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$N(k)$-PARACOMPACT THREE METRIC AS A ETA-RICCI SOLITON

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Abstract

In this paper, we study Eta-Ricci soliton (η-Ricci soliton) on three dimensional $N(k)$-paracontact metric manifolds. We prove that the scalar curvature of an $N(k)$-paracontact metric manifold admitting η-Ricci solitons is constant and the manifold is of constant curvature k. Also, we prove that such manifolds are Einstein. Moreover, we show the condition of that the η-Ricci soliton to be expanding, steady or shrinking. In such a case we prove that the potential vector field is Killing vector field. Also, we show that the potential vector field is an infinitesimal automorphism or it leaves the structure tensor in the direction perpendicular to the Reeb vector field ξ. Finally, we illustrate an example of a three dimensional $N(k)$-paracontact metric manifold admitting an η-Ricci soliton.

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1 Introduction

In 1982, R. S. Hamilton [12] introduced the notion of Ricci flow in Riemannian geometry to find a canonical metric of the aforesaid manifolds which plays an important role to understand its singularities. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g = -2S, \tag{1}$$

where S denotes the Ricci tensor of the Riemannian metric g. Ricci solitons are special solutions of the Ricci flow equation (1) of the form $g = \sigma(t)\psi_t^* g$ with the

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initial condition $g(0) = g$, where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric.
We recall the notion of Ricci soliton according to [5]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called the potential vector field and λ a real scalar, called potential function such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2}$$

where $\pounds_V g$ denotes the Lie derivative of the metric g along V. Metrics satisfying (2) are interesting and useful in physics and are often referred as quasi-Einstein ([6],[7]).
Reffering to the equation (2), a Ricci soliton is said to be shrinking, steady or expanding according whether λ is negative, zero or positive, respectively.
Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [11] who discusses some aspects of it. Recently, the notion of almost Ricci soliton has been introduced in [21] by Piagoli, Riogoli, Rimoldi and Setti.
Recently, Ricci solitons have also been extensively studied in pseudo-Riemannian settings. For wide survey and further information on Riemannian (or pseudo-Riemannian) Ricci solitons, we may refer to ([3], [13], [14], [20], [26], [27]) and many others.
As a generalization of Ricci solitons, the notion of Eta-Ricci solitons(or η-Ricci solitons) was introduced by Cho and Kimura [8]. This notion has also been studied by Calin and Crasmareanu in [5], for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are real constants, and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0. \tag{3}$$

In this connection we mention the works of Blaga ([1], [2]) and Prakasha et al. [22] on η-Ricci solitons. In particular, if $\mu = 0$, then the notion of η-Ricci solitons (g, V, λ, μ) reduces to the notion of Ricci solitons (g, V, λ). If $\mu \neq 0$, then the η-Ricci solitons is called proper η-Ricci solitons. We refer to ([1], [2] [16], [17], [18], [19]) and references therein for a survey and further references on the geometry of η-Ricci solitons on pseudo-Riemannian manifolds.
The notion of almost paracontact manifolds as analogous to almost contact manifolds was introduced by Sato in ([24],[25]). It is eye-catching that an almost contact manifold is always odd dimensional but an almost paracontact manifold could be even dimensional as well. The notion of paracontact metric structures were initiated by Kaneyuki and Williams [15] in 1985. In the last few years, many authors have studied paracontact metric manifolds, highlighting similarities and differences regarding the most investigated research works on contact metric manifolds.

We have inspired to classify η-Ricci solitons on three dimensional $N(k)$-paracontact manifolds by the above studies and its interest to the theoretical physicists.
After introduction, in section 2, we study the basic informations and formulae concerning three dimensional $N(k)$-paracontact metric manifolds. In section 3, we prove that the scalar curvature of an $N(k)$-paracontact metric manifold admitting η-Ricci soliton is constant and the manifold is of constant curvature k. Also, we prove that such manifolds are Einstein. Moreover, we show the condition of that the η-Ricci soliton to be expanding, steady or shrinking. Then we prove that the potential vector field is an infinitesimal automorphism or it leaves the structure tensor in the direction perpendicular to the Reeb vector field ξ.

