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SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH PASCAL DISTRIBUTION SERIES

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Abstract

In this paper, we find the necessary and sufficient conditions and inclusion relations for Pascal distribution series to be in the classes $\mathcal{W}_\delta(\alpha,\gamma,\beta)$ of analytic functions. Further, we consider an integral operator related to Pascal distribution series. Several corollaries and consequences of the main results are also considered.

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1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Further, let $\mathcal{T}_\delta$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n e^{i\delta} \geq 0, \ \ |\delta| < \pi/2, z \in \mathbb{U}. \tag{2}$$

For $\gamma, \beta \geq 0$, $0 \leq \alpha < \cos\delta, |\delta| < \pi/2$ and function $f \in \mathcal{T}_\delta$ is said to be in the

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class $\mathcal{W}_\delta(\alpha,\gamma,\beta)$ if it satisfies the analytic criteria

$$\Re\{e^{i\delta}[(1-\gamma+2\beta)\frac{f(z)}{z}+(\gamma-2\beta)f'(z)+\beta z f''(z)]\}>\alpha, \quad (z\in\mathbb{U}). \tag{3}$$

Remark 1. *The class $\mathcal{W}_0(\alpha,\gamma,\beta)$ is a subclass of the class $\mathcal{W}_\beta(\alpha,\gamma)$ which is defined by Ali et al. [1] (see also [18]). In particular, the class $\mathcal{W}_0(\alpha,\gamma,0)=\mathcal{Q}_\gamma(\alpha)$ was studied by Ding et al. [6], the classes $\mathcal{W}_\delta(\alpha,1,0)=\mathcal{S}(\delta,\alpha)$ and $\mathcal{W}_\delta(\alpha,0,0)=\mathcal{T}(\delta,\alpha)$were introduced and studied by Sudharasan et al. [22].*

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A,B)$, $\tau\in\mathbb{C}\backslash\{0\}$, $-1\le B<A\le 1$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1, \qquad z\in\mathbb{U}.$$

This class was introduced by Dixit and Pal [7].

A variable x is said to be Pascal distribution if it takes the values $0,1,2,3,\ldots$ with probabilities $(1-q)^m$, $\frac{qm(1-q)^m}{1!}$, $\frac{q^2m(m+1)(1-q)^m}{2!}$, $\frac{q^3m(m+1)(m+2)(1-q)^m}{3!}$, $\ldots$, respectively, where q and m are called the parameters, and thus

$$P(x=k)=\binom{k+m-1}{m-1}q^k(1-q)^m, k=0,1,2,3,\ldots.$$

Very recently, El-Deeb et alt. [5] (see also [13, 3]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

$$\Psi_q^m(z):=z+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n,\ z\in\mathbb{U},$$

where $m\ge 1$, $0\le q\le 1$, and we note that, by ratio test the radius of convergence of above series is infinity. We also define the series

$$\Phi_q^m(z):=2z-\Psi_q^m(z)=z-\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n,\ z\in\mathbb{U}. \tag{4}$$

Let consider the linear operator $\mathcal{I}_q^m:\mathcal{A}\to\mathcal{A}$ defined by the convolution or Hadamard product

$$\mathcal{I}_q^m f(z):=\Psi_q^m(z)*f(z)=z+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m a_n z^n,\ z\in\mathbb{U},$$

where $m\ge 1$ and $0\le q\le 1$.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, by using hypergeometric functions (see for example, [4, 9, 10, 20, 21]) and by the recent investigations (see, for example

[2, 8, 12, 14, 15, 16, 17]), in the present paper we determine the necessary and sufficient conditions for Φ_q^m to be in our class $\mathcal{W}_\delta(\alpha,\gamma,\beta)$. We give connections of these subclasses with $\mathcal{R}^\tau(A,B)$, and finally, we give sufficient conditions for the function f such that its image by the integral operator $\mathcal{G}_q^m f(z)=\int_0^z \frac{\Phi_q^m(t)}{t}dt$ belongs to the above class.

2 Preliminary lemmas

Employing the same technique proved by Sekine [19] (see also, [11]) we have the following lemma.

Lemma 1. *A function $f\in\mathcal{T}_\delta$ of the form (2) is in the class $\mathcal{W}_\delta(\alpha,\gamma,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}[n(n-1)\beta+(\gamma-2\beta)n+(1-\gamma+2\beta)]\,|a_n|\le\cos\delta-\alpha. \tag{5}$$

for some $\gamma,\beta\ge0$ and $0\le\alpha<\cos\delta, |\delta|<\pi/2$. The result (5) is sharp.

Furthermore, we also need the following result.

