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GENERALIZATIONS OF THE OSTROWSKI TYPE INEQUALITIES USING DIFFERENT TYPES OF CONVEXITY

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Abstract

In this paper we establish some Ostrowski type inequalities using some classes of convex functions. We will use the following types of convexity: (α, m, h)-convexity, *log*-convexity and the Arithmetic-Harmonic convexity(AH-convexity).

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1 Introduction

Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on $\mathring{I}$, such that $|f'(x)| \leq M$ on $[a, b]$, where $a, b \in I$ and $a < b$. Then f satisfies the following inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]. \tag{1}$$

Inequality (1) is known in literature as *the Ostrowski's inequality.*

In the following, we will present the types of inequalities used in this paper.

Definition 1. *Let $f : I \subset [0, \infty) \to \mathbb{R}_+^*$ be a real function. We say that f is log-convex if:*

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}, \tag{2}$$

for all $x, y \in I$ and $t \in [0, 1]$.

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Definition 2. *Let $f : I \subset [0,\infty) \to \mathbb{R}\backslash\{0\}$ be a real function. The function f is arithmetically-harmonic convex (AH-convex) if:*

$$f(tx+(1-t)y) \leq \frac{1}{t\frac{1}{f(x)}+(1-t)\frac{1}{f(y)}}, \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$.

In article [4], the $(\alpha, m, h)-$convexity was defined as follows:

Definition 3. *Let $h : I \subset \mathbb{R} \to \mathbb{R}$ be a real function. The function $f : I \subset [0,\infty) \to \mathbb{R}$ is $(\alpha, m, h)-$convex if the following relation holds:*

$$f(tx+m(1-t)y) \leq h(t^\alpha)f(x)+mh(1-t^\alpha)f(y), \tag{4}$$

$\forall x, y \in I$, $t \in (0,1)$, $m \in (0,1]$ and $\alpha \in [0,1]$.

Also, we will consider some particular cases of the $(\alpha, m, h)-$convexity, such as:

1. The Jensen convexity:

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \tag{5}$$

   which is obtained by taking $h(t) = t$ and $\alpha = m = 1$.

2. The (α, m)-convexity:

$$f(tx+m(1-t)y) \leq t^\alpha f(x)+m(1-t^\alpha)f(y). \tag{6}$$

   obtained for $h(t) = t$.

3. The h-convexity:

$$f(tx+(1-t)y) \leq h(t)f(x)+h(1-t)f(y) \tag{7}$$

   which is obtained if $\alpha = m = 1$.

4. The (s, m)-convexity:

$$f(tx+m(1-t)y) \leq t^s f(x)+m(1-t)^s f(y). \tag{8}$$

   obtained for $\alpha = 1$ and $h(t) = t^s$.

5. The (s, m)-Godunova-Levin convexity:

$$f(tx+m(1-t)y) \leq \frac{f(x)}{t^s}+m\frac{f(y)}{(1-t)^s}. \tag{9}$$

   obtained if $\alpha = 1$ and $h(t) = t^{-s}$.

6. The m-Godunova-Levin convexity:

$$f(tx+m(1-t)y) \leq \frac{f(x)}{t} + m\frac{f(y)}{1-t}. \tag{10}$$

obtained for considering $s=1$ in the previous case.

7. The (α, s)-Godunova-Levin convexity:

$$f(tx+(1-t)y) \leq \frac{f(x)}{t^{\alpha s}} + \frac{f(y)}{(1-t^{\alpha})^{s}}. \tag{11}$$

obtained by considering $m=1$ and $h(t)=t^{-s}$.

8. The α-Godunova-Levin convexity:

$$f(tx+(1-t)y) \leq \frac{f(x)}{t^{\alpha}} + \frac{f(y)}{1-t^{\alpha}}. \tag{12}$$

obtained taking $m=1$ și $h(t)=t^{-1}$.

9. The (α, s)-convexity:

$$f(tx+(1-t)y) \leq t^{\alpha s} f(x) + (1-t^{\alpha})^{s} f(y). \tag{13}$$

obtained if $m=1$ and $h(t)=t^{s}$.

