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SOME HYPERGEOMETRIC SUMMATIONS FORMULAS AND SERIES IDENTITIES WITH APPLICATIONS

R. GUPTA [1], **M. MENON** [2], **E. MITTAL**[*,3].

Abstract

In the paper we present two incomplete Gaussian hypergeometric formulas in summation form by specific known formulas.We also developed each of these formulas and how they use to derive double series identities in general forms.

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1 Introduction

We begin by recalling the following classical Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$, (see, e.g., [[6], pp. 169-170])

$$p_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\Big(\frac{x-1}{2}\Big)^{n-k}\Big(\frac{x+1}{2}\Big)^{k}$$

$$= \binom{n+\alpha}{n}\ {}_2F_1\Bigg[-n, n+\alpha+\beta+1; \alpha+1; \Big(\frac{1-x}{2}\Big)\Bigg]. \quad (1)$$

The incomplete Gamma functions's families $\gamma(s,y)$ and $\Gamma(s,y)$ defined by [2]

$$\gamma(s,y) = \int_0^y t^{(s-1)} e^{-t} dt, \quad (\mathfrak{Re}(s) > 0). \quad (2)$$

[1]Department of Mathematics, *The IIS* University, Jaipur, India, e-mail: rajni.gupta@iisuniv.ac.in
[2]Department of Mathematics, *The IIS* University, Jaipur, India, e-mail: muditanishant.menon@iisuniv.ac.in
[3*] *Corresponding author*, Department of Mathematics, *The IIS* University, Jaipur, India, e-mail: ekta.mittal@iisuniv.ac.in

$$\Gamma(s,y)=\int_y^{\infty} t^{(s-1)}e^{-t}dt, \quad (\mathfrak{Re}(s)>0)). \tag{3}$$

respectively satisfy following decomposition formula: For $y \geq 0$, we get

$$\gamma(s,y)+\Gamma(s,y)=\Gamma(s) \tag{4}$$

the functions plays an important role in many area where we study analytic solutions of the problems. In this entire paper, N is a sets of positive integers, Z^{-} denotes sets of negative integers, C represent complex number's set, respectively, Recently, Srivastava et al. [10] investigated in systematic manner the two families of generalized incomplete hypergeometric functions:

$${}_p\gamma_q\left[(a_1;y),a_2,...,a_p;b_1,b_2,...,b_q;z\right]=\sum_{k=0}^{\infty}\frac{(a_1;y)_k(a_2)_k....(a_p)_k}{(b_1)_k...(b_q)_k}\frac{z^k}{(k)!} \tag{5}$$

,

$${}_p\Gamma_q\left[(a_1;y),a_2,...,a_p;b_1,b_2,..,b_q;z\right]=\sum_{k=0}^{\infty}\frac{[a_1;y]_k(a_2)_k...(a_p)_k}{(b_1)_k...(b_q)_k}\frac{z^k}{(k)!} \tag{6}$$

, where,$\gamma(a_1,y)$ and $\Gamma(a_1,y)$ are defined by (2) and (3) in form of the incomplete Gamma functions. The incomplete Pochhammer symbols $(a_1;y)_k$ and $[a_1;y]_k$ represented by:

$$(a_1;y)_k=\frac{\gamma(a_1+k,y)}{\Gamma(a_1)} \quad (a_1\in C, Re(a_1)>0, k\in N; y\geq 0) \tag{7}$$

,

$$[a_1;y]_k=\frac{\Gamma(a_1+k,y)}{\Gamma(a_1)} \quad (a_1\in C, Re(a_1)>0, k\in N; y\geq 0) \tag{8}$$

,

so that, obviously, these incomplete Pochhammer symbols $(a_1;y)_k$ and $[a_1;y]_k$ satisfy the relation:

$$(a_1;y)_k+[a_1;y]_k=(a_1)_k \quad (a_1\in C, Re(a_1)>0; y\geq 0). \tag{9}$$

Here, $(a_1)_k$,$(a_1 \in C)$ denotes the Pochhammer symbol which is defined by

$$(a_1)_k=\frac{\Gamma(a_1+k)}{\Gamma(a_1)}, \tag{10}$$

we know that $(0)_0 = 1$ and assumed that the Γ ratio exists [4]. It is observed by Srivastava et al.[10], the definitions (5) and (6) readily yield the following decomposition formula:

$${}_p\gamma_q\left[(a_1;y),a_2,...,a_p;b_1,b_2,..,b_q;z\right]+\ {}_p\Gamma_q\left[(a_1;y),a_2,...,a_p;b_1,b_2,...,b_q;z\right]$$

$$= \; {}_pF_q\left[a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; z\right] \tag{11}$$

