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JENSEN'S TYPE INEQUALITIES FOR THE FINITE HILBERT TRANSFORM OF CONVEX FUNCTIONS

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Abstract

In this paper we obtain some new inequalities for the finite Hilbert transform of convex functions by the use of Jensen's integral inequality. Applications for exponential function are provided as well.

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1 Introduction

Allover this paper, we consider the *finite Hilbert transform* on the open interval (a,b) defined by

$$(Tf)(a,b;t) := \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau$$

for $t \in (a,b)$ and for various classes of functions f for which the above Cauchy Principal Value integral exists, see [13, Section 3.2] or [17, Lemma II.1.1].

For several recent papers devoted to inequalities for the finite Hilbert transform (Tf), see [2]-[10], [14]-[16] and [18]-[19].

Now, if we assume that the mapping $f : (a,b) \to \mathbb{R}$ is convex on (a,b), then it is locally Lipschitzian on (a,b) and then the finite Hilbert transform of f exists in every point $t \in (a,b)$.

The following result concerning upper and lower bounds for the finite Hilbert transform of a convex function holds.

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Theorem 1 (Dragomir et al., 2001 [1])**.** *Let $f:(a,b)\to\mathbb{R}$ be a convex function on (a,b) and $t\in(a,b)$. Then we have*

$$\begin{aligned}&\frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(t)-f(a)+\varphi(t)(b-t)\right]\\&\le(Tf)(a,b;t)\\&\le\frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(b)-f(t)+\varphi(t)(t-a)\right],\end{aligned}\tag{1}$$

where $\varphi(t)\in\left[f'_-(t),f'_+(t)\right]$, $t\in(a,b)$.

Corollary 1. *Let $f:(a,b)\to\mathbb{R}$ be a differentiable convex function on (a,b) and $t\in(a,b)$. Then we have*

$$\begin{aligned}&\frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(t)-f(a)+f'(t)(b-t)\right]\\&\le(Tf)(a,b;t)\\&\le\frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right)+f(b)-f(t)+f'(t)(t-a)\right].\end{aligned}\tag{2}$$

We observe that if we take $t=\frac{a+b}{2}$, then we get from (2) that

$$\begin{aligned}&\frac{1}{\pi}\left[f\left(\frac{a+b}{2}\right)-f(a)+\frac{1}{2}f'\left(\frac{a+b}{2}\right)(b-a)\right]\\&\le(Tf)\left(a,b;\frac{a+b}{2}\right)\\&\le\frac{1}{\pi}\left[f(b)-f\left(\frac{a+b}{2}\right)+\frac{1}{2}f'\left(\frac{a+b}{2}\right)(b-a)\right].\end{aligned}\tag{3}$$

More recently, we have established the following result as well:

Theorem 2 (Dragomir, 2018 [12])**.** *Let $f:(a,b)\to\mathbb{R}$ be a convex function on (a,b) with finite lateral derivatives $f'_+(a)$ and $f_-(b)$. Then for $t\in(a,b)$ we have*

$$\begin{aligned}\frac{1}{\pi}(b-a)f'_+(a)&\le\frac{1}{\pi}(b-a)\frac{f(t)-f(a)}{t-a}\\&\le(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\\&\le\frac{1}{\pi}(b-a)\frac{f(b)-f(t)}{b-t}\le\frac{1}{\pi}(b-a)f_-(b).\end{aligned}\tag{4}$$

In particular,

$$\begin{aligned}\frac{1}{\pi}(b-a)f'_+(a)&\le\frac{2}{\pi}(b-a)\frac{f\left(\frac{a+b}{2}\right)-f(a)}{b-a}\\&\le(Tf)\left(a,b;\frac{a+b}{2}\right)\\&\le\frac{2}{\pi}(b-a)\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-a}\le\frac{1}{\pi}(b-a)f_-(b).\end{aligned}\tag{5}$$

In this paper, by the use of Jensen's celebrated inequality, we obtain some new inequalities for the finite Hilbert transform of convex functions. Applications for exponential function are provided as well.