2 Three dimensional $N(k)$-paracontact metric manifolds

A (2n+1)-dimensional smooth manifold M^{2n+1} has an almost paracontact structure (ϕ, ξ, η) if it admits a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η and and a Riemannian metric g satisfying the following condition [29]:

$$\phi^2 X = X - \eta(X)\xi, \qquad \eta(X) = g(X, \xi), \tag{4}$$

for any smooth vector fields X, Y on M^{2n+1} and consequently we have

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1. \tag{5}$$

An almost paracontact manifold equipped with a pseudo-Riemannian metric g is said to be an almost paracontact metric manifold if the following condition holds:

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{6}$$

for any smooth vector fields X, Y on M^{2n+1}.
The structure (ϕ, ξ, η, g) is named as almost paracontact metric structure. Any almost paracontact structure admits compatible metrics, which, because of (5), have signature $(n+1, n)$. The fundamental 2-form Φ of an almost paracontact metric structure (ϕ, ξ, η, g) is defined by $\Phi(X, Y) = g(X, \phi Y)$, for any smooth vector fields X, Y on M^{2n+1}. If $d\eta = \Phi$, then the manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called a paracontact metric manifold and g the associated metric.
An almost paracontact metric structure (ϕ, ξ, η, g) is said to be normal if

$$[\phi, \phi] - 2d\eta \otimes \xi = 0, \tag{7}$$

where $[\phi, \phi]$ is the Nijenhuis torsion tensor of ϕ, is defined by $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$, for any smooth vector fields X, Y on M^{2n+1}. In a paracontact metric manifold, we define a symmetric trace-free tensor h of type (1,1) by $h = \frac{1}{2}\mathcal{L}_\xi \phi$ satisfying the following relations ([23], [29]):

$$\phi h + h\phi = 0, \quad h\xi = 0, \quad \mathrm{Trh} = \mathrm{Tr}\phi\mathrm{h} = 0, \tag{8}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{9}$$

for any smooth vector fields X, Y on M^{2n+1}.
It is trivial to say that h vanishes if and only if the Reeb vector field ξ is Killing and then (ϕ, ξ, η, g) is said to be K-paracontact structure. An almost paracontact metric manifold is said to be para-Sasakian manifold if and only if

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{10}$$

for any smooth vector fields X, Y on M^{2n+1}.
A normal paracoontact metric manifold is a para-Sasakian manifold and it obeys

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{11}$$

for any smooth vector fields X, Y on M^{2n+1}.
It is well known that every para-Sasakian manifold is K-paracontact, but the converse is not always true, as it is shown in three dimensional case [4].

Definition 1 ([23]). *A paracontact metric manifold is said to be a paracontact (k, μ)-manifold if it satisfy the curvature condition*

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \quad \forall X, Y \in \chi(M^{2n+1}), \tag{12}$$

where R denotes the curvature tensor of M of type (1,3) and k, μ are real constants.

In the case, $\mu = 0$, this manifold is called an $N(k)$-paracontact metric manifold. Then the last equation gives

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{13}$$

for any smooth vector fields X, Y on M^{2n+1}.
In a three dimensional $N(k)$-paracontact metric manifold $(M^3, \phi, \xi, \eta, g)$, the following relations hold for any smooth vector fields $X, Y \in \chi(M^3)$ ([10], [23]):

$$QX = (\frac{r}{2} - k)X + (3k - \frac{r}{2})\eta(X)\xi, \tag{14}$$

$$S(X, Y) = (\frac{r}{2} - k)g(X, Y) + (3k - \frac{r}{2})\eta(X)\eta(Y), \tag{15}$$

$$\begin{aligned} R(X, Y)Z \;=\; & (\frac{r}{2} - 2k)\{g(Y, Z)X - g(X, Z)Y\} \\ & +(3k - \frac{r}{2})\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ & +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}, \end{aligned} \tag{16}$$

$$(\nabla_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \tag{17}$$

$$S(X,\xi) = 2k\eta(X), \tag{18}$$

$$Q\xi = 2k\xi, \tag{19}$$

where Q, S, r denote the Ricci operator, Ricci tensor and scalar curvature respectively.
In a consequence of (9), we get

$$(\nabla_X \eta)Y = g(X, \phi Y) - g(hX, \phi Y), \tag{20}$$

for all smooth vector fields X, Y on M^3

Definition 2. *[9] An infinitesimal automorphism is a smooth vector field such that the Lie derivatives of all objects of some tensor structure along it vanishes.*

For an almost paracontact metric structure, the vector field V is an infinitesimal automorphism if

$$\pounds_V \eta = \pounds_V \xi = \pounds_V \phi = \pounds_V g = 0. \tag{21}$$

3 η-Ricci solitons on three dimensional $N(k)$-paracontact metric manifolds

This section deals with the characterization of η-Ricci solitons on three dimensional $N(k)$-paracontact metric manifolds. Then equation (3) holds, which can also be written as