Lemma 2. *[7] If $f\in\mathcal{R}^\tau(A,B)$ is of the form (1) , then*

$$|a_n|\le(A-B)\frac{|\tau|}{n},\qquad n\in\mathbb{N}-\{1\}.$$

The result is sharp for the function

$$f(z)=\int_0^z(1+(A-B)\frac{\tau t^{n-1}}{1+Bt^{n-1}})dt,\qquad (z\in\mathbb{U};n\in\mathbb{N}-\{1\}).$$

3 Necessary and sufficient conditions for $\Phi_q^m\in\mathcal{W}_\delta(\alpha,\gamma,\beta)$

For convenience throughout in the sequel, we use the following identities for $m>1$ and $0\le q<1$:

$$\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n=\frac{1}{(1-q)^m},\quad \sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n=\frac{1}{(1-q)^{m-1}},$$
$$\sum_{n=0}^{\infty}\binom{n+m}{m}q^n=\frac{1}{(1-q)^{m+1}},\quad \sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n=\frac{1}{(1-q)^{m+2}}.$$

By simple calculations we derive the following relations:

$$\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}=\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n-1=\frac{1}{(1-q)^m}-1,$$
$$\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}=qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n=\frac{qm}{(1-q)^{m+1}},$$

and

$$\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1} = q^2m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n = \frac{q^2m(m+1)}{(1-q)^{m+2}}.$$

Unless otherwise mentioned, we shall assume in this paper that $\gamma, \beta \geq 0$ and $0 \leq \alpha < \cos\delta, |\delta| < \pi/2$, while $m \geq 1$ and $0 \leq q < 1$.

Firstly, we obtain the necessary and sufficient conditions for Φ_q^m to be in the class $\mathcal{W}_\delta(\alpha, \gamma, \beta)$.

Theorem 1. *We have $\Phi_q^m \in \mathcal{W}_\delta(\alpha, \gamma, \beta)$, if and only if*

$$\beta\frac{q^2m(m+1)}{(1-q)^2} + \gamma\frac{qm}{1-q} + (1-(1-q)^m) \leq \cos\delta - \alpha. \tag{6}$$

Proof. Since Φ_q^m is defined by (4), in view of Lemma 1 it is sufficient to show that

$$\sum_{n=2}^{\infty}[n(n-1)\beta+(\gamma-2\beta)n+(1-\gamma+2\beta)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq \cos\delta-\alpha. \tag{7}$$

Writing in left hand side of (7)

$$n = (n-1)+1,$$
$$n^2 = (n-1)(n-2)+3(n-1)+1,$$

we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(n-1)\beta + (\gamma-2\beta)n + (1-\gamma+2\beta)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
= &\sum_{n=2}^{\infty}[\beta n^2 + n(\gamma-3\beta) + (1-\gamma+2\beta)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
= &\beta\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&+\gamma\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
= &\beta\frac{q^2m(m+1)}{(1-q)^2} + \gamma\frac{qm}{1-q} + (1-(1-q)^m)
\end{aligned}$$

but this last expression is upper bounded by $\cos\delta - \alpha$ if and only if (6) holds. □

4 Sufficient conditions for $\mathcal{I}_q^m\left(\mathcal{R}^\tau(A,B)\right) \subset \mathcal{W}_\delta(\alpha,\gamma,\beta)$

Making use of Lemma 2, we will study the action of the Pascal distribution series on the class $\mathcal{W}_\delta(\alpha,\gamma,\beta)$.

Theorem 2. *Let $m > 1$. If $f \in \mathcal{R}^\tau(A,B)$ and the inequality*

$$(A-B)|\tau| \left\{ \beta \frac{qm}{(1-q)} + (\gamma - 2\beta)\left(1-(1-q)^m\right) + \frac{(1-\gamma+2\beta)}{q(m-1)}\left[(1-q)-(1-q)^m-(m-1)q(1-q)^m\right] \right\} \le \cos\delta - \alpha. \tag{8}$$

is satisfied then $\mathcal{I}_q^m f \in \mathcal{W}_\delta(\alpha,\gamma,\beta)$.

