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DRAZIN INVERSE: REPRESENTATION, APPROXIMATION, CONTINUITY AND ILLUSTRATIONS

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Abstract

In this paper, we present some characteristics and expressions of the Drazin inverse for matrices and bounded linear operators in Banach spaces. We give a survey of some of results on the continuity of the Moore-Penrose and Drazin inverse, direct technics for computing the Drazin inverse are discussed, they are based on Euler-Knopp Method and characterized in terms of a limiting process. The examples presented are for illustrative purposes, some of which are provided for testing the considered iterative processes.

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1 Introduction and preliminary results

The main theme of this paper is the study of a generalized inverse introduced by M. P. Drazin, and its generalization due to J. J. Koliha. $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ matrices, with complex entries, equipped with the Euclidean norm. Suppose that $(A_j)_{j\in\mathbb{N}}$ is a sequence of $m\times n$ matrices, and $A \in \mathbb{C}^{m\times n}$, then $(A_j)_{j\in\mathbb{N}}$ converges to A if and only if the entries of A_j converge to the corresponding entries of A, as $j \to \infty$. Let $A \in \mathbb{C}^{m\times n}$, Penrose has proved in [15] that the system:

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA$$

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has a unique solution, which he called the Moore-Penrose inverse of A and denoted by $X = A^{\dagger}$. Let us recall that for any $A \in \mathbb{C}^{m\times n}$, we have $(A^{\dagger})^{\dagger} = A$, $(A^*)^{\dagger} = (A^{\dagger})^*$, $(A^*A)^{\dagger} = A^{\dagger}(A^{\dagger})^*$, but in general, $A^{\dagger}A \neq AA^{\dagger}$.

Contrary to the usual inverse of a square matrix, it is well known that the Moore-Penrose inverse of a matrix is not necessarily a continuous function of the elements of the matrix. Indeed, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, for each $\varepsilon > 0$, we have $(A+\varepsilon B)^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, hence $(A+\varepsilon B) \to A$, but $(A+\varepsilon B)^{-1}$ has no limit when $\varepsilon \to 0^+$.

The following theorem gives necessary and sufficient conditions for the continuity of the Moore-Penrose inverse of matrix.

Theorem 1. *([15]) Let A, A_j, $j \in \mathbb{N}$, be $m \times n$ complex matrices such that $\lim\limits_{j\to\infty} A_j = A$. Then, $\lim\limits_{j\to\infty} A_j^{\dagger} = A^{\dagger}$ if and only if there exists $q \in \mathbb{N}$ for which $rankA_j = rankA$, for all $j \geq q$.*

Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on a complex Banach space X, $\|.\|$ represents the norm of X. For $A \in \mathcal{B}(X)$, write $\mathcal{N}(A)$, $\mathcal{R}(A)$, $\sigma(A)$, $\rho(A)$ and $r(A)$, as the null space, the range, the spectrum, the resolvent set and the spectral radius of A, respectively. For $\lambda \in \rho(A)$, $(\lambda - A)^{-1}$ is the resolvent operator of A. I denotes the identity operator on X and A^* is the adjoint operator of A. By $\sigma'(A)$ we denote the set of all non-zero elements of $\sigma(A)$. By $iso\sigma(A)$ we define the set of all isolated spectral points of A. If M is a subspace of X, then $A_{|M}$ denotes the restriction of A to M. It is well known (see e.g. [3]) that $A \in \mathcal{B}(X)$ has closed range if and only if there exists a unique operator $A^{\dagger} \in \mathcal{B}(X)$, the Moore-Penrose inverse of A, which satisfies the following properties:

$$AA^{\dagger}A = A,\ A^{\dagger}AA^{\dagger} = A^{\dagger},\ (AA^{\dagger})^* = AA^{\dagger} \text{ and } (A^{\dagger}A)^* = A^{\dagger}A.$$

In general, if $\lim\limits_{j\to\infty} A_j = A$ uniformly, $(A_j)_{j\in\mathbb{N}}$, $A \in \mathcal{B}(X)$, and each A_j has a closed range, then A need not have a closed range.

Example 1. *Let $X = l_2$ the space of square-summable complex sequences. Define A_j, $j \in \mathbb{N}^*$, and A on l_2 by:*

$$A_j(x_1, x_2, ..., x_j, x_{j+1}, ...) = \left(x_1, \frac{x_2}{2}, ..., \frac{x_j}{j}, 0, 0, ...\right),$$
$$A(x_1, x_2, ..., x_j, x_{j+1}, ...) = \left(x_1, \frac{x_2}{2}, ..., \frac{x_j}{j}, \frac{x_{j+1}}{j+1}, ...\right).$$

Each A_j is of finite rank and thus with closed range and Moore-Penrose invertible operator. It is also clear that the sequence $(A_j)_{j\in\mathbb{N}^*}$ is uniformly convergent to A as $j \to \infty$. But the limit A does not have a closed range in l_2 because A is a compact operator.

The continuity of the Moore-Penrose inverse of an operator on Hilbert spaces has been studied by Izumino [7]. If X is a Hilbert space, let $(A_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{B}(X)$, $A\in\mathcal{B}(X)$, and $\lim\limits_{j\to\infty} A_j = A$. If $A_j^{\dagger}$, for all $j\in\mathbb{N}$, and $A^{\dagger}$ exist, it is shown in [7] that the following four conditions are equivalent:

$$(1)\ \lim_{j\to\infty} A_j^{\dagger} = A^{\dagger}.\ (2)\ \sup_{j\in\mathbb{N}} \left\|A_j^{\dagger}\right\| < \infty.$$

$$(3)\ \lim_{j\to\infty} A_j^{\dagger}A_j = A^{\dagger}A.\ (4)\ \lim_{j\to\infty} A_jA_j^{\dagger} = AA^{\dagger}.$$

The Drazin inverses of the elements of $\mathbb{C}^{n\times n}$ and $\mathcal{B}(X)$ have been intensively studied by authors such as Ben-Israel and Greville [1] and King [8], one may also refer to [[5], [17]] for some interesting applications about the Drazin inverse in singular differential and difference equations, Markov chain, numerical analysis, ... In this paper, we shall give, by analogy with the Moore-Penrose inverse, representation and computational procedures for the Drazin inverse. Let's recall some well-known results obtained for the Drazin inverse of a square matrix and the Drazin inverse of a bounded linear operator.