2 Main results

In this section we will present the Ostrowski type inequalities we have obtained using the (α, m, h)-convexity. Also, we will present some particular cases of these results obatined by using different types of (α, m, h)-convexity.

2.1 Ostrowski type inequalities

In order to prove the inequalities obtained in this paper we will need the following lemma, used in [2], which is a more general case of the result obtained by Dragomir S.S., Agarwal R.P. and Cerone P. in [3].

Lemma 1. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$, be a real and differentiable function on I. If $f' \in L^1[ma, mb]$, where $m \in (0,1]$ and $a, b \in I$ with $a < b$, then the following equality holds:*

$$\begin{aligned} mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du = \frac{(x-ma)^2}{b-a}\int_0^1 tf'(tx+m(1-t)a)dt \\ + \frac{(mb-x)^2}{b-a}\int_0^1 tf'(tx+m(1-t)b)dt. \end{aligned} \tag{14}$$

In the following Theorem we present the first Ostrowski type inequality we obtained:

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$, $\forall x \in [ma, mb]$, with $ma, mb \in I$, $m \in (0,1]$ and $a < b$. If $|f'|^q$ is (α, m, h)-convex on $[ma, mb]$ for $(\alpha, m) \in [0,1] \times (0,1]$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then we have:*

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left[\int_0^1 h(t^\alpha)dt + m\int_0^1 h(1-t^\alpha)dt\right]^{\frac{1}{q}}\left[\frac{(x-ma)^2 + (mb-x)^2}{b-a}\right] \quad (15)$$

Proof. First, we shall apply the modulus function to relation (14):

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| = \left|\frac{(x-ma)^2}{b-a}\int_0^1 tf'(tx + m(1-t)a)dt + \frac{(mb-x)^2}{b-a}\int_0^1 tf'(tx+m(1-t)b)dt\right| \leq$$
$$\leq \frac{(x-ma)^2}{b-a}\int_0^1 t|f'(tx+m(1-t)a)|dt + \frac{(mb-x)^2}{b-a}\int_0^1 t|f'(tx+m(1-t)b)|dt. \quad (16)$$

Now, by applying Hölder's inequality to the two integrals on the right hand side term from above we will get:

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq \frac{(x-ma)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+m(1-t)a)|^q dt\right)^{\frac{1}{q}} + $$
$$+ \frac{(x-mb)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tx+m(1-t)b)|^q dt\right)^{\frac{1}{q}}. \quad (17)$$

Using the fact that $|f'|^q$ is $(\alpha, m, h)-$convex, we get:

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg[\frac{(x-ma)^2}{b-a}\left(\int_0^1 \left(h(t^\alpha)|f'(x)|^q + mh(1-t^\alpha)|f'(a)|^q\right)dt\right)^{\frac{1}{q}} + $$
$$+ \frac{(x-mb)^2}{b-a}\left(\int_0^1 \left(h(t^\alpha)|f'(x)|^q + mh(1-t^\alpha)|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\Bigg]. \quad (18)$$

But, $|f'(x)| \leq M$ so $|f'(x)|^q \leq M^q$, therefore we have:

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq$$
$$\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Big[\frac{(x-ma)^2}{b-a}\left(M^q\int_0^1 \left(h(t^\alpha)+mh(1-t^\alpha)\right)dt\right)^{\frac{1}{q}}+$$
$$+\frac{(x-mb)^2}{b-a}\Big(M^q\int_0^1 (h(t^\alpha)+mh(1-t^\alpha)dt)^{\frac{1}{q}}\Big)\Big]. \quad (19)$$

Now, rearranging the terms we get:

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\times$$
$$\left[\int_0^1 h(t^\alpha)dt + m\int_0^1 h(1-t^\alpha)dt\right]^{\frac{1}{q}}\left[\frac{(x-ma)^2+(mb-x)^2}{b-a}\right], \quad (20)$$

which ends our proof. □

Remark 1. *Using the fact that for* $p > 1$

$$\frac{1}{2} \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \leq 1, \quad (21)$$

and applying this inequality in the result of Theorem 1, we get

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq$$
$$M\left[\int_0^1 h(t^\alpha)dt + m\int_0^1 h(1-t^\alpha)dt\right]^{\frac{1}{q}}\times\left[\frac{(x-ma)^2+(mb-x)^2}{b-a}\right]. \quad (22)$$

The following result is obtained by applying the power mean inequality.