Proof. According to Lemma 1 it is sufficient to show that

$$\sum_{n=2}^{\infty} [n(n-1)\beta + (\gamma-2\beta)n + (1-\gamma+2\beta)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m |a_n| \le \cos\delta - \alpha.$$

Since $f \in \mathcal{R}^\tau(A,B)$, using Lemma 2 we have

$$|a_n| \le \frac{(A-B)|\tau|}{n}, \ n \in \mathbb{N}\setminus\{1\},$$

therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty}([\beta n(n-1) + (\gamma-2\beta)n + (1-\gamma+2\beta)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m |a_n| \\
\le\ & (A-B)|\tau| \Bigg[\sum_{n=2}^{\infty} \Big[\beta(n-1) + (\gamma-2\beta) \\
&\qquad + \frac{1}{n}(1-\gamma+2\beta) \Big] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \Bigg] \\
=\ & (A-B)|\tau|(1-q)^m \Bigg[\beta \sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} q^{n-1} \\
& + (\gamma-2\beta) \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} + (1-\gamma+2\beta) \sum_{n=2}^{\infty} \frac{1}{n} \binom{n+m-2}{m-1} q^{n-1} \Bigg] \\
=\ & (A-B)|\tau|(1-q)^m \Bigg\{ \beta \frac{qm}{(1-q)^{m+1}} + (\gamma-2\beta)\left(\frac{1}{(1-q)^m} - 1\right) \\
&\qquad + \frac{(1-\gamma+2\beta)}{q(m-1)} \left[\sum_{n=0}^{\infty} \binom{n+m-2}{m-2} q^n - 1 - (m-1)q \right] \Bigg\} \\
=\ & (A-B)|\tau|(1-q)^m \Bigg\{ \beta \frac{qm}{(1-q)^{m+1}} + (\gamma-2\beta)\left(\frac{1}{(1-q)^m} - 1\right) \\
&\qquad + \frac{(1-\gamma+2\beta)}{q(m-1)} \left[\frac{1}{(1-q)^{m-1}} - 1 - (m-1)q \right] \Bigg\}
\end{aligned}$$

$$= (A-B)\,|\tau| \left\{ \beta \frac{qm}{(1-q)} + (\gamma-2\beta)\,(1-(1-q)^m) \right.$$
$$\left. \frac{(1-\gamma+2\beta)}{q(m-1)}\,[(1-q)-(1-q)^m-(m-1)q(1-q)^m] \right\}.$$

But this last expression is upper bounded by $\cos\delta-\alpha$ if (8) holds, which completes our proof. □

5 Properties of a special function

Theorem 3. *Let $m>1$. If the function $\mathcal{G}_q^m$ is given by*

$$\mathcal{G}_q^m(z) := \int_0^z \frac{\Phi_q^m(t)}{t}dt,\ z\in\mathbb{U}, \tag{9}$$

then $\mathcal{G}_q^m \in \mathcal{W}_\delta(\alpha,\gamma,\beta)$, if and only if

$$\begin{aligned} & \frac{qm\beta}{(1-q)} + (\gamma-2\beta)\,(1-(1-q)^m) \\ & + \frac{(1-\gamma+2\beta)}{q(m-1)}\,[(1-q)-(1-q)^m-(m-1)q(1-q)^m] \\ \leq\ & \cos\delta-\alpha. \end{aligned} \tag{10}$$

holds.

Proof. According to (4) it follows that

$$\mathcal{G}_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \frac{z^n}{n},\ z\in\mathbb{U}.$$

Using Lemma 1, the function $\mathcal{G}_q^m(z)$ belongs to $\mathcal{W}_\delta(\alpha,\gamma,\beta)$ if and only if

$$\sum_{n=2}^{\infty}[n(n-1)\beta+(\gamma-2\beta)n+(1-\gamma+2\beta)]\frac{1}{n}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq \cos\delta-\alpha.$$

By a similar proof like those of Theorem 2 we get that $\mathcal{G}_q^m f \in \mathcal{WT}(\alpha,\gamma,\beta)$ if and only if (10) holds. □

6 Corollaries and consequences

By specializing the parameters $\beta=0$ and $\delta=0$ in Theorem 1, Theorem 2, and Theorem 3 we obtain the following special cases for the subclass $\mathcal{QT}_\gamma(\alpha) := \mathcal{Q}_\gamma(\alpha)\cap\mathcal{T}_o$.

Corollary 1. *We have $\Phi_q^m \in \mathcal{QT}_\gamma(\alpha)$, if and only if*

$$\gamma\frac{qm}{1-q} \leq (1-q)^m-\alpha. \tag{11}$$

Corollary 2. *Let $m > 1$. If $f \in \mathcal{R}^{\tau}(A, B)$ and the inequality*

$$(A-B)|\tau|\Big[\gamma\left(1-(1-q)^{m}\right)+\frac{(1-\gamma)}{q(m-1)}\Big[(1-q)-(1-q)^{m} -(m-1)q(1-q)^{m}\Big]\Big] \leq 1-\alpha.$$

is satisfied then $\mathcal{I}_q^m f \in \mathcal{QT}_{\gamma}(\alpha)$.