Let $A\in\mathbb{C}^{n\times n}$, with $k = ind(A)$ the smallest positive number such that $rankA^{k+1} = rankA^k$, the Drazin inverse of A is defined to be the unique matrix $X\in\mathbb{C}^{n\times n}$ that satisfying the following equations:

$$XAX = X, AX = XA \text{ and } A^{k+1}X = A^k. \tag{1}$$

It is denoted by $X = A^D$. In particular, a square matrix always has Drazin inverse and if $ind(A) = 0$, then A is invertible and $A^D = A^{\dagger} = A^{-1}$. Campbell and Meyer gave in [5] an explicit expression of the Drazin inverse of a square matrix via its canonical form representation.

Theorem 2. *Let $A\in\mathbb{C}^{n\times n}$ is such that $ind(A) = k > 0$, then there exists a non-singular matrix P such that:*

$$A = P\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} P^{-1} \tag{2}$$

where C is non-singular and N is nilpotent of index k.

Furthermore, if P, C and N are any matrices satisfying the above conditions, then:

$$A^D = P\begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}. \tag{3}$$

Proof. Let $\mathcal{B} = \{e_1, ..., e_r, e_{r+1}, ..., e_n\}$ be a basis for $\mathbb{C}^n$ such that $\{e_1, ..., e_r\}$ is a basis for $\mathcal{R}(A^k)$ and $\{e_{r+1}, ..., e_n\}$ is a basis for $\mathcal{N}(A^k)$, $k = ind(A) > 0$. $\mathcal{R}(A^k)$ and $\mathcal{N}(A^k)$ are invariant subspaces for A and $A^k\left(\mathcal{N}(A^k)\right) = \{0\}$, on the other hand A restricted to $\mathcal{R}(A^k)$, $C = A_{|\mathcal{R}(A^k)}$, is invertible and its restriction to $\mathcal{N}(A^k)$, $N = A_{|\mathcal{N}(A^k)}$, is nilpotent of degree k. So, we obtain the block form for A if $P = [e_1, ..., e_n]$. We can easily verify that the matrix A^D given by (3) satisfies the three equations (1) and then by uniqueness A^D is necessarily the Drazin inverse of A. $\square$

A first application of this representation is the following result:

Proposition 1. *Let $A \in \mathbb{C}^{n\times n}$, then $AA^DA = A$ if and only if $ind(A) \leq 1$.*

Proof. If $ind(A) = 0$, then A is invertible, $A^D = A^{-1}$ and $AA^DA = AA^{-1}A = A$.

Suppose that $in(A) \geq 1$. Then by virtue of the canonical form representation (2) of A and the corresponding expression (3) of A^D, we have $AA^DA = A$ if and only if $N = 0$. But $N = 0$ is equivalent to $ind(A) = 1$. □

Several methods and efficient algorithms have been given for computation of Moore-Penrose inverse of singular matrices and bounded linear operators with closed range, where this generalized inverse can be represented as the limit of a sequence of matrices or operators, respectively. The principle is to construct an iterative process of computing a sequence that converges to the generalized inverse. In what follows we explain this process for the Drazin inverse of a matrix and for a bounded linear operator and we estimate the corresponding error bounds. We also study the continuity of the generalized Drazin inverse.

The purpose of Section 1 is to present definitions and results that will be used throughout the paper. In Section 2, we recall and discuss some computational procedures for the Drazin inverse of a square matrix. The aim of Section 3 is to investigate the continuity of Drazin inverse in finite-dimensional case. Necessary and sufficient conditions for the continuity of the Drazin inverse of a matrix are given. In Section 4, we present the Drazin inverse (generalized Drazin inverse) of an operator, its uniqueness, existence, and some basic properties. We give also a representation theorem and a computational procedure of the Drazin inverse. In Section 5, we discuss some remarkable properties of the generalized Drazin inverse and we study its continuity in $\mathcal{B}(X)$. The results of each section are described in detail and interpreted by interesting examples.

2 Computational procedure for the Drazin inverse of a matrix

For $A \in \mathbb{C}^{n\times n}$, A^D can be computed recursively by the well known algorithms [6]. We have chosen to explain here the procedure developed by [17] for the computational of Drazin inverse and corresponding error bound. Wei and Wu found in [17] a specific expression and computational procedures for Drazin inverse, they have established the following formula for $A \in \mathbb{C}^{n\times n}$ with real spectrum and $ind(A) = k$:

$$
\begin{aligned}
A^D &= \lim_{j\to\infty} S_j\left(\widetilde{A}\right)A^l, \qquad (4)\\
\frac{\left\|A^D - S_j\left(\widetilde{A}\right)A^l\right\|_P}{\|A^D\|_P} &\leq \max_{x\in\sigma\left(\widetilde{A}\right)} |S_j(x)x - 1| + \mathcal{O}(\varepsilon),\ \varepsilon > 0,
\end{aligned}
$$

where $(S_j(x))_{j\in\mathbb{N}}$ is a family of continuous real valued functions on an open set Ω such that $\sigma\left(\widetilde{A}\right) \subset \Omega \subset]0,\infty[$, with $\lim_{j\to\infty} S_j(x) = \frac{1}{x}$ uniformly on the spectrum

$\sigma\left(\widetilde{A}\right)$ of $\widetilde{A}=\left(A^{l+1}\right)_{|\mathcal{R}(A^k)}$ for $l\geq k$, P is an invertible matrix such that $P^{-1}AP$ is the Jordan canonical form of A and $\|A\|_P=\left\|P^{-1}AP\right\|$.