Theorem 2. *Let* $I \subset \mathbb{R}$ *be an open interval such that* $[0,\infty) \subset I$. *Let* $f : I \to \mathbb{R}$ *be a differentiable function which satisfies* $|f'(x)| \leq M$, $\forall x \in [ma, mb]$, *where* $ma, mb \in I$ *and* $a < b$. *If* $|f'|^q$ *is a* (α, m, h)*-convex function on* $[ma, mb]$ *for* $(\alpha, m) \in [0,1]\times(0,1]$ *and* $q \geq 1$, *then we have the following inequality*

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq$$
$$M\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\int_0^1 th(t^\alpha)dt + m\int_0^1 th(1-t^\alpha)dt\right]^{\frac{1}{q}}\left[\frac{(x-ma)^2+(mb-x)^2}{b-a}\right] \quad (23)$$

Proof. We will treat separately the case $q = 1$.

As before, we shall apply the modulus function to relation (14):

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| =$$
$$\left|\frac{(x-ma)^2}{b-a}\int_0^1 tf'(tx+m(1-t)a)dt + \frac{(mb-x)^2}{b-a}\int_0^1 tf'(tx+m(1-t)b)dt\right| \leq$$
$$\leq \frac{(x-ma)^2}{b-a}\int_0^1 t|f'(tx+m(1-t)a)|dt + \frac{(mb-x)^2}{b-a}\int_0^1 t|f'(tx+m(1-t)b)|dt. \tag{24}$$

Using the fact that $|f'|$ is (α, m, h)-convex, we get

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq \frac{(x-ma)^2}{b-a}\int_0^1 t(h(t^\alpha)|f'(x)| + mh(1-t^\alpha)|f'(a)|)dt$$
$$+ \frac{(mb-x)^2}{b-a}\int_0^1 t(h(t^\alpha)|f'(x)| + mh(1-t^\alpha)|f'(b)|)dt. \tag{25}$$

Now, using $|f'(x)| \leq M$ and rearranging the terms, we obtain:

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq$$
$$M\left[\int_0^1 th(t^\alpha)dt + m\int_0^1 th(1-t^\alpha)dt\right]\left[\frac{(x-ma)^2 + (mb-x)^2}{b-a}\right], \tag{26}$$

which completes the proof for $q = 1$.

We consider now $q > 1$, and we will rewrite inequality (24) as follows

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq \frac{(x-ma)^2}{b-a}\int_0^1 t^{1-\frac{1}{q}} \cdot t^{\frac{1}{q}}|f'(tx+m(1-t)a)|dt +$$
$$\frac{(mb-x)^2}{b-a}\int_0^1 t^{1-\frac{1}{q}} \cdot t^{\frac{1}{q}}|f'(tx+m(1-t)b)|dt. \tag{27}$$

Now, by applying Hölder's inequality to the inequality above, we get

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq$$
$$\frac{(x-ma)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|f'(tx+m(1-t)a)|^q dt\right)^{\frac{1}{q}} +$$
$$+ \frac{(mb-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|f'(tx+m(1-t)b)|^q dt\right)^{\frac{1}{q}}.$$

Now, using the fact that $|f'|^q$ is (α, m, h)-convex, $|f'(x)| \leq M$ and $\int_0^1 t dt = \frac{1}{2}$, we obtain

$$\left| mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u) du \right| \leq$$
$$M \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\int_0^1 th(t^\alpha) dt + m \int_0^1 th(1-t^\alpha) dt\right]^{\frac{1}{q}} \left[\frac{(x-ma)^2 + (mb-x)^2}{b-a}\right], \tag{28}$$

which completes the proof. □

In the following results we consider $|f'|^q$ to be a *log*-convex function, respectively AH-convex.