2 Main Results

We have:

Theorem 3. *Let $f:(a,b)\to\mathbb{R}$ be a convex function on (a,b) and $x:(a,b)\to\mathbb{R}$ a locally absolutely continuos function on (a,b) with $x'(s)\in(a,b)$ for almost every $s\in(a,b)$. Then we have*

$$\begin{aligned} & f\left(\frac{1}{b-a}\left[\pi(Tx)(a,b;t)-x(t)\ln\left(\frac{b-t}{t-a}\right)\right]\right) \\ & \leq \frac{1}{b-a}\left[\pi T\left(\int_a f\circ x'(s)\,ds\right)(a,b;t)-\left(\int_a^t f\circ x'(s)\,ds\right)\ln\left(\frac{b-t}{t-a}\right)\right], \end{aligned} \tag{6}$$

for any $t\in(a,b)$.

In particular,

$$f\left(\frac{\pi}{b-a}(Tx)\left(a,b;\frac{a+b}{2}\right)\right)\leq\frac{\pi}{b-a}T\left(\int_a f\circ x'(s)\,ds\right)\left(a,b;\frac{a+b}{2}\right). \tag{7}$$

Proof. Using Jensen's integral inequality for the convex function f we have

$$f\left(\frac{x(\tau)-x(t)}{\tau-t}\right)=f\left(\frac{\int_t^\tau x'(s)\,ds}{\tau-t}\right)\leq\frac{\int_t^\tau f\circ x'(s)\,ds}{\tau-t} \tag{8}$$

for any τ, $t\in(a,b)$ with $\tau\neq t$.

Let $t\in(a,b)$ and $t-a>\varepsilon>0$, then by integrating (8) over $\tau\in[a,t-\varepsilon]$ we get

$$\int_a^{t-\varepsilon} f\left(\frac{x(\tau)-x(t)}{\tau-t}\right)d\tau\leq\int_a^{t-\varepsilon}\left(\frac{\int_t^\tau f\circ x'(s)\,ds}{\tau-t}\right)d\tau. \tag{9}$$

Using Jensen's integral inequality for the convex function f we get

$$f\left(\frac{1}{t-\varepsilon-a}\int_a^{t-\varepsilon}\frac{x(\tau)-x(t)}{\tau-t}d\tau\right)\leq\frac{1}{t-\varepsilon-a}\int_a^{t-\varepsilon} f\left(\frac{x(\tau)-x(t)}{\tau-t}\right)d\tau. \tag{10}$$

By using (9) and (10) we get

$$\begin{aligned} & (t-\varepsilon-a)\,f\left(\frac{1}{t-\varepsilon-a}\int_a^{t-\varepsilon}\frac{x(\tau)-x(t)}{\tau-t}d\tau\right) \\ & \qquad\leq\int_a^{t-\varepsilon}\left(\frac{\int_t^\tau f\circ x'(s)\,ds}{\tau-t}\right)d\tau \end{aligned} \tag{11}$$

for $t\in(a,b)$ and $t-a>\varepsilon>0$.

Let $t \in (a, b)$ and $b - t > \varepsilon > 0$, then by integrating (8) over $\tau \in [t + \varepsilon, b]$ we obtain

$$\int_{t+\varepsilon}^{b} f\left(\frac{x(\tau) - x(t)}{\tau - t}\right) d\tau \leq \int_{t+\varepsilon}^{b} \left(\frac{\int_t^\tau f \circ x'(s)\, ds}{\tau - t}\right) d\tau. \tag{12}$$

Using Jensen's integral inequality for the convex function f we get

$$f\left(\frac{1}{b-t-\varepsilon}\int_{t+\varepsilon}^{b} \frac{x(\tau) - x(t)}{\tau - t}\right) d\tau \leq \frac{1}{b-t-\varepsilon}\int_{t+\varepsilon}^{b} f\left(\frac{x(\tau) - x(t)}{\tau - t}\right) d\tau. \tag{13}$$

On utilising (12) and (13) we deduce

$$(b-t-\varepsilon) f\left(\frac{1}{b-t-\varepsilon}\int_{t+\varepsilon}^{b} \frac{x(\tau) - x(t)}{\tau - t}\right) d\tau \leq \int_{t+\varepsilon}^{b} \left(\frac{\int_t^\tau f \circ x'(s)\, ds}{\tau - t}\right) d\tau \tag{14}$$

for $t \in (a, b)$ and $b - t > \varepsilon > 0$.