Corollary 3. *Let $m > 1$. If the function $\mathcal{G}_q^m$ is given by (9), then $\mathcal{G}_q^m \in \mathcal{QT}_{\gamma}(\alpha)$ if and only if*

$$\gamma\left(1-(1-q)^{m}\right)+\frac{(1-\gamma)}{q(m-1)}\left[(1-q)-(1-q)^{m}-(m-1)q(1-q)^{m}\right] \leq 1-\alpha.$$

Concluding Remarks. Specializing the parameter β and γ we can state various interesting inclusion results (as proved in above theorems) for the subclasses $\mathcal{S}(\delta, \alpha)$ and $\mathcal{T}(\delta, \alpha)$ as stated in Remark 1.

References

[1] Ali, R.M., Badghaish, A., Ravichandran, V. and Swaminathan, A., *Starlikeness of integral transforms and duality*, J. Math. Anal. Appl. **385** (2012), no. 2, 808–822.

[2] El-Ashwah, R.M. and Kota, W.Y., *Some condition on a Poisson distribution series to be in subclasses of univalent functions*, Acta Univ. Apulensis Math. Inform., **51** (2017), 89–103.

[3] Çakmak, S., Yalçın, S. and Altınkaya, Ş., *Some connections between various classes of analytic functions associated with the power series distribution*, Sakarya Univ. J. Sci., **23** (2019), no. 5, 982–985.

[4] Cho, N.E., Woo. S.Y. and Owa, S., *Uniform convexity properties for hypergeometric functions*, Fract. Calc. Appl. Anal. **5** (2002), no. 3, 303–313.

[5] El-Deeb, S.M., Bulboacă, T. and Dziok, J., *Pascal distribution series connected with certain subclasses of univalent functions*, Kyungpook Math. J. **59** (2019), 301–314.

[6] Ding, S.S., Ling, Y. and Bao, G.J., *Some properties of a class of analytic functions*, J. Math. Anal. Appl. **195** (1995), no. 1, 71–81.

[7] Dixit, K.K. and Pal, S.K., *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math. **26** (1995), no. 9, 889-896.

[8] Frasin, B.A., *On certain subclasses of analytic functions associated with Poisson distribution series*, Acta Univ. Sapientiae Math. **11** (2019). no. 1, 78–86.

[9] Frasin, B.A., Al-Hawary, T. and Yousef, F., *Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions*, Afr. Mat. **30** (2019), no. 1-2, 223–230.

[10] Merkes, E. and Scott, B.T., *Starlike hypergeometric functions*, Proc. Amer. Math. Soc. **12** (1961), 885–888.

[11] Murugusundaramoorthy, G., *Studies on classes of analytic functions with negative coefficients*, Ph.D. Thesis, University of Madras, 1994.

[12] Murugusundaramoorthy, G., *Subclasses of starlike and convex functions involving Poisson distribution series*, Afr. Mat. **28** (2017), 1357–1366.

[13] Murugusundaramoorthy, G., Frasin, B.A. and Al-Hawary, T., *Uniformly convex spiral functions and uniformly spirallike function associated with Pascal distribution series*, arXiv:2001.07517 [math.CV].

[14] Murugusundaramoorthy, G., Vijaya, K. and Porwal, S., *Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series*, Hacettepe J. Math. Stat., **45** (2016), no. 4, 1101–1107.

[15] Porwal, S., *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal. (2014), Art. ID 984135, 1–3.

[16] Porwal, S., *Mapping properties of generalized Bessel functions on some subclasses of univalent functions*, An. Univ. Oradea Fasc. Mat. **20** (2013), no. 2, 51–60.

[17] Porwal, S., and Kumar, M., *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat. **27** (2016), no.5, 1021–1027.

[18] Ramachandran, C., Vanitha, L., *Certain aspect of subordination for a class of analytic function*, International Journal of Mathematical Analysis, **9**(20) (2015), 979 - 984.

[19] Sekine, T., *A generalization of certain class of analytic functions with negative coefficients*, Math. Japonica **36** (1991), no. 1, 13-19.

[20] Silverman, H., *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl. **172** (1993), 574–581.

[21] Srivastava, H.M. Murugusundaramoorthy, G. and Sivasubramanian, S., *Hypergeometric functions in the parabolic starlike and uniformly convex domains*, Integral Transforms Spec. Funct. **18** (2007), 511–520.

[22] Sudharasan, T.V., Subramanian, K.G. and Balasubrahmanyam, P., *On two generalized classes of analytic functions with negative coefficient*, Soochow Journal of Mathematics **25** (1999), no. 1, 11-17.