Theorem 3. *Let $I \subset \mathbb{R}$ be an open interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable nonconstant function such that $|f'(x)| \leq M$, $\forall x \in [a, b]$, with $a, b \in I$ and $a < b$. If $|f'|^q$ is log-convex on $[a, b]$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq M \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\frac{(x-a)^2 + (b-x)^2}{b-a}\right] \tag{29}$$

Proof. The proof is similar to that of Theorem 1. Hence, after applying the modulus function and Hölder inequality to the relation from Lemma 1 with $m = 1$, we get

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{(x-a)^2}{b-a} \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx + (1-t)a)|^q dt\right)^{\frac{1}{q}} +$$
$$+ \frac{(x-b)^2}{b-a} \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx + (1-t)b)|^q dt\right)^{\frac{1}{q}}. \tag{30}$$

Now, using the fact that $|f'|^q$ is *log*-convex and $|f'(x)| \leq M$, which, due to $q - qt > 0$, implies that $|f'(x)|^{qt} \leq M^{qt}$ and $|f'(x)|^{q-qt} \leq M^{q-qt}$, we obtain

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq$$
$$\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\frac{(x-a)^2}{b-a} \left(\underbrace{\int_0^1 M^{qt} M^{q-qt} dt}_{=M^q}\right)^{\frac{1}{q}} + \frac{(x-b)^2}{b-a} \left(\underbrace{\int_0^1 M^{qt} M^{q-qt} dt}_{=M^q}\right)^{\frac{1}{q}}\right], \tag{31}$$

which concludes the proof. □

Remark 2. *For $p > 1$, using inequality (21) in relation (36), we get*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{M}{b-a}\left[(x-a)^2 + (b-x)^2\right]. \tag{32}$$

Theorem 4. *Let $I \subset \mathbb{R}$ be an open interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable nonconstant function which satisfies $|f'(x)| \leq M$, $\forall x \in [a, b]$, with $a, b \in I$ and $a < b$. If $|f'|^q$ is log-convex on $[a, b]$ and $q \geq 1$, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]. \tag{33}$$

Proof. The proof is similar to that of Theorem 2. Therefore, we apply the modulus function to the relation from Lemma 14, and then we will rewrite the inequality as in (27) with $m = 1$. As before, we shall treat the case $q = 1$ separately.

Consequently, for $q = 1$, after using the fact that $|f'|$ is a *log*-convex bounded function, we get

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 t|f'(x)|^t|f'(a)|^{1-t}dt + \frac{(x-b)^2}{b-a}\int_0^1 t|f'(x)|^t|f'(b)|^{1-t}dt \leq \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 tM^tM^{1-t}dt + \frac{(x-b)^2}{b-a}\int_0^1 tM^tM^{1-t}dt = \\
&= M\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]\int_0^1 tdt = \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]
\end{aligned} \tag{34}$$

which is the desired result for $q = 1$.

For the case $q > 1$, applying the Hölder inequality to relation (27) and using the fact that $|f'|^q$ is *log*-convex and $|f'|$ is bounded, we get

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{(x-a)^2}{b-a}\left(\int_0^1 t[|f'(x)|^q]^t[|f'(a)|^q]^{1-t}dt\right)^{\frac{1}{q}} + \right. \\
&\left. + \frac{(x-b)^2}{b-a}\left(\int_0^1 t[|f'(x)|^q]^t[|f'(b)|^q]^{1-t}dt\right)^{\frac{1}{q}}\right] \leq \\
&\leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{(x-a)^2}{b-a}\left(\int_0^1 tM^{qt}M^{q-qt}dt\right)^{\frac{1}{q}} + \frac{(x-b)^2}{b-a}\left(\int_0^1 tM^{qt}M^{q-qt}\right)^{\frac{1}{q}}\right] = \\
&= \left(\frac{1}{2}\right)^{1-\frac{1}{q}} M\left[(x-a)^2 + (b-x)^2\right]\left(\int_0^1 tdt\right)^{\frac{1}{q}} = \\
&M\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[(x-a)^2 + (b-x)^2\right]\left(\frac{1}{2}\right)^{\frac{1}{q}} = \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]
\end{aligned} \tag{35}$$

which concludes the proof. □

Remark 3. *The inequality in Theorem 4 is no other than Ostrowski's inequality.*

In the following results we will present some Ostrowski type inequalities for AH-convex functions.