Consider the following sequence with parameter $\alpha>0$:

$$S_j(x)=\alpha\sum_{m=0}^{j}(1-\alpha x)^m\,, \tag{5}$$

which can be viewed as the Euler-Knopp transform of the series $\sum\limits_{j=0}^{\infty}(1-x)^j$. Clearly, $\lim\limits_{j\to\infty}S_j(x)=\frac{1}{x}$ uniformly on any compact subset of the set:

$$E_\alpha=\{x:|1-\alpha x|<1\}=\left\{x:0<x<\frac{2}{\alpha}\right\}. \tag{6}$$

Lemma 1. *If $A\in\mathbb{C}^{n\times n}$ with $ind(A)=k$, $\widetilde{A}=\left(A^{l+1}\right)_{|\mathcal{R}(A^k)}$ for $l\geq k$, and the spectrum of A is real, then $\sigma\left(\widetilde{A}\right)\subset\left[\frac{1}{\|A^D\|^{l+1}},\|A\|^{l+1}\right]$.*

Proof. Let $\{\lambda_1,...,\lambda_r\}$ be the nonzero eigenvalues of A where $rank(A^k)=r$. Hence, $0<\lambda_i^{l+1}\in\sigma\left(\widetilde{A}\right)\subseteq\sigma\left(A^{l+1}\right)$, $i=1,...,r$. It is clear that $ind(A^{l+1})=1$ and $\frac{1}{\lambda_i^{l+1}}\in\sigma\left(\left(A^D\right)^{l+1}\right)$ and thus $\frac{1}{\lambda_i^{l+1}}\leq\left\|A^D\right\|^{l+1}$, $i=1,...,r$. Therefore, $\lambda\geq\frac{1}{\|A^D\|^{l+1}}$ for every $\lambda\in\sigma\left(\widetilde{A}\right)$. Moreover, since $\left\|A^{l+1}\right\|=\left\|\left(A^{l+1}\right)_{|\mathcal{R}(A^k)}\right\|$, we have $\left\|\widetilde{A}\right\|\leq\|A\|^{l+1}$ and then $\lambda\leq\|A\|^{l+1}$ for every $\lambda\in\sigma\left(\widetilde{A}\right)$. □

Since, by Lemma 1, $\sigma\left(\widetilde{A}\right)\subset\left]0,\|A\|^{l+1}\right]$, if we choose the parameter α such that $0<\alpha<\frac{2}{\|A\|^{l+1}}$, and $\left]0,\|A\|^{l+1}\right]\subseteq E_\alpha$ then we have the representation:

$$A^D=\alpha\sum_{j=0}^{\infty}\left(I-\alpha A^{l+1}\right)^jA^l. \tag{7}$$

Let us pose:

$$A_j=\alpha\sum_{m=0}^{j}\left(I-\alpha A^{l+1}\right)^mA^l. \tag{8}$$

Then $A_0=\alpha A^l$, $A_{j+1}=\left(I-\alpha A^{l+1}\right)A_j+A_0$, $j\in\mathbb{N}$, and by construction $\lim\limits_{j\to\infty}A_j=A^D$. Now, remark that the sequence of functions $\left(S_j(x)\right)_{j\in\mathbb{N}}$ satisfies:

$$xS_{j+1}(x)-1=(1-\alpha x)\left(xS_j(x)-1\right),\ j\in\mathbb{N}, \tag{9}$$

hence,

$$|xS_j(x)-1|=|1-\alpha x|^{j+1}\leq\left[\max\left(\left|1-\alpha\|A\|^{l+1}\right|,\left|1-\frac{\alpha}{\|A^D\|^{l+1}}\right|\right)\right]^{j+1} \tag{10}$$

which tends to 0 as $j \to \infty$ if $x \in \sigma\left(\widetilde{A}\right)$ and $0 < \alpha < \frac{2}{\|A\|^{l+1}}$, and estimates by virtue of (4) the variation of the errors. So, we have shown the following approximation result.

Theorem 3. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$, $\widetilde{A} = \left(A^{l+1}\right)_{|\mathcal{R}(A^k)}$ for $l \geq k$, and the spectrum of A is real. Then the sequence $(A_j)_{j\in\mathbb{N}}$ defined by (8) converges to the Drazin inverse A^D of A if $0 < \alpha < \frac{2}{\|A\|^{l+1}}$. Furthermore, the error bound is:*

$$\frac{\left\|A_j - A^D\right\|_P}{\left\|A^D\right\|_P} \leq \beta^{j+1} + \mathcal{O}\left(\varepsilon\right), \varepsilon > 0$$

where $\beta = \max\left(\left|1 - \alpha\|A\|^{l+1}\right|, \left|1 - \frac{\alpha}{\|A^D\|^{l+1}}\right|\right) < 1.$

Remark 1. *We can significantly improve the convergence speed of this process using other iterative methods as Newton–Raphson method, Newton–Gregory interpolation formula, Hermite interpolation, ... (see e.g. [17]).*

Example 2. *We use the iterative algorithm previously developed to compute the Drazin inverse of the singular matrix:*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It is easy to see that the minimal polynomial of A is given by $\lambda^3 - 3\lambda^2$, hence $\sigma(A) = \{0, 0, 3\}$ and $A^2 = 3A$. An easy computations show that $ind(A) = 1$ and that the Drazin inverse of A is given by $\frac{1}{9}A$. So we can choose $\alpha = 10^{-2}$ and $l = 1$, since $\|A\|^2 = 27$. Then the iterations are given by:

$$\begin{aligned} A_0 &= \alpha A = \begin{pmatrix} 0,01 & 0,01 & 0,01 \\ 0,01 & 0,01 & 0,01 \\ 0,01 & 0,01 & 0,01 \end{pmatrix}, \\ A_{j+1} &= BA_j + A_0,\ B = \left(I - \alpha A^2\right) = \left(I - 3A_0\right),\ j \in \mathbb{N}. \end{aligned}$$

Thus,

$$B = (I - 3A_0) = \begin{pmatrix} 0,97 & -0,03 & -0,03 \\ -0,03 & 0,97 & -0,03 \\ -0,03 & -0,03 & 0,97 \end{pmatrix},$$

and

$$A_j = \left(B^j + B^{j-1} + ... + I\right)A_0 = \sum_{m=0}^{j} B^m A_0,\ j \in \mathbb{N}.$$

The matrix B is diagonalizable, $B = S^{-1}DS$, where S is a non-singular matrix and D is diagonal:

$$S = \begin{pmatrix} -0,3333 & 0,6667 & -0,3333 \\ -0,3333 & -0,3333 & 0,6667 \\ 0,3333 & 0,3333 & 0,3333 \end{pmatrix}, \; S^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\text{and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0,91 \end{pmatrix}.$$

$$\begin{aligned} B^m &= S^{-1}D^m S, \; m \in \{0,1,...,j\}, \\ A_j &= S^{-1}\sum_{m=0}^{j} D^m S A_0 = S^{-1}\begin{pmatrix} (j+1) & 0 & 0 \\ 0 & (j+1) & 0 \\ 0 & 0 & \alpha_j \end{pmatrix} S A_0, \end{aligned}$$

where $\alpha_j = \sum\limits_{m=0}^{j} (0,91)^m = \frac{1-(0,91)^{j+1}}{0,09}$, $j \in \mathbb{N}$.