. Where ${}_pF_q\left[a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; z\right]$ is known as generalized hypergeometric function and for the specific values of p and q as $p = 2$ and $q = 1$ equation (11) becomes

$${}_2\gamma_1\left[(a_1; y), a_2; b_1; z\right] + \; {}_2\Gamma_1\left[(a_1; y), a_2; b_1; z\right] = \; {}_2F_1\left[a_1, a_2; b_1; z\right] \tag{12}$$

2 Representaion of two Gaussian hypergeometic summation Formula

We start with classical polynomials $p_n^{(\alpha,\beta)}(x)$ called Jacobi polynomial in terms of incomplete Gamma function such as

$$p_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\Big(\frac{x-1}{2}\Big)^{n-k}\Big(\frac{x+1}{2}\Big)^{k}$$

$$= \binom{n+\alpha}{n} \; {}_2\gamma_1\left[(-n; y), n+\alpha+\beta+1; \alpha+1; \Big(\frac{1-x}{2}\Big)\right]$$

$$+ \binom{n+\alpha}{n} \; {}_2\Gamma_1\left[[-n; y], n+\alpha+\beta+1; \alpha+1; \Big(\frac{1-x}{2}\Big)\right]$$

$$= \binom{n+\alpha}{n}\Bigg[\sum_{k=0}^{\infty} \frac{(-n; y)_k (n+\alpha+\beta+1)_k}{(\alpha+1)_k} \frac{(\frac{1-x}{2})^k}{k!} + \sum_{k=0}^{\infty} \frac{[-n; y]_k (n+\alpha+\beta+1)_k}{(\alpha+1)_k} \frac{(\frac{1-x}{2})^k}{k!}\Bigg],$$

$$= \binom{n+\alpha}{n}\Bigg[\sum_{k=0}^{\infty} \frac{(n+\alpha+\beta+1)_k}{(\alpha+1)_k} \frac{(\frac{1-x}{2})^k}{k!}\bigg(\frac{\gamma(-n+k, y)}{\Gamma(-n)} + \frac{\Gamma(-n+k, y)}{\Gamma(-n)}\bigg)\Bigg], \tag{13}$$

where $(-n; y)_k$ and $[-n; y]_k$ are incomplete Pochhammer symbols satisfying the folllowing relation:

$$(-n; y)_k + [-n; y]_k = (-n)_k \quad (-n \in C, Re(-n) > 0; y \geq 0).$$

Here $(-n)_k$, $(-n \in C, Re(-n) > 0)$, denotes the Pochhammer symbols (or the shifted factorial) which is defined in general by $(-n)_k = \frac{(-1)^k n!}{(n-k)!}$.

Hence using result (12) in equation (13) , we obtain

$$p_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1\Big[-n, n+\alpha+\beta+1; \alpha+1; \Big(\frac{1-x}{2}\Big)\Big],$$

${}_2F_1(-)$ denotes the Gaussian hypergeometric function. It is well known that the familiar Chebyshev polynomial $T_n(x)$ first kind and $U_n(x)$ second kind and respectively are special cases of (1) with the following relationships:

$$T_n(x) = \frac{1}{\binom{n-\frac{1}{2}}{n}} p_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)$$

$$= \frac{1}{\binom{n-\frac{1}{2}}{n}} \binom{n-\frac{1}{2}}{n} \Big[{}_2\gamma_1\Big[(-n;y), n; \frac{1}{2}; (\frac{1-x}{2})\Big] + {}_2\Gamma_1\Big[(-n;y), n; \frac{1}{2}; (\frac{1-x}{2})\Big]\Big], \tag{14}$$

and

$$U_n(x) = \frac{1}{2\binom{n+\frac{1}{2}}{n+1}} p_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x)$$

$$= \frac{1}{2\binom{n+\frac{1}{2}}{n+1}} \binom{n+\frac{1}{2}}{n} \Big[{}_2\gamma_1\Big[(-n;y), n+2; \frac{3}{2}; (\frac{1-x}{2})\Big] + {}_2\Gamma_1\Big[(-n;y), n+2; \frac{3}{2}; (\frac{1-x}{2})\Big]\Big], \tag{15}$$