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{22}$$

for any smooth vector fields X, Y on M^3.
Applying (15) in (22), we get

$$(\pounds_V g)(X,Y) = (2k - 2\lambda - r)g(X,Y) + (r - 6k - 2\mu)\eta(X)\eta(Y). \tag{23}$$

Taking covariant derivative to both sides of (23) with respect to an arbitrary vector field W and then using (21), we have

$$\begin{aligned}(\nabla_W \pounds_V g)(X,Y)) &= -dr(W)\{g(X,Y) - \eta(X)\eta(Y)\} \\ &\quad +(r - 6k - 2\mu)\{g(W,\phi X)\eta(Y) - g(hW,\phi X)\eta(Y) \\ &\quad +g(W,\phi Y)\eta(X) - g(hW,\phi Y)\eta(X)\}. \end{aligned}\tag{24}$$

According to yano [28],

$$\begin{aligned}(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) &= -g((\pounds_V \nabla)(X,Y),Z) \\ &\quad -g((\pounds_V \nabla)(X,Z),Y). \end{aligned}\tag{25}$$

Making use of parallelism of the pseudo-Riemanian metric g, in the above equation yields

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(Z,X),Y). \tag{26}$$

Due to the symmetricity of $\pounds_V \nabla$, the equation (26) implies that

$$\begin{aligned} 2g((\pounds_V \nabla)(X,Y),Z) &= (\nabla_X \pounds_V g)(Y,Z) + (\nabla_Y \pounds_V g)(X,Z) \\ &\quad -(\nabla_Z \pounds_V g)(X,Y). \end{aligned} \tag{27}$$

By the virtue of (24) and (27), we find that

$$\begin{aligned} 2g((\pounds_V \nabla)(X,Y),Z) &= -dr(X)\{g(Y,Z) - \eta(Y)\eta(Z)\} \\ &\quad -dr(Y)\{g(X,Z) - \eta(X)\eta(Z)\} \\ &\quad +dr(Z)\{g(X,Y) - \eta(X)\eta(Y)\} \\ &\quad +(r - 6k - 2\mu)\{g(X,\phi Y)\eta(Z) \\ &\quad -g(hX,\phi Y)\eta(Z) + g(X,\phi Z)\eta(Y) \\ &\quad -g(hX,\phi Z)\eta(Y) + g(Y,\phi Z)\eta(X) \\ &\quad -g(hY,\phi Z)\eta(X) + g(Y,\phi X)\eta(Z) \\ &\quad -g(hY,\phi X)\eta(Z) - g(Z,\phi X)\eta(Y) \\ &\quad +g(hZ,\phi X)\eta(Y) - g(Z,\phi Y)\eta(X) \\ &\quad +g(hZ,\phi Y)\eta(X)\}. \end{aligned} \tag{28}$$

As ϕ is skew-symmetric, the equation (28) gives

$$\begin{aligned} 2g((\pounds_V \nabla)(X,Y),Z) &= -dr(X)\{g(Y,Z) - \eta(Y)\eta(Z)\} \\ &\quad -dr(Y)\{g(X,Z) - \eta(X)\eta(Z)\} \\ &\quad +dr(Z)\{g(X,Y) - \eta(X)\eta(Y)\} \\ &\quad +(r - 6k - 2\mu)\{2g(X,\phi Z)\eta(Y) \\ &\quad +2g(Y,\phi Z)\eta(X) - 2g(hX,\phi Y)\eta(Z)\}, \end{aligned} \tag{29}$$

from which it follows that

$$\begin{aligned} 2(\pounds_V \nabla)(X,Y) &= -dr(X)\{Y - \eta(Y)\xi\} \\ &\quad -dr(Y)\{X - \eta(X)\xi\} \\ &\quad +Dr\{g(X,Y) - \eta(X)\eta(Y)\} \\ &\quad -2(r - 6k - 2\mu)\{\eta(Y)\phi X \\ &\quad +\eta(X)\phi Y - g(hX,\phi Y)\xi\}, \end{aligned} \tag{30}$$

and hence

$$\begin{aligned}(\pounds_V\nabla)(X,Y) &= -\frac{1}{2}dr(X)\{Y-\eta(Y)\xi\} \\ &\quad -\frac{1}{2}dr(Y)\{X-\eta(X)\xi\} \\ &\quad +\frac{1}{2}Dr\{g(X,Y)-\eta(X)\eta(Y)\} \\ &\quad -(r-6k-2\mu)\{\eta(Y)\phi X \\ &\quad +\eta(X)\phi Y - g(hX,\phi Y)\xi\}. \qquad (31)\end{aligned}$$