$$A_j = \begin{pmatrix} -(j+1) & -(j+1) & \alpha_j \\ (j+1) & 0 & \alpha_j \\ 0 & (j+1) & \alpha_j \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0,01 & 0,01 & 0,01 \end{pmatrix} = \frac{\alpha_j}{100} A.$$

Since $\lim\limits_{j\to\infty} \alpha_j = \sum\limits_{m=0}^{\infty} (0,91)^m = \frac{100}{9}$, *it follows that* $\lim\limits_{j\to\infty} A_j = \frac{1}{9}A = A^D$.

3 Continuity of the Drazin inverse of a matrix

The Drazin inverse of a matrix is not necessarily a continuous function of the elements of the matrix. In particular, It is easy to produce examples to show that Theorem 1 is not valid for the Drazin inverse (see [4]).

Example 3. *1) Let:*

$$A_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{j} \\ 0 & 0 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, $\lim\limits_{j\to\infty} A_j = A$, $\lim\limits_{j\to\infty} A_j^D = A^D$, *but* $rank(A_j) > rank(A)$, *for all* $j \in \mathbb{N}^*$.

2) Let:

$$A_j = \begin{pmatrix} \frac{1}{j} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$A_j^D = \begin{pmatrix} j & j^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^D = 0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\lim_{j\to\infty} A_j = A$, $rank(A_j) = rank(A)$ *and* $ind(A_j) = ind(A)$ *for all* $j \in \mathbb{N}^*$, *but* $\lim_{j\to\infty} A_j^D \neq A^D$.

Define the core-rank of $A \in \mathbb{C}^{n\times n}$ as rank of the matrix A^k where $k = ind(A)$. It is showed in [4] an analogous result, to that of Moore Penrose (Theorem 1), for the Drazin inverse of a square matrix.

Theorem 4. *([4]) Suppose that* $A_j, A \in \mathbb{C}^{n\times n}$, $j \in \mathbb{N}$, *are such that* $\lim_{j\to\infty} A_j = A$. *Then,* $\lim_{j\to\infty} A_j^D = A^D$ *if and only if there is* $q \in \mathbb{N}$ *such that core-rank of* A_j *is equal to core-rank of* A *for all* $j \in \mathbb{N}$, $j \geq q$.

Similarly, from the Jordan canonical form for A_j and A, $A = C + N$, $A_j = C_j + N_j$, where $C, N, C_j, N_j \in \mathbb{C}^{n\times n}$, C and C_j are non-singular and N and N_j are nilpotent, for every $j \in \mathbb{N}$, it is immediate to give necessary and sufficient conditions for the continuity of the Drazin inverse of a matrix. If $\lim_{j\to\infty} A_j = A$,

$$\lim_{j\to\infty} A_j^D = A^D \Longleftrightarrow \exists q \in \mathbb{N} : rank(C_j) = rank(C) \text{ for } j \geq q.$$

4 Remarkable properties and computational procedure for the GD-inverse

Recall that if $A \in \mathcal{B}(X)$, then $a(A)$ and $d(A)$, respectively the ascent and the descent of A, is the smallest non-negative integer k such that $\mathcal{N}(A^k) = \mathcal{N}(A^{k+1})$ and $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$. If no such k exists, then $a(A) = \infty$ and $d(A) = \infty$. It is well known that if the ascent and the descent of an operator are finite, then they are equal. $A^D \in \mathcal{B}(X)$ is the Drazin inverse of $A \in \mathcal{B}(X)$ if $A^D A = AA^D$, $A^D AA^D = A^D$ and $AA^D A = A + Q$ where Q is a nilpotent operator on X. A^D is unique. The concept of Drazin invertible operators has been generalized by Koliha [9] by replacing the nilpotent operator Q in the equation $AA^D A = A + Q$ by a quasi-nilpotent operator. In this case, A^D is called a generalized Drazin inverse (GD-inverse) of A and noted A^{GD}. Invertible operators, right invertible operators and left invertible operators are GD-invertible operators. We define the Drazin index of A by:

$$ind(A) = \begin{cases} 0 & \text{if } \; A \text{ is invertible} \\ k & \text{if } \; Q = A\left(A^D A - I\right) \text{ is nilpotent of index } k \\ \infty & \text{if } \; Q = A\left(A^D A - I\right) \text{ is quasi-nilpotent.} \end{cases}$$

Note that a square matrix always has Drazin inverse. But, if X is an infinite-dimensional complex Banach space, then it is well known that an operator $A \in \mathcal{B}(X)$ has a Drazin inverse A^D if and only if it has finite ascent and descent (in such a case, the index of A is equal to the ascent of A), or equivalently if and only if 0 is a pole of its resolvent operator, which is also equivalent to the fact that $A = R \oplus N$ where R is invertible and N is nilpotent. A has a GD-inverse if and only if 0 is an isolated point of its spectrum or $A = R \oplus Q$ where R is invertible and Q is quasi-nilpotent, or equivalently if and only if there is a bounded projection P on X (the generalized Drazin idempotent of A) such that $R = A_{|\mathcal{R}(P)}$ is invertible operator and $Q = A_{|\mathcal{N}(P)}$ is quasi-nilpotent operator (see [[11], [9], [12], [14], [13]]). $\oplus$ denotes the algebraic direct sum. Note that the GD-inverse A^{GD} of A, if it exists, is uniquely determined by $A^{GD} = R^{-1} \oplus 0$. Moreover, by the spectral mapping theorem we have that:

$$\sigma(A^{GD}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma'(A) \right\} \cup \{0\}, \ \sigma'(A^{GD}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma'(A) \right\}$$

and if $r(A) > 0$, then $dist(\sigma'(A), 0) = \frac{1}{r(A^{GD})}$, where $dist$ is the distance between a bounded subset of $\mathbb{C}$ and 0.