further setting $x = Cos(\theta)$ in (14) and (15) we get

$$T_n(Cos(\theta)) = Cos(n\theta), \tag{16}$$

$$U_n(Cos(\theta)) = \frac{Sin((n+1)\theta)}{Sin(\theta)}, \tag{17}$$

where

$$\theta = \left\{ \begin{array}{ll} arctan\left(\frac{b}{a}\right) & (a, b \in R^+) \\ \pi - arctan\left(\frac{b}{|a|}\right) & (a \in R^-; b \in R^+) \\ arctan\left(\frac{|b|}{|a|} - \pi\right) & (a, b \in R^-) \\ -\, arctan\left(\frac{b}{|a|}\right) & (a \in R^+; b \in R^-) \end{array} \right\}. \tag{18}$$

3 Representation of function in terms of incomplete Gamma function

For the Chebyshev polynomials $T_n(x)$ and $U_n(x)$, we recall the following result ([7] p,467) and using relation (12).

$$ {}_2\gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2};\frac{1}{2};z\Big] +_2\Gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2};\frac{1}{2};z\Big] = T_n(\sqrt{1-z}), \tag{19} $$

$$ {}_2\gamma_1\Big[(-\frac{n}{2};y),\frac{1-n}{2};\frac{1}{2};z\Big]+_2\Gamma_1\Big[(-\frac{n}{2};y),\frac{1-n}{2};\frac{1}{2};z\Big] = (1-z)^{\frac{n}{2}}T_n(\frac{1}{\sqrt{1-z}}), \tag{20} $$

$$ {}_2\gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2}+1;\frac{3}{2};z\Big] +_2\Gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2}+1;\frac{3}{2};z\Big] = \frac{1}{n+1}U_n(\sqrt{1-z}), \tag{21} $$

$$ \begin{aligned} & {}_2\gamma_1\Big[(-\frac{n}{2};y),\frac{1-n}{2}+1;\frac{3}{2};z\Big] +_2\Gamma_1\Big[(-\frac{n}{2};y),\frac{1-n}{2}+1;\frac{3}{2};z\Big] \\ = \quad & \frac{(1-z)^{\frac{n}{2}}}{n+1}U_n(\frac{1}{\sqrt{(1-z)}}, \end{aligned} $$

and Euler's transformation formula (see, e.g., [[1], p. 60])

$$ {}_2\gamma_1\Big[(a;y),b;c;z\Big] +_2\Gamma_1\Big[(a;y),b;c;z\Big] $$

$$ = (1-z)^{-a}\Bigg[{}_2\gamma_1\Big[(a;y),c-b;c;\frac{-z}{1-z}\Big] + {}_2\Gamma_1\Big[(a;y),c-b;c;\frac{-z}{1-z}\Big]\Bigg], $$

$$ (|z|<1,|\frac{z}{1-z}|<1). \tag{22} $$

3.1 Theorem 1:

Let $n \in N_0$ and a, b, θ be given in equation (18), then

$$ {}_2\gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2};\frac{1}{2};\frac{b^2}{a^2+b^2}\Big] +_2\Gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2};\frac{1}{2};\frac{b^2}{a^2+b^2}\Big] = Cos(n\theta) $$

$$ (n\theta \neq \frac{2k+1}{2}\pi, k \in z). \tag{23} $$

and

$$ \begin{aligned} & {}_2\gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2}+1;\frac{3}{2};\frac{b^2}{a^2+b^2}\Big] +_2\Gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2}+1;\frac{3}{2};\frac{b^2}{a^2+b^2}\Big] \\ = \quad & \frac{1}{n+1}\frac{Sin[(n+1)\theta]}{Sin\theta} \end{aligned} $$

$$((n+1)\theta \neq \pi, k \in z). \tag{24}$$

Proof: Setting $z=\frac{b^2}{a^2+b^2}$ in equation (19) we have

$$ {}_2\gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2};\frac{1}{2};\frac{b^2}{a^2+b^2}\Big] +_2\Gamma_1\Big[(-\frac{n}{2};y),\frac{n}{2};\frac{1}{2};\frac{b^2}{a^2+b^2}\Big] $$
$$ = T_n\Big[\frac{|a|}{\sqrt{a^2+b^2}}\Big] = T_n(Cos\theta), $$

which, further using equation (16) leads to the required result (24).
Similarly setting $z=\frac{b^2}{a^2+b^2}$ in equation (21) and further using equation (17), we obtain the required result (25).