Theorem 5. *Let $I \subset \mathbb{R}$ be an open interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function with the property $|f'(x)| \le M$, $\forall x \in [a, b]$, with $a, b \in I$ and $a < b$. If $|f'|^q$ is AH- convex on $[a, b]$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \le M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2 + (b-x)^2}{b-a}\right] \tag{36}$$

Proof. The proof of this result is similar to that of Theorem 3. We shall apply the AH-convexity condition to a function $|f'|^q$ and for the boundedness we shall use the following reasoning:

$$|f'(x)| \le M \Rightarrow |f'(x)|^q \le M^q \Rightarrow \frac{1}{|f'(x)|^q} \ge \frac{1}{M^q} \text{ și } t \ge 0 \Rightarrow \frac{t}{|f'(x)|^q} \ge \frac{t}{M^q},$$

$$|f'(y)| \le M \Rightarrow |f'(y)|^q \le M^q \Rightarrow \frac{1}{|f'(y)|^q} \ge \frac{1}{M^q} \text{ și } 0 \le t \le 1 \Rightarrow \frac{1-t}{|f'(x)|^q} \ge \frac{1-t}{M^q}.$$

Now, by summing the last two inequalities above we will obtain:

$$\frac{t}{|f'(x)|^q} + \frac{1-t}{|f'(x)|^q} \ge \frac{t}{M^q} + \frac{1-t}{M^q} = \frac{1}{M^q},$$

which gives:

$$\frac{1}{t\frac{1}{|f'(x)|^q} + (1-t)\frac{1}{|f'(y)|^q}} \le M^q.$$

□

2.2 Some particular cases for Ostrowski type inequalities

In this section we will study some particular cases of the (α, m, h)-convexity applied to the Ostrowski type inequalities.

Remark 4. *As a first particular case of Ostrowski type inequality, we shall consider $h(t) = t$ in Theorem 1 and we will get the following inequality*

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \le M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\alpha m + 1}{\alpha + 1}\right)^{\frac{1}{q}}\frac{(x-ma)^2 + (mb-x)^2}{b-a}, \tag{37}$$

which is in fact an inequality given by Özdemir, Kavurmaci and Set in [2].

Now, we shall consider the types of convexity presented in the beginning of the paper.

1. The (s, m)−convexity $(h(t) = t^s, \alpha = 1)$:

Corollary 1. *Let $I \subset \mathbb{R}$ be an open interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$, $\forall x \in [ma, mb]$, with $ma, mb \in I$ and $a < b$. If $|f'|^q$ is $(s,m)-$convex on $[ma, mb]$ for $m \in (0,1]$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then:*

$$\left| mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u)du \right| \leq$$
$$\leq M \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{m+1}{s+1} \right)^{\frac{1}{q}} \left[\frac{(x-ma)^2 + (mb-x)^2}{b-a} \right]. \quad (38)$$

Proof. In Theorem 1, taking $h(t) = t^s$ and $\alpha = 1$ we get:

$$\left| mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u)du \right| \leq$$

$$\leq M \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{(x-ma)^2 + (mb-x)^2}{b-a} \right] \left[\int_0^1 t^s dt + m \int_0^1 (1-t)^s dt \right]^{\frac{1}{q}}.$$

And, from the calculations we obtain:

$$\left[\int_0^1 t^s dt + m \int_0^1 (1-t)^s dt \right]^{\frac{1}{q}} = \left(\frac{m+1}{s+1} \right)^{\frac{1}{q}}$$

□

Corollary 2. *Let $I \subset \mathbb{R}$ be an open interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function which satisfies $|f'(x)| \leq M$, $\forall x \in [ma, mb]$, with $ma, mb \in I$ and $a < b$. If $|f'|^q$ is $(s,m)-$convex on $[ma, mb]$ for $m \in (0,1]$, $q \geq 1$, then:*

$$\left| mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u)du \right| \leq$$
$$\leq M \left(\frac{1}{2} \right)^{\frac{q-1}{q}} \left(\frac{s+m+1}{(s+1)(s+2)} \right)^{\frac{1}{q}} \left[\frac{(x-ma)^2 + (mb-x)^2}{b-a} \right]. \quad (39)$$