Example 4. *1) Every Drazin invertible operator is GD-invertible. It is also clear from the definition of a GD-inverse that every quasi-nilpotent operator is GD-invertible with a generalized Drazin inverse* 0.

2) Every quasi-nilpotent operator which is not nilpotent (for example Volterra operator) is GD-invertible and cannot be Drazin invertible. Indeed, suppose that $A \in \mathcal{B}(X)$ is quasi-nilpotent but not nilpotent and Drazin invertible with $A^D = B$. We have seen before that A is GD-invertible with GD-inverse 0. *By the uniqueness of the GD-inverse we must have that $B = 0$. Also, because $A - ABA$ is nilpotent, we have that A is also nilpotent. This is a contradiction.*

The following result gives an interesting characterization of Drazin invertible operators.

Proposition 2. *([9]) Let $A \in \mathcal{B}(X)$ and $k \in \mathbb{N}$, $k \geq 1$. Then the following assertions are equivalent:*

1) A is Drazin invertible and $ind(A) = k$.

2) $a(A) = d(A) = k$.

3) The resolvent operator $(\lambda - A)^{-1}$ has a pole of order k at $\lambda = 0$.

Example 5. *Let's check that if $A \in \mathcal{B}(X)$ is normal, $A^*A = AA^*$, and if $0 \in iso\sigma(A)$, then A is Drazin invertible and $ind(A) = 1$. Indeed, if P is the spectral projection associated with* 0. *We know that A and P commute and then $A(\mathcal{R}(P)) \subseteq \mathcal{R}(P)$ and $\sigma\left(A_{|\mathcal{R}(P)}\right) = \{0\}$. On the other hand, since $A_{|\mathcal{R}(P)}$ is normal operator on $\mathcal{R}(P)$, we have $\left\|A_{|\mathcal{R}(P)}\right\| = r\left(A_{|\mathcal{R}(P)}\right) = 0$, so it follows that $AP = 0$. The Laurent series expansion around* 0 *of the resolvent $(\lambda - A)^{-1}$ is given by:*

$$(\lambda - A)^{-1} = \sum_{j=1}^{\infty} \frac{P_j}{\lambda^j} + \sum_{j=0}^{\infty} Q_j \lambda^j, \ 0 < |\lambda| < \epsilon \tag{11}$$

where the coefficients P_j and Q_j are given by the formulas:

$$P_j = \frac{1}{2\pi i} \oint\limits_{0<|\lambda|<\epsilon} \lambda^{j-1}(\lambda - A)^{-1} d\lambda \text{ and } Q_j = \frac{1}{2\pi i} \oint\limits_{0<|\lambda|<\epsilon} \lambda^{-j-1}(\lambda - A)^{-1} d\lambda. \quad (12)$$

It follows from (12), *immediately using the functional calculus, that:*

$$P_1 = P \text{ and } P_j = A^{j-1}P, \; j \in \mathbb{N}^*,$$

hence 0 *is a simple pole of* $(\lambda - A)^{-1}$. *Proposition* 2 *shows that* A *is Drazin invertible and* $ind(A) = 1$.

Drazin inverses are not symmetric in general. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$, A is Drazin invertible with $A^D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\left(A^D\right)^D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq A$. Our next result gives the expression of the element $\left(A^{GD}\right)^{GD}$.

Theorem 5. *Let $A \in \mathcal{B}(X)$. Suppose that A is GD-invertible and that P is the generalized Drazin idempotent of A. Then $\left(A^{GD}\right)^{GD} = A\left(I - P\right)$.*

Proof. Suppose that A is generalized Drazin invertible and that P is the generalized Drazin idempotent of A. If A is invertible then $P = 0$ and the result obviously holds.

If $0 \in iso\sigma(A)$, then P is the spectral projection of A corresponding to 0 and $P = I - A^{GD}A$. We show that $B = A(I - P)$ is the generalized Drazin inverse of A^{GD}. Using the fact that $AP = PA$, we have that $A^{GD}B = A^{GD}A(I - P) = A(I - P)A^{GD} = BA^{GD}$. We also have that:

$$\begin{aligned} BA^{GD}B &= A(I-P)A^{GD}A(I-P) = A(I-P)A^{GD}A\left(A^{GD}A\right) \\ &= A(I-P)\left(A^{GD}AA^{GD}\right)A = A(I-P)A^{GD}A \\ &= A(I-P)\left(I-P\right) = A(I-P) = B, \end{aligned}$$

and

$$\begin{aligned} A^{GD} - A^{GD}BA^{GD} &= A^{GD} - A^{GD}A(I-P)A^{GD} = A^{GD} - A^{GD}A(A^{GD}A)A^{GD} \\ &= A^{GD} - (A^{GD}AA^{GD})AA^{GD} = A^{GD} - A^{GD}AA^{GD} = 0 \end{aligned}$$

is quasi-nilpotent. Hence, $\left(A^{GD}\right)^{GD} = B = A\left(I - P\right)$. This completes the proof. □

As 0 is a simple pole of $(\lambda - A)^{-1}$ if and only if $AP = 0$, we obtain:

Corollary 1. *Let $A \in \mathcal{B}(X)$ and $0 \in iso\sigma(A)$. Then $\left(A^{GD}\right)^{GD} = A$ if and only if 0 is a simple pole of $(\lambda - A)^{-1}$.*

In the following we give a representation theorem for the Drazin inverse of a linear operator in Banach space and the corresponding error bound.