Putting $Y=\xi$ in (31) and using the fact that $\xi r=0$, as ξ is a Killing vector field, we obtain

$$(\pounds_V\nabla)(X,\xi) = -(r-6k-2\mu)\phi X. \qquad (32)$$

Taking covariant derivative of the preceeding equation with respect to an arbitrary smooth vector field W and applying (17), we find that

$$\begin{aligned}(\nabla_W\pounds_V\nabla)(X,\xi) &= -(Wr)\phi X-(r-6k-2\mu)[-g(W,X)\xi \\ &\quad +g(hW,X)\xi+\eta(X)(W-hW)]. \qquad (33)\end{aligned}$$

It is well known [28] that for any smooth vector field X, Y, Z,

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z)-(\nabla_Y\pounds_V\nabla)(X,Z), \qquad (34)$$

which shows in light of (33) that

$$\begin{aligned}(\pounds_V R)(X,Y)\xi &= -(Xr)\phi Y+(Yr)\phi X \\ &\quad +(r-6k-2\mu)[\eta(X)Y-\eta(Y)X \\ &\quad +\eta(Y)hX-\eta(X)hY]. \qquad (35)\end{aligned}$$

Taking inner product of the above equation with an arbitrary smooth vector field Z yields

$$\begin{aligned}g((\pounds_V R)(X,Y)\xi,Z) &= -(Xr)g(\phi Y,Z)+(Yr)g(\phi X,Z) \\ &\quad +(r-6k-2\mu)[\eta(X)g(Y,Z)-\eta(Y)g(X,Z) \\ &\quad +\eta(Y)g(hX,Z)-\eta(X)g(hY,Z)]. \qquad (36)\end{aligned}$$

Let us assume that $\{e_i : i=1,2,3\}$ be a local orthonormal frame. Then on contraction of X,Z, from (36) it follows that

$$\begin{aligned}(\pounds_V S)(Y,\xi) &= -g(\phi Y,Dr)+(Yr)\mathrm{Tr}\phi \\ &\quad +(r-6k-2\mu)[\eta(Y)-3\eta(Y) \\ &\quad +\eta(Y)\,\mathrm{Tr}\,h-g(hY,\xi)], \qquad (37)\end{aligned}$$

which follows that

$$(\pounds_V S)(Y,\xi) = g(Y,\phi Dr)-2(r-6k-2\mu)\eta(Y). \qquad (38)$$

Taking Lie derivative of (18) along V and using (20), we get

$$(\pounds_V S)(Y,\xi) = 2k[g(V,\phi Y) - g(hV,\phi Y)]. \tag{39}$$

Comparing (38) and (39), we find that

$$\begin{aligned} & g(Y,\phi Dr) - 2(r-6k-2\mu)\eta(Y) \\ & = 2k[g(V,\phi Y) - g(hV,\phi Y)]. \end{aligned} \tag{40}$$

Replacing Y by ξ in the last equation, we have

$$r = 6k + 2\mu, \tag{41}$$

which implies that the scalar curvature of the aforesaid manifold is constant and hence we are in a position to state the following:

Theorem 1. *Let $(M^3, g, V, \lambda, \mu)$ be an η-Ricci soliton on a three dimensional $N(k)$- paracontact metric manifold. Then the scalar curvature of the manifold is constant.*

In view of Theorem 1, equation (16) takes the form

$$\begin{aligned} R(X,Y)Z &= (k+\mu)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad -\mu\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \end{aligned} \tag{42}$$

Taking inner product of (42) with W and then contracting X, W, we get

$$S(Y,Z) = (2k+\mu)g(Y,Z) - \mu\eta(Y)\eta(Z), \tag{43}$$

which permits

$$S(Y,\xi) = (2k+\mu)\eta(Y). \tag{44}$$

The equations (18) and (44) together allow the following equation

$$\mu = 0. \tag{45}$$

Hence we can conclude the following:

Theorem 2. *In a three dimensional $N(k)$-paracontact metric manifold, η-Ricci soliton $(M^3, g, V, \lambda, \mu)$ reduces to a Ricci soliton (M^3, g, V, λ).*