If we add the inequalities (11) and (14) we get

$$\begin{aligned} &(t-\varepsilon-a) f\left(\frac{1}{t-\varepsilon-a}\int_{a}^{t-\varepsilon} \frac{x(\tau) - x(t)}{\tau - t} d\tau\right) \\ &+ (b-t-\varepsilon) f\left(\frac{1}{b-t-\varepsilon}\int_{t+\varepsilon}^{b} \frac{x(\tau) - x(t)}{\tau - t}\right) d\tau \\ &\leq \int_{a}^{t-\varepsilon} \left(\frac{\int_t^\tau f \circ x'(s)\, ds}{\tau - t}\right) d\tau + \int_{t+\varepsilon}^{b} \left(\frac{\int_t^\tau f \circ x'(s)\, ds}{\tau - t}\right) d\tau \end{aligned} \tag{15}$$

for $t \in (a, b)$ and $\min\{b - t, t - a\} > \varepsilon > 0$.

By the convexity of f we also have

$$\begin{aligned} &(t-\varepsilon-a) f\left(\frac{1}{t-\varepsilon-a}\int_{a}^{t-\varepsilon} \frac{x(\tau) - x(t)}{\tau - t} d\tau\right) \\ &\qquad + (b-t-\varepsilon) f\left(\frac{1}{b-t-\varepsilon}\int_{t+\varepsilon}^{b} \frac{x(\tau) - x(t)}{\tau - t}\right) d\tau \\ &\geq (b-a-2\varepsilon) f\left(\frac{1}{b-a-2\varepsilon}\left[\int_{a}^{t-\varepsilon} \frac{x(\tau) - x(t)}{\tau - t} d\tau + \int_{t+\varepsilon}^{b} \frac{x(\tau) - x(t)}{\tau - t}\right]\right) \end{aligned} \tag{16}$$

for $t \in (a, b)$ and $\min\{b - t, t - a\} > \varepsilon > 0$.

Therefore, by (15) and (16) we get

$$\begin{aligned} &(b-a-2\varepsilon) f\left(\frac{1}{b-a-2\varepsilon}\left[\int_{a}^{t-\varepsilon} \frac{x(\tau) - x(t)}{\tau - t} d\tau + \int_{t+\varepsilon}^{b} \frac{x(\tau) - x(t)}{\tau - t}\right]\right) \\ &\leq \int_{a}^{t-\varepsilon} \left(\frac{\int_t^\tau f \circ x'(s)\, ds}{\tau - t}\right) d\tau + \int_{t+\varepsilon}^{b} \left(\frac{\int_t^\tau f \circ x'(s)\, ds}{\tau - t}\right) d\tau \\ &= \int_{a}^{t-\varepsilon} \left(\frac{\int_a^\tau f \circ x'(s)\, ds - \int_a^t f \circ x'(s)\, ds}{\tau - t}\right) d\tau \\ &+ \int_{t+\varepsilon}^{b} \left(\frac{\int_a^\tau f \circ x'(s)\, ds - \int_a^\tau f \circ x'(s)\, ds}{\tau - t}\right) d\tau \end{aligned} \tag{17}$$

for $t \in (a,b)$ and $\min\{b-t, t-a\} > \varepsilon > 0$.

Taking the limit over $\varepsilon \to 0+$ in (17) we get

$$f\left(\frac{1}{b-a}PV\left(\int_a^b \frac{x(\tau)-x(t)}{\tau-t}d\tau\right)\right) \leq \frac{1}{b-a}PV\int_a^b \left(\frac{\int_a^\tau f\circ x'(s)\,ds - \int_a^t f\circ x'(s)\,ds}{\tau-t}\right)d\tau, \tag{18}$$

for $t \in (a,b)$, which is an inequality of interest in itself.