Proof. In Theorem 2, taking $h(t) = t^s$ and $\alpha = 1$ we have:

$$\left| mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u)du \right| \leq$$

$$\leq M \left(\frac{1}{2} \right)^{\frac{q-1}{q}} \left[\frac{(x-ma)^2 + (mb-x)^2}{b-a} \right] \left[\int_0^1 t \cdot t^s dt + m \int_0^1 t \cdot (1-t)^s dt \right]^{\frac{1}{q}}$$

Now, by integration, we obtain

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq$$

$$\leq M\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left[\frac{(x-ma)^2+(mb-x)^2}{b-a}\right]\left[\frac{1}{s+2}+\frac{m}{(s+1)(s+2)}\right]^{\frac{1}{q}} =$$

$$= M\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left[\frac{(x-ma)^2+(mb-x)^2}{b-a}\right]\left[\frac{s+m+1}{(s+1)(s+2)}\right]^{\frac{1}{q}}$$

□

2. The (s,m)-Godunova-Levin functions ($h(t)=t^{-s}$, $\alpha=1$):

Corollary 3. *Let $I \subset \mathbb{R}$ be an open interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function which satisfies $|f'(x)| \leq M$, $\forall x \in [ma, mb]$, with $ma, mb \in I$ and $a < b$. If $|f'|^q$ is a (s,m)-Godunova-Levin function on $[ma, mb]$ for $m \in (0,1]$, $q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then:*

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq$$

$$\leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{m+1}{1-s}\right)^{\frac{1}{q}}\left[\frac{(x-ma)^2+(mb-x)^2}{b-a}\right]. \quad (40)$$

Proof. In Theorem 1, taking $h(t)=t^{-s}$ and $\alpha=1$, we get:

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq$$

$$\leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-ma)^2+(mb-x)^2}{b-a}\right]\left[\int_0^1 t^{-s}dt + m\int_0^1 (1-t)^{-s}dt\right]^{\frac{1}{q}},$$

and by calculations we get the conclusion. □

Corollary 4. *Let $I \subset \mathbb{R}$ be an open interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function which satisfies $|f'(x)| \leq M$, $\forall x \in [ma, mb]$, with $ma, mb \in I$ and $a < b$. If $|f'|^q$ is a (s,m)-Godunova-Levin function on $[ma, mb]$ for $m \in (0,1]$, $q \geq 1$, then:*

$$\left|mf(x) - \frac{1}{b-a}\int_{ma}^{mb} f(u)du\right| \leq$$

$$\leq M\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\frac{m-s+1}{(1-s)(2-s)}\right)^{\frac{1}{q}}\left[\frac{(x-ma)^2+(mb-x)^2}{b-a}\right]. \quad (41)$$

Proof. In Theorem 2, taking $h(t)=t^{-s}$ and $\alpha=1$, and then, by doing the calculations we get the conclusion. □

3. The (α, s)-convexity ($h(t) = t^s, m = 1$):

Corollary 5. *Let $I \subset \mathbb{R}$ be an open interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function which satisfies $|f'(x)| \leq M$, $\forall x \in [a, b]$, with $a, b \in I$ and $a < b$. If $|f'|^q$ is (α, s)-convex on $[a, b]$ for $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$
$$\leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha s+1} + \frac{1}{\alpha}\frac{\Gamma(s+1)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(1+s+\frac{1}{\alpha}\right)}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right], \tag{42}$$

where Γ is the Euler's Gamma function.

Proof. In Theorem 1, if we consider $h(t) = t^s$ and $m = 1$ then:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$
$$\leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\int_0^1 t^{\alpha s}dt + \int_0^1 (1-t^a)^s dt\right]^{\frac{1}{q}}$$

Now, if we use the change of variable, $t = u^{\frac{1}{\alpha}}$ for the second integral, on the right hand side term of the inequality, we get:

$$\int_0^1 (1-t^a)^s dt = \frac{1}{\alpha}\int_0^1 (1-u)^s u^{\frac{1-\alpha}{\alpha}} du = \frac{1}{\alpha}\int_0^1 (1-u)^s u^{\frac{1}{\alpha}-1} du =$$
$$= \frac{1}{\alpha}\beta\left(s+1, \frac{1}{q}\right) = \frac{1}{\alpha}\frac{\Gamma(s+1)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(1+s+\frac{1}{\alpha}\right)}$$