With the help of (45), from (42) we obtain

$$R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}. \tag{46}$$

Thus we can say that

Theorem 3. *Let $(M^3, g, V, \lambda, \mu)$ be an η-Ricci soliton on a three dimensional $N(k)$- paracontact metric manifold. Then the manifold is of constant curvature k.*

Moreover, from (43) and (45), we observe that

$$S(Y, Z) = 2kg(Y, Z). \tag{47}$$

Therefore, we have the following:

Theorem 4. *Let $(M^3, g, V, \lambda, \mu)$ be an η-Ricci soliton on a three dimensional $N(k)$- paracontact metric manifold. Then the manifold is Einstein with constant scalar curvature $6k$.*

With the help of Theorem 4, the soliton equation (22) can also be written as

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + (4k + 2\lambda)g(X, Y) = 0. \tag{48}$$

Substituting $V = \xi$ in the previous equation and then using (9), we have

$$2g(\phi hX, Y) + 2(2k + \lambda)g(X, Y) = 0. \tag{49}$$

Putting $X = Y = \xi$ in (49), we get,

$$\lambda = -2k. \tag{50}$$

Analogously, from (50), we can easily can state the following:

Theorem 5. *In a three dimensional $N(k)$-paracontact metric manifold, η-Ricci soliton $(M^3, g, V, \lambda, \mu)$ is shrinking, steady or expanding accordingly k is positive, zero or nagetive.*

As $\lambda = -2k$ and $\mu = 0$, from (23), it follows that

$$\pounds_V g = 0, \tag{51}$$

that is, V is a Killing vector field and then one can easily observe that $\pounds_V \eta = 0$ and $\pounds_V \xi = 0$. Also from $d\eta(X, Y) = g(X, \phi Y)$, we deduce

$$(\pounds_V d\eta)(X, Y) = -g(\pounds_V X, \phi Y) - g(X, (\pounds_V \phi)Y). \tag{52}$$

Setting $X = \xi$ in the last equation, we derive

$$g(\pounds_V \phi)Y, \xi) = 0, \tag{53}$$

and hence we have the following statement:

Theorem 6. *Let $(M^3, g, V, \lambda, \mu)$ be an η-Ricci soliton on a three dimensional $N(k)$- paracontact metric manifold. Then the potential vector field is an infinitesimal automorphism or it leaves the structure tensor ϕ in the direction perpendicular to the Reeb vector field ξ.*

4 Example

In the present section, we illustrate an example of a three dimensional $N(-1)$-paracontact metric manifold admitting an η-Ricci soliton.

Example 1. *In this section we illustrate an example of an $\eta-$Ricci soliton on a 3-dimensional $N(-1)$-paracontact metric manifold $(M, g, V, -9, 4)$. We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the standard certesian co-ordinates of $\mathbb{R}^3$. We consider the vector fields*

$$\phi e_2 = e_1, \;\; \phi e_1 = e_2, \;\; \phi e_3 = 0,$$

where

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point of the manifold M The 1-form $\eta = dz$ defines an almost paracontact structure on M with characteristic vector field $\xi = e_3$.
Let g be a pseudo-Riemannian metric defined by:

$$g = (g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.
Using Koszul's formula we have

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

The components of the curvature tensor are:

$$R(e_1, e_2)e_1 = e_2, \;\; R(e_1, e_2)e_2 = e_1, \;\; R(e_1, e_2)e_3 = 0,$$

$$R(e_1, e_3)e_1 = e_3, \;\; R(e_1, e_3)e_2 = 0, \;\; R(e_1, e_3)e_3 = -e_1,$$

$$R(e_2, e_3)e_1 = 0, \;\; R(e_2, e_3)e_2 = -e_3, \;\; R(e_2, e_3)e_3 = -e_2.$$

It is very easy to check that $R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y]$, for $k = -1$ and hence this is an example of a 3-dimensional $N(-1)-$paracontact metric manifold.
The components of Ricci tensor are:

$$S = (S_{ij}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which implies that the scalar curvature $r = 2$.
Using $r = 2$ *and* $k = -1$ *in the relation (41), we have* $\mu = 4$.
Therefore, the Ricci soliton equation

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0$$

with $V = e_1$ *gives* $\lambda = -9$.
Thus this an example of a three dimensional $N(-1)$*-paracontact metric manifold admitting an* η*-Ricci soliton of type* $(M^3, g, V, -9, 4)$.

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