Theorem 6. *Let $A \in \mathcal{B}(X)$ be Drazin invertible of index k and $\mathcal{R}(A^k)$ is closed. Define $\widetilde{A} = \left(A^k A^{*2k+1} A^{k+1}\right)_{|\mathcal{R}(A^k)}$. If Ω is an open set such that $\sigma\left(\widetilde{A}\right) \subset \Omega \subset]0, \infty[$ and $\left(S_j(x)\right)_{j\in\mathbb{N}}$ is a sequence of continuous real valued functions on Ω with $\lim\limits_{j\to\infty} S_j(x) = \frac{1}{x}$ uniformly on $\sigma\left(\widetilde{A}\right)$, then:*

$$A^D = \lim_{j\to\infty} S_j\left(\widetilde{A}\right) A^k A^{*2k+1} A^k.$$

Furthermore, for any $\varepsilon > 0$, there is an operator norm $\|.\|_*$ on X such that:

$$\frac{\left\|S_j\left(\widetilde{A}\right) A^k A^{*2k+1} A^k - A^D\right\|_*}{\|A^D\|_*} \leq \max_{x\in\sigma\left(\widetilde{A}\right)} |S_j(x)x - 1| + \mathcal{O}\left(\varepsilon\right).$$

Proof. It's clear that $\sigma\left(\widetilde{A}\right) = \sigma\left(A^k A^{*2k+1} A^{k+1}\right) = \sigma\left(\left(A^{2k+1}\right)^* \left(A^{2k+1}\right)\right) \subset]0, \infty[$, since $\widetilde{A}$ is positive and boundedly invertible. Using functional calculus, we have:

$$\lim_{j\to\infty} S_j\left(\widetilde{A}\right) = \widetilde{A}^{-1}.$$

It then follows from [2] that:

$$\lim_{j\to\infty} S_j\left(\widetilde{A}\right) A^k A^{*2k+1} A^k = \widetilde{A}^{-1} A^k A^{*2k+1} A^k = A^D.$$

To obtain the error bound, we note that:

$$\begin{aligned} A^k A^{*2k+1} A^k &= \widetilde{A} A^D, \\ S_j\left(\widetilde{A}\right) A^k A^{*2k+1} A^k - A^D &= \left(S_j\left(\widetilde{A}\right) \widetilde{A} - I\right) A^D. \end{aligned}$$

We also know that for any $\varepsilon > 0$, one can define a new norm $\|.\|_*$ on X with the formula:

$$\|x\|_* = \sqrt{\sum_{j=0}^{m} \left(\frac{\|A^j x\|}{M^j}\right)^2},$$

where $M = r(A) + \varepsilon$ and $m \in \mathbb{N}$ has been chosen as the first integer such that $\|A^m\|^{1/m} < M$. It is easy to see that this norm is equivalent to the original norm and it induces a norm on $\mathcal{B}(X)$ such that $\|A\|_* < r(A) + \varepsilon$. Thus,

$$\begin{aligned} \left\|S_j\left(\widetilde{A}\right) A^k A^{*2k+1} A^k - A^D\right\|_* &\leq \left\|S_j\left(\widetilde{A}\right) \widetilde{A} - I\right\|_* \left\|A^D\right\|_* \\ &\leq \left(\max_{x\in\sigma\left(\widetilde{A}\right)} |S_j(x)x - 1| + \mathcal{O}\left(\varepsilon\right)\right) \left\|A^D\right\|_*. \end{aligned}$$

□

Now we explain the Euler-Knopp Method for computational of the Drazin inverse of a bounded operator and the way to get the corresponding error bound. First we need the following result concerning lower and upper bounds for $\sigma\left(\widetilde{A}\right)$.

Lemma 2. *Let $A \in \mathcal{B}(X)$ be Drazin invertible of index k and $\mathcal{R}(A^k)$ is closed. Then, for all $\lambda \in \sigma\left(\widetilde{A}\right)$:*

$$\frac{1}{\left\|(A^{2k+1})^{\dagger}\right\|^{2}} \leq \lambda \leq \|A\|^{4k+2}.$$

Proof. For all $\lambda \in \sigma\left(\widetilde{A}\right)$, $\lambda > 0$ and $\lambda \in \sigma\left(\left(A^{2k+1}\right)^{*}\left(A^{2k+1}\right)\right)$. Furthermore, it's clear that $ind\left(\left(A^{2k+1}\right)^{*}A^{2k+1}\right) = 1$ and

$$\frac{1}{\lambda} \in \sigma\left(\left(\left(A^{2k+1}\right)^{*}A^{2k+1}\right)^{\dagger}\right) = \sigma\left(\left(A^{2k+1}\right)^{\dagger}\left(\left(A^{2k+1}\right)^{\dagger *}\right)\right).$$

Thus,

$$\frac{1}{\lambda} \leq \left\|\left(A^{2k+1}\right)^{\dagger}\left(\left(A^{2k+1}\right)^{\dagger *}\right)\right\| = \left\|\left(A^{2k+1}\right)^{\dagger}\right\|^{2} \text{ and } \lambda \geq \frac{1}{\left\|(A^{2k+1})^{\dagger}\right\|^{2}}.$$

On the other hand, since $\left\|(A^{k}A^{*2k+1}A^{k+1})_{|\mathcal{R}(A^k)}\right\| \leq \left\|A^{k}A^{*2k+1}A^{k+1}\right\|$, we obtain $\left\|\widetilde{A}\right\| \leq \|A\|^{4k+2}$ so what $\lambda \leq \left\|\widetilde{A}\right\| \leq \|A\|^{4k+2}$ for all $\lambda \in \sigma\left(\widetilde{A}\right)$. □

Consider now the sequence $(S_p(x))_{p\in\mathbb{N}}$ and the set E_α, $\alpha > 0$, defined respectively in (5) and (6). By Lemma 2, we get $\sigma\left(\widetilde{A}\right) \subseteq \left]0, \|A\|^{4k+2}\right]$. So, if we choose the parameter α, $0 < \alpha < \frac{2}{\|A\|^{4k+2}}$ such that $\sigma\left(\widetilde{A}\right) \subseteq \left]0, \|A\|^{4k+2}\right] \subset E_\alpha$, then we obtain the following representation of the Drazin inverse A^D of A :

$$A^{D} = \alpha \sum_{j=0}^{\infty}\left(I - \alpha A^{k}A^{*2k+1}A^{k+1}\right)^{j} A^{k}A^{*2k+1}A^{k}.$$

Setting $A_j = \alpha \sum\limits_{m=0}^{j}\left(I - \alpha A^{k}A^{*2k+1}A^{k+1}\right)^{m} A^{k}A^{*2k+1}A^{k}$, we have the following iterative procedure for the Drazin inverse:

$$A_0 = \alpha A^{k}A^{*2k+1}A^{k} \text{ and } A_{j+1} = \left(I - \alpha A^{k}A^{*2k+1}A^{k+1}\right)A_j,\ j \in \mathbb{N}.$$