Therefore, after doing the calculations we will obtain:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$
$$\leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left(\frac{1}{\alpha s+1} + \frac{1}{\alpha}\frac{\Gamma(s+1)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(1+s+\frac{1}{\alpha}\right)}\right)^{\frac{1}{q}}$$

□

Corollary 6. *Let $I \subset \mathbb{R}$ be an open interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$, $\forall x \in [a, b]$, with $a, b \in I$ and $a < b$. If $|f'|^q$ is (α, s)-convex on $[a, b]$ for $q \geq 1$, then:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$
$$\leq M\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\frac{1}{\alpha s+2} + \frac{1}{\alpha}\frac{\Gamma(s+1)\Gamma\left(\frac{2}{\alpha}\right)}{\Gamma\left(1+s+\frac{2}{\alpha}\right)}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]. \tag{43}$$

Proof. In Theorem 2 taking $h(t) = t^s$ and $m = 1$ we get:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$

$$\leq M\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\int_0^1 t\cdot t^{\alpha s}dt + \int_0^1 t(1-t^\alpha)^s dt\right]^{\frac{1}{q}}$$

Now, using the substituion $t = u^{\frac{1}{\alpha}}$, on the right hand side, we show that:

$$\int_0^1 t(1-t^\alpha)^s dt = \frac{1}{\alpha}\frac{\Gamma(s+1)\Gamma\left(\frac{2}{\alpha}\right)}{\Gamma\left(1+s+\frac{2}{\alpha}\right)}$$

then, by calculations we shall obtain the mentioned inequality. □

4. The (α, s)-Godunova-Levin functions $(h(t) = t^{-s}, m = 1)$:

Corollary 7. *Let $I \subset \mathbb{R}$ be an open interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differntiable function such that $|f'(x)| \leq M$, $\forall x \in [a, b]$, with $a, b \in I$ and $a < b$. If $|f'|^q$ is a (α, s)-Godunova-Levin function on $[a, b]$ for $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$
$$\leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{1-\alpha s} + \frac{1}{\alpha}\frac{\Gamma(1-s)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(1-s+\frac{1}{\alpha}\right)}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]. \tag{44}$$

Proof. In Theorem 1 if we take $h(t) = t^{-s}$ and $m = 1$ we get:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$

$$\leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\int_0^1 t^{-\alpha s}dt + \int_0^1 (1-t^\alpha)^{-s}dt\right]^{\frac{1}{q}}$$

Again, by using the substitution: $t = u^{\frac{1}{\alpha}}$, on the right hand side, we will obtain:

$$\int_0^1 t(1-t)^s dt = \frac{1}{\alpha}\frac{\Gamma(1-s)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(1-s+\frac{1}{\alpha}\right)}.$$

Now, by doing the calculations we will get the inequality in the hypothesis. □

Corollary 8. *Let $I \subset \mathbb{R}$ be an open interval such that $[0,\infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentibale function such that $|f'(x)| \leq M$, $\forall x \in [a,b]$, with $a, b \in I$ and $a < b$. If $|f'|^q$ is an (α, s)-Godunova-Levin function on $[a,b]$ for $q \geq 1$ then:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$
$$\leq M\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left(\frac{1}{2-\alpha s} + \frac{1}{\alpha}\frac{\Gamma(1-s)\Gamma\left(\frac{2}{\alpha}\right)}{\Gamma\left(1-s+\frac{2}{\alpha}\right)}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]. \quad (45)$$

Proof. In Theorem 2 taking $h(t) = t^{-s}$ and $m = 1$ then:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq$$
$$\leq M\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\int_0^1 t \cdot t^{-\alpha s}dt + \int_0^1 t(1-t^\alpha)^{-s}dt\right]^{\frac{1}{q}}$$

Now, by using the substitution $t = u^{\frac{1}{\alpha}}$ we get:

$$\int_0^1 t(1-t^\alpha)^{-s}dt = \frac{1}{\alpha}\frac{\Gamma(1-s)\Gamma\left(\frac{2}{\alpha}\right)}{\Gamma\left(1-s+\frac{2}{\alpha}\right)}$$

and by doing the calculations we will get the inequality mentioned. □

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