As for the mapping , $\mathbf{1}(t) = 1$, $t \in (a,b)$, we have

$$\begin{aligned}(T\mathbf{1})(a,b;t) &= \frac{1}{\pi}PV\int_a^b \frac{1}{\tau-t}d\tau \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{1}{\tau-t}d\tau + \int_{t+\varepsilon}^b \frac{1}{\tau-t}d\tau\right] \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\ln|\tau-t|\big|_a^{t-\varepsilon} + \ln(\tau-t)\big|_{t+\varepsilon}^b\right] \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\ln\varepsilon - \ln(t-a) + \ln(b-t) - \ln\varepsilon\right] \\ &= \frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right), \quad t \in (a,b).\end{aligned}$$

Then, obviously, for $g : (a,b) \to \mathbb{R}$ we have

$$\begin{aligned}(Tg)(a,b;t) &= \frac{1}{\pi}PV\int_a^b \frac{g(\tau)-g(t)+g(t)}{\tau-t}d\tau \\ &= \frac{1}{\pi}PV\int_a^b \frac{g(\tau)-g(t)}{\tau-t}d\tau + \frac{g(t)}{\pi}PV\int_a^b \frac{1}{\tau-t}d\tau\end{aligned}$$

from where we get the equality

$$(Tg)(a,b;t) - \frac{g(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) = \frac{1}{\pi}PV\int_a^b \frac{g(\tau)-g(t)}{\tau-t}d\tau \tag{19}$$

for all $t \in (a,b)$.

Using the equality (19) we have

$$PV\left(\int_a^b \frac{x(\tau)-x(t)}{\tau-t}d\tau\right) = \pi(Tx)(a,b;t) - x(t)\ln\left(\frac{b-t}{t-a}\right)$$

and

$$\begin{aligned}&PV\int_a^b \left(\frac{\int_a^\tau f\circ x'(s)\,ds - \int_a^t f\circ x'(s)\,ds}{\tau-t}\right)d\tau \\ &= \pi\left(T\left(\int_a f\circ x'(s)\,ds\right)\right)(a,b;t) - \int_a^t f\circ x'(s)\,ds\ln\left(\frac{b-t}{t-a}\right)\end{aligned}$$

and by (18) we get the desired result (6). □

We also have the reverse inequality:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $x:[a,b]\to\mathbb{R}$ an absolutely continuos function on $[a,b]$ with $x'(s)\in[a,b]$ for almost every $s\in(a,b)$. Then we have*

$$\begin{aligned}&\frac{1}{b-a}\left[\pi T\left(\int_a f\circ x'(s)\,ds\right)(a,b;t)-\left(\int_a^t f\circ x'(s)\,ds\right)\ln\left(\frac{b-t}{t-a}\right)\right] \quad (20)\\&\le\frac{f(a)}{b-a}\left[b-\frac{1}{b-a}\left(\pi(Tx)(a,b;t)-x(t)\ln\left(\frac{b-t}{t-a}\right)\right)\right]\\&+\frac{f(b)}{b-a}\left[\frac{1}{b-a}\left(\pi(Tx)(a,b;t)-x(t)\ln\left(\frac{b-t}{t-a}\right)\right)-a\right]\end{aligned}$$

for any $t\in(a,b)$.

In particular,

$$\begin{aligned}&\frac{\pi}{b-a}T\left(\int_a f\circ x'(s)\,ds\right)\left(a,b;\frac{a+b}{2}\right) \quad (21)\\&\le\frac{f(a)}{b-a}\left[b-\frac{\pi}{b-a}(Tx)\left(a,b;\frac{a+b}{2}\right)\right]\\&+\frac{f(b)}{b-a}\left[\frac{\pi}{b-a}(Tx)\left(a,b;\frac{a+b}{2}\right)-a\right].\end{aligned}$$

Proof. By the convexity of f we have

$$\begin{aligned}f\left(x'(s)\right)&=f\left(\frac{(b-x'(s))\,a+(x'(s)-a)\,b}{b-a}\right) \quad (22)\\&\le\frac{b-x'(s)}{b-a}f(a)+\frac{x'(s)-a}{b-a}f(b)\end{aligned}$$

for almost every $s\in[a,b]$.