Therefore, $\lim\limits_{j\to\infty} A_j = A^D$. For the error bound, we note that from (9) and (10), we have $|xS_j(x) - 1| = |1 - \alpha x|^{j+1} \leq \beta^{j+1} \underset{j\to\infty}{\longrightarrow} 0$, if $x \in \sigma\left(\widetilde{A}\right)$, $0 < \alpha < \frac{2}{\|A\|^{4k+2}}$

and $\beta = \max\left\{\left|1-\alpha\left\|A\right\|^{4k+2}\right|,\left|1-\frac{\alpha}{\left\|\left(A^{2k+1}\right)^{\dagger}\right\|^{2}}\right|\right\} < 1$. It follows from the above inequality and Theorem 6, the error bound:

$$\frac{\left\|A_j - A^D\right\|_*}{\left\|A^D\right\|_*} \leq \beta^{j+1} + \mathcal{O}\left(\varepsilon\right),\ \varepsilon > 0.$$

Note that this approximation generalizes to infinite-dimensional case the result obtained on the square matrices in Theorem 3.

Example 6. *Let $A \in \mathcal{B}(X)$ be selfadjoint, $A^* = A$, and $0 \in iso\sigma(A)$. Then A has closed range and is Drazin invertible with $ind(A) = 1$. Let's use the iterative procedure developed previously with $0 < \alpha < \frac{2}{\|A\|^6}$ and $\beta = \max\left\{\left|1-\alpha\left\|A\right\|^{6}\right|,\left|1-\frac{\alpha}{\left\|\left(A^{3}\right)^{\dagger}\right\|^{2}}\right|\right\} < 1$. So, $\lim\limits_{j\to\infty} A_j = A^D$ where $A_0 = \alpha A^5$, $A_{j+1} = \left(I - \alpha A^6\right)A_j$, $j \in \mathbb{N}$, and the error bound $\frac{\left\|A_j - A^D\right\|_*}{\left\|A^D\right\|_*} \leq \beta^{j+1} + \mathcal{O}\left(\varepsilon\right),\ \varepsilon > 0$.*

5 Continuity of the GD-Drazin inverse

Drazin inversion is not continuous in general, we illustrate this in the following example.

Example 7. *Let $A \in \mathcal{B}(l_2)$ be a weighted shift with weight sequence:*

$$0, 0, -1, 0, 0, -1, 0, 0, -1, ...$$

so that A is nilpotent of index 3. Then A is Drazin invertible and with $A^D = 0$. Let $A_j = A + \frac{1}{j}I$, for all $j \in \mathbb{N}^*$. Then, for each $j \in \mathbb{N}^*$:

$$A_j\left(jI - j^2A + j^3A^2\right) = \left(jI - j^2A + j^3A^2\right)A_j = I.$$

Thus A_j is invertible and hence Drazin invertible with:

$$A_j^D = A_j^{-1} = \left(jI - j^2A + j^3A^2\right), j \in \mathbb{N}^*.$$

It is clear that $A_j \longrightarrow A$ as $j \to \infty$ in $\mathcal{B}(l_2)$, but the unbounded sequence $\left(A_j^D\right)_{j\in\mathbb{N}^*}$ does not converge to $A^D = 0$.

Rakocevic investigate in [16] the continuity of the Drazin inverse of a bounded linear operator on Banach space, i.e. the continuity of the maps $A \longrightarrow A^D$ and $A \longrightarrow A^{GD}$, $A \in B(X)$, he generalized the continuity result of [7] to Drazin inverse in the following way. Let $(A_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{B}(X)$, and let $\lim\limits_{j\to\infty} A_j = A$. Suppose that A and A_j, have Drazin inverses A^D and A_j^D respectively. Then the following conditions are equivalent:

(1) $\lim\limits_{j\to\infty} A_j^D = A^D$. *(2)* $\sup\limits_{j\in\mathbb{N}}\left\|A_j^D\right\| < \infty$. *(3)* $\lim\limits_{j\to\infty} A_j^D A_j = A^D A$.

Furthermore, by virtue of Banach-Steinhaus theorem, we can easily deduce, as a generalization, an equivalent result when $(A_j)_{j\in\mathbb{N}}$ converges to A strongly.

It is interesting to study the continuity of the GD-inverse. We are now ready to present the main result of this section, it is due to Koliha and Rakocevic [10], nevertheless, the proof below is direct and of a technical nature.

Theorem 7. *Let $(A_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{B}(X)$, and let $\lim\limits_{j\to\infty} A_j = A$. Suppose that A and A_j have generalized Drazin inverses A^{GD} and A_j^{GD}, and let P and P_j be the spectral projections corresponding to 0, of A and A_j, respectively, for every $j \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) $\lim\limits_{j\to\infty} A_j^{GD} = A^{GD}$.

(ii) $\sup\limits_{j\in\mathbb{N}} \left\| A_j^{GD} \right\| < \infty$.

(iii) $\sup\limits_{j\in\mathbb{N}} \left\| r\left(A_j^{GD}\right) \right\| < \infty$.

(iv) $\inf\limits_{j\in\mathbb{N}} dist(\sigma'(A_j), 0) > 0$.

(v) There exists an $r > 0$ such that $\widetilde{B}(0,r) \subseteq \rho(A) \cap \bigcap\limits_{j=0}^{\infty} \rho(A_j)$, where $\widetilde{B}(0,r)$ is the open ball excluding the center 0 and with radius r.

(vi) $\lim\limits_{j\to\infty} A_j^{GD} A_j = A^{GD} A$.

(vii) $\lim\limits_{j\to\infty} P_j = P$.

Proof. (i) $\Longrightarrow$ (ii). Follows from the fact that convergence implies boundedness.

(ii) $\Longrightarrow$ (iii). Suppose that (ii) holds. Since $r\left(A_j^{GD}\right) \le \left\| A_j^{GD} \right\| \le \sup\limits_{j\in\mathbb{N}} \left\| A_j^{GD} \right\| < \infty$, for all $j \in \mathbb{N}$, we obtain that $\sup\limits_{j\in\mathbb{N}} r\left(A_j^{GD}\right) \le \sup\limits_{j\in\mathbb{N}} \left\| A_j^{GD} \right\| < \infty$.

