mathbb{N}$. Note that from (3.32), we have

$$\begin{aligned} & \rho_j^{-k+1} \frac{H'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \\ = \; & h_j'(\zeta) + \rho_j^{-k+1} \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p^2(\omega_j + z_j + \rho_j\zeta)} H(\omega_j + z_j + \rho_j\zeta) \\ = \; & h_j'(\zeta) + \rho_j \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} h_j(\zeta). \end{aligned} \tag{3.36}$$

Now from (3.32), (3.34) and (3.36), we observe that

$$\rho_j^{-k+1} \frac{H'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h'(\zeta).$$

Suppose

$$\rho_j^{-k+l} \frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(l)}(\zeta).$$

Let

$$G_j(\zeta) = \rho_j^{-k+l} \frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}.$$

Then $G_j(\zeta) \to h^{(l)}(\zeta)$. Note that

$$\begin{aligned} & \rho_j^{-k+l+1} \frac{H^{(l+1)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \\ = \; & G_j'(\zeta) + \rho_j^{-k+l+1} \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p^2(\omega_j + z_j + \rho_j\zeta)} H^{(l)}(\omega_j + z_j + \rho_j\zeta) \\ = \; & G_j'(\zeta) + \rho_j \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} G_j(\zeta). \end{aligned} \tag{3.37}$$

So from (3.34) and (3.37), we see that

$$\rho_j^{-k+l+1}\frac{H^{(l+1)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)}\to G_j'(\zeta),$$

$$\text{i.e.,}\qquad \rho_j^{-k+l+1}\frac{H^{(l+1)}(\omega_j+z_n+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)}\to h_j^{(l+1)}(\zeta).$$

Then by mathematical induction we get desired result (3.35). Let

$$(\hat{h}_j(\zeta))^{(k)}=\frac{\hat{H}^{(k)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)}. \tag{3.38}$$

From (3.22), we have

$$\frac{H^{(k)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)}\frac{\hat{H}^{(k)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)}\equiv 1$$

and so from (3.35) and (3.38), we get

$$(h_j(\zeta))^{(k)}(\hat{h}_j(\zeta))^{(k)}\equiv 1. \tag{3.39}$$

Suppose $k=0$. Therefore from (3.35) and (3.39), we can deduce that $\hat{h}_j(\zeta)\to\hat{h}(\zeta)$, spherically locally uniformly in $\mathbb{C}$, where $\hat{h}(\zeta)$ is some non-constant holomorphic function in $\mathbb{C}$.

Suppose $k\in\mathbb{N}$. Now from (3.35), (3.39) and the formula of higher derivatives we can deduce that $\hat{h}_j(\zeta)\to\hat{h}(\zeta)$, spherically locally uniformly in $\mathbb{C}$, where $\hat{h}(\zeta)$ is some non-constant holomorphic function in $\mathbb{C}$. Thus in either cases we can deduce that

$$\hat{h}_j(\zeta)\to\hat{h}(\zeta),\ \text{i.e.,}\ \frac{\hat{H}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)}\to\hat{h}(\zeta), \tag{3.40}$$

spherically locally uniformly in $\mathbb{C}$, where $\hat{h}(\zeta)$ is some non-constant holomorphic function in $\mathbb{C}$. By Hurwitz's theorem we see that $\hat{h}(\zeta)\neq 0$. Therefore (3.40) can be rewritten as

$$(\hat{h}_j(\zeta))^{(k)}\to(\hat{h}(\zeta))^{(k)} \tag{3.41}$$

spherically locally uniformly in $\mathbb{C}$. From (3.35), (3.39) and (3.41), we get

$$(h(\zeta))^{(k)}(\hat{h}(\zeta))^{(k)}\equiv 1. \tag{3.42}$$

Since $\rho(h)\leq 1$, from (3.42), we see that $\rho(h)=\rho(h^{(k)})=\rho(\hat{h}^{(k)})=\rho(\hat{h})\leq 1$. Since h and $\hat{h}$ are non-constant entire functions such that $h\neq 0$ and $\hat{h}\neq 0$, so we can take $h=e^{\alpha_2}$ and $\hat{h}=e^{\beta_2}$, where α_2 and β_2 are non-constants entire functions. As $\rho(h)\leq 1$ and $\rho(\hat{h})\leq 1$, α_2 and β_2 must be polynomials such that $\deg(\alpha_2)=1$ and $\deg(\beta_2)=1$ Therefore we can take

$$h(z)=\hat{c}_1e^{\hat{c}z}\text{ and }\hat{h}(z)=\hat{c}_2e^{-\hat{c}z}, \tag{3.43}$$

where $\hat{c}, \hat{c}_1, \hat{c}_2 \in \mathbb{C}\setminus\{0\}$ such that $(-1)^k(\hat{c}_1\hat{c}_2)(\hat{c})^{2k} = 1$. Also from (3.43), we have

$$\frac{h_j'(\zeta)}{h_j(\zeta)} = \rho_j \frac{F'(w_j + z_j + \rho_j\zeta)}{F(w_j + z_j + \rho_j\zeta)} \to \frac{h'(\zeta)}{h(\zeta)} = \hat{c}, \tag{3.44}$$

spherically locally uniformly in $\mathbb{C}$. From (3.33) and (3.44), we get

$$\begin{aligned}\rho_j \left|\frac{F'(\omega_j + z_j)}{F(\omega_j + z_j)}\right| = \frac{1 + |F(\omega_j + z_j)|^2}{|F'(\omega_j + z_j)|}\frac{|F'(\omega_j + z_j)|}{|F(\omega_j + z_j)|} &= \frac{1 + |F(\omega_j + z_j)|^2}{|F(\omega_j + z_j)|} \\ &\to \left|\frac{h'(0)}{h(0)}\right| = |\hat{c}|.\end{aligned}$$

This shows that $\lim_{j\to\infty} F(\omega_j + z_j) \neq 0, \infty$ and so from (3.32) we get

$$h_j(0) = \rho_j^{-k} F(\omega_j + z_j) \to \infty. \tag{3.45}$$

Again from (3.32) and (3.43), we have

$$h_j(0) \to h(0) = \hat{c}_1. \tag{3.46}$$

Now from (3.45) and (3.46), we arrive at a contradiction. □

4 Proofs of the Theorems

Proof of Theorem 1. Let $F = \frac{(P_1(f_1)F_1)^{(k)}}{p}$ and $G = \frac{(P_1(g_1)G_1)^{(k)}}{p}$. Then F and G share $(1, 2)$ except for the zeros of p. When $H \not\equiv 0$, we follow the proof of Theorem 1.3 [10] while for $H \equiv 0$ we follow Lemmas 6, 7 and 12. So we omit the detail proof. □

Proof of Theorem 2. Let F and G be defined as in Theorem 1. Then F and G share $(1, 1)$ except for the zeros of p. When $H \not\equiv 0$, we follow the proof of Theorem 1.4 [10] while for $H \equiv 0$ we follow Lemmas 6, 7 and 12. So we omit the detail proof. □

Proof of Theorem 3. Let F and G be defined as in Theorem 1. Then F and G share $(1, 0)$ except for the zeros of p. When $H \not\equiv 0$, we follow the proof of Theorem 1.5 [10] while for $H \equiv 0$ we follow Lemmas 6, 7 and 12. So we omit the detail proof. □

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