Let $t,\,\tau\in[a,b]$ with $t\neq\tau$. Taking the integral mean $\frac{1}{\tau-t}\int_t^\tau$, which is a linear and isotonic functional, in the inequality (22), we get

$$\begin{aligned}&\frac{1}{\tau-t}\int_t^\tau f\left(x'(s)\right)ds \quad (23)\\&\le\frac{1}{\tau-t}\int_t^\tau\left[\frac{b-x'(s)}{b-a}f(a)+\frac{x'(s)-a}{b-a}f(b)\right]ds\\&=\frac{b-\frac{1}{\tau-t}\int_t^\tau x'(s)\,ds}{b-a}f(a)+\frac{\frac{1}{\tau-t}\int_t^\tau x'(s)\,ds-a}{b-a}f(b)\\&=\frac{b-\frac{x(\tau)-x(t)}{\tau-t}}{b-a}f(a)+\frac{\frac{x(\tau)-x(t)}{\tau-t}-a}{b-a}f(b)\\&=\frac{f(a)}{b-a}\left[b-\frac{x(\tau)-x(t)}{\tau-t}\right]+\frac{f(b)}{b-a}\left[\frac{x(\tau)-x(t)}{\tau-t}-a\right]\end{aligned}$$

for any $t,\,\tau\in[a,b]$ with $t\neq\tau$.

Let $t \in (a,b)$ and $t-a > \varepsilon > 0$, then by integrating (23) over $\tau \in [a, t-\varepsilon]$ we get

$$\begin{aligned}
&\int_a^{t-\varepsilon} \left(\frac{\int_t^\tau (f \circ x')(s)\, ds}{\tau - t} \right) d\tau \qquad (24)\\
&\leq \int_a^{t-\varepsilon} \left(\frac{f(a)}{b-a} \left[b - \frac{x(\tau)-x(t)}{\tau - t} \right] + \frac{f(b)}{b-a} \left[\frac{x(\tau)-x(t)}{\tau - t} - a \right] \right) d\tau \\
&= \frac{f(a)}{b-a} \left[b(t-\varepsilon-a) - \int_a^{t-\varepsilon} \frac{x(\tau)-x(t)}{\tau-t} d\tau \right] \\
&+ \frac{f(b)}{b-a} \left[\int_a^{t-\varepsilon} \frac{x(\tau)-x(t)}{\tau-t} d\tau - a(t-\varepsilon-a) \right].
\end{aligned}$$

Let $t \in (a,b)$ and $b-t > \varepsilon > 0$, then by integrating (23) over $\tau \in [t+\varepsilon, b]$ we obtain

$$\begin{aligned}
&\int_{t+\varepsilon}^{b} \left(\frac{\int_t^\tau (f \circ x')(s)\, ds}{\tau - t} \right) d\tau \qquad (25)\\
&\leq \int_{t+\varepsilon}^{b} \left(\frac{f(a)}{b-a} \left[b - \frac{x(\tau)-x(t)}{\tau - t} \right] + \frac{f(b)}{b-a} \left[\frac{x(\tau)-x(t)}{\tau - t} - a \right] \right) d\tau \\
&= \frac{f(a)}{b-a} \left[b(b-t-\varepsilon) - \int_{t+\varepsilon}^{b} \frac{x(\tau)-x(t)}{\tau-t} d\tau \right] \\
&+ \frac{f(b)}{b-a} \left[\int_{t+\varepsilon}^{b} \frac{x(\tau)-x(t)}{\tau-t} d\tau - a(b-t-\varepsilon) \right].
\end{aligned}$$

For $t \in (a,b)$ and $\min\{b-t, t-a\} > \varepsilon > 0$, we have by adding the inequalities (24) and (25) that

$$\begin{aligned}
&\int_a^{t-\varepsilon} \left(\frac{\int_t^\tau (f \circ x')(s)\, ds}{\tau - t} \right) d\tau + \int_{t+\varepsilon}^{b} \left(\frac{\int_t^\tau (f \circ x')(s)\, ds}{\tau - t} \right) d\tau \qquad (26)\\
&\leq \frac{f(a)}{b-a} \left[b(t-\varepsilon-a) - \int_a^{t-\varepsilon} \frac{x(\tau)-x(t)}{\tau-t} d\tau \right. \\
&\left. + b(b-t-\varepsilon) - \int_{t+\varepsilon}^{b} \frac{x(\tau)-x(t)}{\tau-t} d\tau \right] \\
&+ \frac{f(b)}{b-a} \left[\int_a^{t-\varepsilon} \frac{x(\tau)-x(t)}{\tau-t} d\tau - a(t-\varepsilon-a) \right. \\
&\left. + \int_{t+\varepsilon}^{b} \frac{x(\tau)-x(t)}{\tau-t} d\tau - a(b-t-\varepsilon) \right].
\end{aligned}$$