(iii) $\Longrightarrow$ (iv). Suppose that $k = \sup\limits_{j\in\mathbb{N}} r\left(A_j^{GD}\right) < \infty$. We distinguish the following three cases:

Case I: $r\left(A_j^{GD}\right) = 0$ for all $j \in \mathbb{N}$. Then $\sigma\left(A_j^{GD}\right) = \{0\}$ and hence $\sigma\left(A_j\right) = \{0\}$, so $dist(\sigma'(A_j), 0) = \infty$, for all $j \in \mathbb{N}$. It then follows that $\inf\limits_{j\in\mathbb{N}} dist(\sigma'(A_j), 0) = \infty > 0$.

Case II: If $r\left(A_j^{GD}\right) > 0$ for all $j \in \mathbb{N}$, then $k > 0$ and $\left(r\left(A_j^{GD}\right)\right)^{-1} \ge k^{-1}$. Hence, $r\left(A_j^{GD}\right) = r\left(A_j^{GD} A_j A_j^{GD}\right) \le \left(r\left(A_j^{GD}\right)\right)^2 r\left(A_j\right)$, so that $r\left(A_j\right) \ge \left(r\left(A_j^{GD}\right)\right)^{-1} \ge k^{-1} > 0$, for all $j \in \mathbb{N}$. Or, for all $j \in \mathbb{N}$, $dist(\sigma'(A_j), 0) = \left(r\left(A_j^{GD}\right)\right)^{-1} \ge k^{-1} > 0$, and hence $\inf\limits_{j\in\mathbb{N}} dist(\sigma'(A_j), 0) \ge k^{-1} > 0$.

Case III: There is at least one $j \in \mathbb{N}$ such that $r\left(A_j^{GD}\right) > 0$ and possibly some other j' for which $r\left(A_{j'}^{GD}\right) = 0$. By case II, we have that $dist(\sigma'(A_j), 0) = \left(r\left(A_j^{GD}\right)\right)^{-1} \ge k^{-1} > 0$ for all $j \in \mathbb{N}$ satisfying $r\left(A_j^{GD}\right) > 0$. By case I,

$dist(\sigma'(A_{j'}),0)=\infty$ for all $j'\in\mathbb{N}$ such that $r\left(A_{j'}^{GD}\right)=0$. Let $k_j=dist(\sigma'(A_j),0)$. Then,

$$\inf\left\{dist(\sigma'(A_j),0):j\in\mathbb{N}\right\}=\inf\left\{k_j:j\in\mathbb{N}\text{ satisfies }r\left(A_j^{GD}\right)>0\right\}\geq k^{-1}>0,$$

hence the result follows.

(iv) $\Longrightarrow$ (v). Suppose that $M=\inf\limits_{j\in\mathbb{N}}dist(\sigma'(A_j),0)>0$. Let $r=\min(m,M)$ where $m=dist(\sigma'(A),0)$. By the choice of r and the fact that $\bigcap\limits_{j=0}^{\infty}\rho(A_j)=\mathbb{C}\backslash\bigcup\limits_{j=0}^{\infty}\sigma(A_j)$, we have that $\widetilde{B}(0,r)\subseteq\bigcap\limits_{j=0}^{\infty}\rho(A_j)$ and $\widetilde{B}(0,r)\subseteq\rho(A)$.

(v) $\Longrightarrow$ (vi). Suppose that there exists an $r>0$ such that $\widetilde{B}(0,r)\subseteq\rho(A)\cap\bigcap\limits_{j=0}^{\infty}\rho(A_j)$ and show that $A_j^{GD}A_j\longrightarrow A^{GD}A$ as $j\to\infty$.

If A is invertible, so A_j is too for all sufficiently large $j\in\mathbb{N}$ and $A_j^{GD}=A_j^{-1}\longrightarrow A^{-1}=A^{GD}$ as $j\to\infty$. By the continuity of multiplication in $\mathcal{B}(X)$, it follows that $A_j^{GD}A_j\longrightarrow A^{GD}A$ as $j\to\infty$.

Suppose now that $0\in iso\sigma(A)$ and P is the spectral projection of A corresponding to 0. Let:

$$\Omega_1=\left\{\lambda\in\mathbb{C}:|\lambda|<\frac{r}{3}\right\}\text{ and }\Omega_2=\left\{\lambda\in\mathbb{C}:|\lambda|>\frac{2r}{3}\right\}.$$

By hypothesis, Ω_1 and Ω_2 are open sets containing $\{0\}$ and $\sigma'(A)$ respectively, and hence $\Omega=\Omega_1\cup\Omega_2$ is an open set containing $\sigma(A)$. Define $f:\Omega\longrightarrow\mathbb{C}$ by:

$$f(\lambda)=\begin{cases}1 & \text{if}\quad \lambda\in\Omega_1\\ 0 & \text{if}\quad \lambda\in\Omega_2\end{cases}.$$

Then, f is holomorphic on Ω and $P=f(A)$. The spectral projection P_j of A_j corresponding to 0, might be 0 for several $j\in\mathbb{N}$. Also, by hypothesis, Ω_2 is an open set containing $\sigma'(A_j)$ for all $j\in\mathbb{N}$, so that $P_j=f(A_j)$ for all $j\in\mathbb{N}$. It follows that $P_j=f(A_j)\longrightarrow f(A)=P$ as $j\to\infty$ and since $A^{GD}=(A+P)^{-1}(I-P)$ and $A_j^{GD}=(A_j+P_j)^{-1}(I-P_j)$, we have $A_j^{GD}A_j\longrightarrow A^{GD}A$ as $j\to\infty$.

From the above, it is clear that (vi) is equivalent to assertion (vii).

Finally, we prove that (vii) $\Longrightarrow$ (i). Suppose that (vii) holds. Since A_j+P_j and $A+P$ are invertible in $\mathcal{B}(X)$, for all $j\in\mathbb{N}$, and $(A_j+P_j)\longrightarrow A+P$ as $j\to\infty$, it then follows that $(A_j+P_j)^{-1}\longrightarrow(A+P)^{-1}$ as $j\to\infty$. Hence, by the continuity of multiplication in $\mathcal{B}(X)$, $A_j^{GD}=(A_j+P_j)^{-1}(I-P_j)\longrightarrow(A+P)^{-1}(I-P)=A^{GD}$ as $j\to\infty$. This completes the proof. □

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