4 Application

By putting $a=\frac{-n}{2}$,$b=\frac{n}{2}$,$c=\frac{1}{2}$, and $z=\frac{b^2}{a^2+b^2}$ in equation (19), we obtain

$$ {}_2\gamma_1\Big[(-\frac{n}{2};y),-\frac{n}{2}+\frac{1}{2};\frac{1}{2};-\frac{b^2}{a^2}\Big] +_2\Gamma_1\Big[(-\frac{n}{2};y),-\frac{n}{2}+\frac{1}{2};\frac{1}{2};-\frac{b^2}{a^2}\Big] $$

$$ = \frac{(a^2+b^2)^{\frac{n}{2}}}{a^n}\,{}_2F_1\Big[-\frac{n}{2},\frac{n}{2};\frac{1}{2};\frac{b^2}{a^2+b^2}\Big] $$
$$ (n\in N_0, a,b\in R \quad except \quad 0). \tag{25}$$

Now applying result (24) to the ${}_2\gamma_1$, ${}_2\Gamma_1$ in the right side of equation (26), we obtain the following known identity $(see \quad [[8], equation(18)])$:

$$ {}_2\gamma_1\Big[(-\frac{n}{2};y),-\frac{n}{2}+\frac{1}{2};\frac{1}{2};-\frac{b^2}{a^2}\Big]+{}_2\Gamma_1\Big[(-\frac{n}{2};y),-\frac{n}{2}+\frac{1}{2};\frac{1}{2};-\frac{b^2}{a^2}\Big] = \frac{(a^2+b^2)^{\frac{n}{2}}}{a^n}Cos(n\theta) $$

,

$$ (n\in N_0, n,\theta \neq \frac{2k+1}{2}\pi, k\in z). \tag{26}$$

where a, b and θ are given in equation (18).
Now again setting $a=\frac{-n}{2}$,$b=\frac{-n}{2}+\frac{1}{2}$,$c=\frac{3}{2}$, and $z=\frac{-b^2}{a^2}$ in equation (23), we get

$$ {}_2\gamma_1\Big[(-\frac{n}{2};y),-\frac{n}{2}+\frac{1}{2};\frac{3}{2};-\frac{b^2}{a^2}\Big] +_2\Gamma_1\Big[(-\frac{n}{2};y),-\frac{n}{2}+\frac{1}{2};\frac{3}{2};-\frac{b^2}{a^2}\Big] $$

$$ = \frac{(a^2+b^2)^{\frac{n}{2}}}{a^n}\,{}_2\gamma_1\Big[(-\frac{n}{2},y),-\frac{n}{2}+1;\frac{3}{2};\frac{b^2}{a^2+b^2}\Big] $$
$$ +\frac{(a^2+b^2)^{\frac{n}{2}}}{a^n}\,{}_2\Gamma_1\Big[[-\frac{n}{2},y],-\frac{n}{2}+1;\frac{3}{2};\frac{b^2}{a^2+b^2}\Big], $$

$$ (n\in N_0, a,b\in R \quad except \quad 0). \tag{27}$$

Using equation (12) and equation (25) in the R.H.S of equation (28), we obtained

$$ {}_2\gamma_1\Big[(-\frac{n}{2};y),-\frac{n}{2}+\frac{1}{2};\frac{3}{2};-\frac{b^2}{a^2}\Big] +{}_2\,\Gamma_1\Big[(-\frac{n}{2};y),-\frac{n}{2}+\frac{1}{2};\frac{3}{2};-\frac{b^2}{a^2}\Big] $$

$$ =\frac{(a^2+b^2)^{\frac{n}{2}}}{a^n(n+1)}\frac{Sin(n+1)\theta}{Sin\theta} $$

,

$$ (n\in N_0, a, b\in R \quad except \quad 0). \tag{28} $$

5 A general family of double-series identities

The Srivastava-Daoust hypergeometric function of two variables [[3], p.64 Equation (1.7, p. 18) with n =2] used by many author's ([11],[12]etc.) to developed a number of double-series identities and their applications. In this section, we apply the hypergeometric summation formulas.