If we take $\varepsilon \to 0+$ in (26), then we obtain

$$\begin{aligned}
& PV\int_a^b \left(\frac{\int_t^\tau (f\circ x')(s)\,ds}{\tau - t}\right) d\tau \\
& \le \frac{f(a)}{b-a}\left[b(t-a)+b(b-t) - PV\int_a^b \frac{x(\tau)-x(t)}{\tau - t}d\tau\right] \\
& + \frac{f(b)}{b-a}\left[PV\int_a^b \frac{x(\tau)-x(t)}{\tau - t}d\tau - a(t-a) - a(b-t)\right] \\
& = \frac{f(a)}{b-a}\left[b(b-a) - PV\int_a^b \frac{x(\tau)-x(t)}{\tau - t}d\tau\right] \\
& + \frac{f(b)}{b-a}\left[PV\int_a^b \frac{x(\tau)-x(t)}{\tau - t}d\tau - a(b-a)\right],
\end{aligned}$$

which produces the following inequality of interest

$$\begin{aligned}
PV\int_a^b \left(\frac{\int_t^\tau (f\circ x')(s)\,ds}{\tau - t}\right) d\tau & \le f(a)\left[b - \frac{1}{b-a}PV\int_a^b \frac{x(\tau)-x(t)}{\tau - t}d\tau\right] \\
& + f(b)\left[\frac{1}{b-a}PV\int_a^b \frac{x(\tau)-x(t)}{\tau - t}d\tau - a\right],
\end{aligned} \tag{27}$$

for any $t \in (a,b)$. Now, since

$$PV\left(\int_a^b \frac{x(\tau)-x(t)}{\tau - t}d\tau\right) = \pi(Tx)(a,b;t) - x(t)\ln\left(\frac{b-t}{t-a}\right)$$

and

$$\begin{aligned}
& PV\int_a^b \left(\frac{\int_a^\tau f\circ x'(s)\,ds - \int_a^t f\circ x'(s)\,ds}{\tau - t}\right) d\tau \\
& = \pi\left(T\left(\int_a f\circ x'(s)\,ds\right)\right)(a,b;t) - \int_a^t f\circ x'(s)\,ds \ln\left(\frac{b-t}{t-a}\right),
\end{aligned}$$

hence by (27) we get

$$\begin{aligned}
& \pi\left(T\left(\int_a f\circ x'(s)\,ds\right)\right)(a,b;t) - \int_a^t f\circ x'(s)\,ds \ln\left(\frac{b-t}{t-a}\right) \\
& \le f(a)\left[b - \frac{1}{b-a}\left(\pi(Tx)(a,b;t) - x(t)\ln\left(\frac{b-t}{t-a}\right)\right)\right] \\
& + f(b)\left[\frac{1}{b-a}\left(\pi(Tx)(a,b;t) - x(t)\ln\left(\frac{b-t}{t-a}\right)\right) - a\right],
\end{aligned}$$

which proves (20). □

3 Examples

If we consider the function $\exp t = e^t$, $t \in (a,b)$ a real interval, then

$$(T\exp)(a,b;t) = \frac{\exp(t)}{\pi}\left[E_i(b-t) - E_i(a-t)\right], \tag{28}$$

where E_i is defined by

$$E_i(x) := PV\int_{-\infty}^{x}\frac{\exp(s)}{s}ds, \quad x \in \mathbb{R}.$$

Indeed, we have

$$\begin{aligned} E_i(b-t) - E_i(a-t) &= PV\int_{a-t}^{b-t}\frac{\exp(s)}{s}ds = PV\int_a^b\frac{\exp(\tau - t)}{\tau - t}ds \\ &= \exp(-t)\,\pi\,(T\exp)(a,b;t) \end{aligned}$$

and the equality (28) is proved.

Let $f:(a,b)\to\mathbb{R}$ be a convex function on (a,b). Then we have by (6) and (20) for $x(t) = \exp t$, that

$$\begin{aligned} & f\left(\frac{\exp(t)}{b-a}\left[E_i(b-t) - E_i(a-t) - \ln\left(\frac{b-t}{t-a}\right)\right]\right) \\ & \le \frac{1}{b-a}\left[\pi T\left(\int_a f\circ\exp(s)\,ds\right)(a,b;t) - \left(\int_a^t f\circ\exp(s)\,ds\right)\ln\left(\frac{b-t}{t-a}\right)\right] \\ & \le \frac{f(a)}{b-a}\left[b - \frac{\exp(t)}{b-a}\left(E_i(b-t) - E_i(a-t) - \ln\left(\frac{b-t}{t-a}\right)\right)\right] \\ & + \frac{f(b)}{b-a}\left[\frac{\exp(t)}{b-a}\left(E_i(b-t) - E_i(a-t) - \ln\left(\frac{b-t}{t-a}\right)\right) - a\right] \end{aligned} \tag{29}$$

for $t \in (a,b)$.

In particular, we have

$$\begin{aligned} & f\left(\frac{\exp\left(\frac{a+b}{2}\right)}{b-a}\left[E_i\left(\frac{b-a}{2}\right) - E_i\left(-\frac{b-a}{2}\right)\right]\right) \\ & \le \frac{1}{b-a}\left[\pi T\left(\int_a f\circ\exp(s)\,ds\right)\left(a,b;\frac{a+b}{2}\right)\right] \\ & \le \frac{f(a)}{b-a}\left[b - \frac{\exp\left(\frac{a+b}{2}\right)}{b-a}\left(E_i\left(\frac{b-a}{2}\right) - E_i\left(-\frac{b-a}{2}\right)\right)\right] \\ & + \frac{f(b)}{b-a}\left[\frac{\exp\left(\frac{a+b}{2}\right)}{b-a}\left(E_i\left(\frac{b-a}{2}\right) - E_i\left(-\frac{b-a}{2}\right)\right) - a\right]. \end{aligned} \tag{30}$$

If we take $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\exp(\alpha x)$, $\alpha\neq 0$, then for an absolutely continuous function $x:[a,b]\subset\mathbb{R}\to\mathbb{R}$ we have from (6) and (20) that

$$\begin{aligned}
&\exp\left(\frac{\alpha}{b-a}\left[\pi(Tx)(a,b;t)-x(t)\ln\left(\frac{b-t}{t-a}\right)\right]\right)\\
&\leq\frac{1}{b-a}\left[\pi T\left(\int_a\exp\left(\alpha x'(s)\right)ds\right)(a,b;t)-\left(\int_a^t\exp\left(\alpha x'(s)\right)ds\right)\ln\left(\frac{b-t}{t-a}\right)\right]\\
&\qquad\leq\frac{\exp a}{b-a}\left[b-\frac{1}{b-a}\left(\pi(Tx)(a,b;t)-x(t)\ln\left(\frac{b-t}{t-a}\right)\right)\right]\\
&\qquad\quad+\frac{\exp b}{b-a}\left[\frac{1}{b-a}\left(\pi(Tx)(a,b;t)-x(t)\ln\left(\frac{b-t}{t-a}\right)\right)-a\right]
\end{aligned}\tag{31}$$

for any $t\in(a,b)$.

In particular,

$$\begin{aligned}
&\exp\left(\frac{\alpha}{b-a}\left[\pi(Tx)\left(a,b;\frac{a+b}{2}\right)\right]\right)\\
&\qquad\leq\frac{1}{b-a}\left[\pi T\left(\int_a\exp\left(\alpha x'(s)\right)ds\right)\left(a,b;\frac{a+b}{2}\right)\right]\\
&\leq\frac{\exp a}{b-a}\left[b-\frac{\pi}{b-a}(Tx)\left(a,b;\frac{a+b}{2}\right)\right]+\frac{\exp b}{b-a}\left[\frac{\pi}{b-a}(Tx)\left(a,b;\frac{a+b}{2}\right)-a\right].
\end{aligned}\tag{32}$$

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