5.1 Theorem 2:

If $\{\Omega_n\}_{n=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers and θ defined by equation (18), then

$$ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\Big(\Omega^{\gamma}_{m+2n}+\Omega^{\Gamma}_{m+2n}\Big)\frac{(ax)^m}{m!}\frac{(bx)^{2n+1}}{(2n+1)!} $$

$$ =x\sqrt{a^2+b^2}\sum_{n=0}^{\infty}\Big(\Omega^{\gamma}_{n}+\Omega^{\Gamma}_{n}\Big)\frac{Sin[(n+1)\theta]}{(n+1)!}(x\sqrt{a^2+b^2})^n, $$

$$ ((n+1)\theta\neq k\pi; k\in Z; n\in N_0). \tag{29} $$

and

$$ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\Big(\Omega^{\gamma}_{m+2n}+\Omega^{\Gamma}_{m+2n}\Big)\frac{(ax)^m}{m!}\frac{(bx)^{2n}}{(2n)!} $$

$$ =\sum_{n=0}^{\infty}\Big(\Omega^{\gamma}_{n}+\Omega^{\Gamma}_{n}\Big)\frac{Con[(n)\theta]}{(n)!}(x\sqrt{a^2+b^2})^n,\ n\theta\neq\frac{2k+1}{2}\pi; k\in Z; n\in N_0; \tag{30} $$

where

$$ \Omega^{\Gamma}_{m+2n}=\frac{[\alpha_1;x]_{m+2n}......(\alpha_p)_{m+2n}}{(\beta_1)_{m+2n}.......(\beta_q)_{m+2n}}, $$

$$ \Omega^{\gamma}_{m+2n}=\frac{(\alpha_1;x)_{m+2n}......(\alpha_p)_{m+2n}}{(\beta_1)_{m+2n}.......(\beta_q)_{m+2n}}, $$

each of these series involved here are absolutely convergent.

Proof- For Convenience, we suppose L.H.S of equation (30) by T, Shifting index m to $m-2n$

$$T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\Big(\Omega^{\gamma}_{m+2n}+\Omega^{\Gamma}_{m+2n}\Big)\frac{(ax)^m}{m!}\frac{(bx)^{2n+1}}{(2n+1)!}$$

$$= \sum_{m=0}^{\infty}\Big(\Omega^{\gamma}_{m}+\Omega^{\Gamma}_{m}\Big)\sum_{n=0}^{|\frac{m}{2}|}(-1)^n\frac{(ax)^{m-2n}}{m-2n!}\frac{(bx)^{2n+1}}{(2n+1)!}.$$

Using duplication formula of gamma function

$$= bx\sum_{m=0}^{\infty}\Big(\Omega^{\gamma}_{m}+\Omega^{\Gamma}_{m}\Big)\frac{(ax)^m}{m!}\sum_{n=0}^{|\frac{m}{2}|}(-1)^n\frac{(-m)_{2n}}{n!(\frac{3}{2})_n}\Big(\frac{b}{2a}\Big)^n,$$

$$= bx\sum_{m=0}^{\infty}\Big(\Omega^{\gamma}_{m}+\Omega^{\Gamma}_{m}\Big)\frac{(ax)^m}{m!}\sum_{n=0}^{|\frac{m}{2}|}\frac{-(\frac{m}{2})_n\Big(\frac{(1-m)}{2}\Big)_n}{n!(\frac{3}{2})_n}\Big(\frac{-b^2}{a^2}\Big)^n,$$

$$= bx\sum_{m=0}^{\infty}\Big(\Omega^{\gamma}_{m}+\Omega^{\Gamma}_{m}\Big)\frac{(ax)^m}{m!}\,{}_2F_1\Big(-\frac{m}{2},\frac{(1-m)}{2};\frac{3}{2};-\frac{b^2}{a^2}\Big), \tag{31}$$

Now using equation (29)and (12), equation (32) becomes

$$T = bx\sum_{m=0}^{\infty}\Big(\Omega^{\gamma}_{m}+\Omega^{\Gamma}_{m}\Big)\frac{(ax)^m}{m!}\frac{(a^2+b^2)^{\frac{(m+1)}{2}}}{(m+1)a^m b}Sin[(m+1)\theta],$$

$$T = x\sqrt{a^2+b^2}\sum_{m=0}^{\infty}\Big(\Omega^{\gamma}_{m}+\Omega^{\Gamma}_{m}\Big)\frac{Sin[(m+1)\theta]}{(m+1)!}(x\sqrt{a^2+b^2})^m, \tag{32}$$

which is the right-hand side of the equation (30) of theorem 2.
Second part of theorem 2 can be proven similarly by appealing to the hypergeometric summation formula (27) in place of equation (29).

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