Bulletin of the *Transilvania* University of Braşov • Vol 13(62), No. 2 - 2020
Series III: Mathematics, Informatics, Physics, 451-462
https://doi.org/10.31926/but.mif.2020.13.62.2.7

ON THE SYMMETRIES OF THREE-DIMENSIONAL GENERALIZED SYMMETRIC SPACES

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Abstract

In this work we consider the three-dimensional generalized symmetric space, equipped with the left-invariant pseudo-Riemannian metric. We determine Killing vector fields and affine vectors fields. Also we obtain a full classification of Ricci, curvature and matter collineations.

2000 *Mathematics Subject Classification:* 53C50, 53B30.
Key words: Generalized symmetric spaces, left-invariant metrics, Killing vector fields, affine vector fields, pseudo-Riemannian metrics, curvature and matter collineations.

1 Introduction

Let (M, g) be a pseudo-Riemannian manifold, a Killing vector field is a vector field on (M, g) that preserves the metric. Killing fields are the infinitesimal generators of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold. More simply, the flow generates a symmetry, in the sense that moving each point on an object the same distance in the direction of the Killing vector will not distort distances on the object. Specifically, a vector field X is a Killing field if the Lie derivative with respect to X of the metric g vanishes: $\mathcal{L}_X g = 0$. In terms of the Levi-Civita connection, this is equivalent to $g(\nabla_Y X, Z) = -g(\nabla_Z X, Y)$ for all vector fields Y, Z. Therefore, it is sufficient to establish it in a preferred coordinate system in order to have it hold in all coordinate systems. The Killing fields on a manifold M form a Lie subalgebra of vector fields on M. This is the Lie algebra of the isometry group of the manifold if M is complete.

A typical use of the Killing field is to express a symmetry in General relativity (in which the geometry of spacetime as distorted by gravitational fields is viewed

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as a 4-dimensional pseudo-Riemannian manifold). In a static configuration, in which nothing changes with time, the time vector will be a Killing vector, and thus the Killing field will point in the direction of forward motion in time.

On the other hand, a vector field X tangent to (M, g) is said to be affine if it satisfies $\mathcal{L}_X \nabla = 0$, where ∇ is the Levi-Civita connection of (M, g) (or equivalently, if $[X, \nabla_Y Z] = \nabla_{[X,Y]} Z + \nabla_Y [X, Z]$ for all vector fields Y, Z) which means that the local fluxes of X given by affine maps. Obviously Killing vector field is also affine. However, the converse does not hold in general. In particular, if (M, g) is a simply connected spacetime, the existence of a non Killing affine vector field implies the existence of a second-order covariantly constant symmetric tensor, nowhere vanishing, not proportional to g. As a consequence, the holonomy group of the manifold is reducible.

A curvature (resp. Ricci) collineations is a vector field X which preserves the Riemann tensor R (resp. the Ricci tensor Ric) in the sense that, $\mathcal{L}_X R = 0$ (resp. $\mathcal{L}_X Ric = 0$), where $\mathcal{L}$ denotes the Lie derivative. The set of all smooth curvature collineations forms a Lie algebra under the Lie bracket operation, which may be infinite-dimensional. Every affine vector field is a curvature collineations.

A matter collineations is a vector field X that satisfies the condition $\mathcal{L}_X T = 0$, where T is the energy-momentum tensor given by $T = Ric - \frac{1}{2}\tau g$ with τ denotes the scalar curvature. The relation between geometry and physics may be highlighted here, as the vector field X is regarded as preserving certain physical quantities along the flow lines of X, this being true for any two observers. In connection with this, it may be shown that every Killing vector field is a matter collineations (by the Einstein field equations, with or without cosmological constant). Thus, a vector field that preserves the metric necessarily preserves the corresponding energy-momentum tensor. When the energy-momentum tensor represents a perfect fluid, every Killing vector field preserves the energy density, pressure and the fluid flow vector field. When the energy-momentum tensor represents an electromagnetic field, a Killing vector field does not necessarily preserve the electric and magnetic fields.

More general, a collineations or a symmetry of a tensor field S on a pseudo-Riemannian manifold (M, g) is a one-parameter group of diffeomorphisms of (M, g), which leaves S invariant. Therefore, each symmetry corresponds to a vector field X which satisfies $\mathcal{L}_X S = 0$. Symmetries of the metric tensor g which correspond to the Killing vector fields. Symmetries of the Levi-Civita connection ∇ which correspond to the affine vector fields. Since symmetries are more significant from physical aspects, they have been studied on several kinds of space-times (see [11, 12], [10, 8, 9],[14]).

The aim of this paper, is to study symmetries of the three-dimensional generalized symmetric space, equipped with a left-invariant pseudo-Riemannian metric. The paper is organized in the following way. In Section 3, we shall report some basic information about three-dimensional generalized space and its left-invariant pseudo-Riemannian metrics in global coordinates, we shall describe their Levi-Civita connection, the curvature and the Ricci tensor. In Section 4, affine, homothetic and Killing vector fields of three-dimensional generalized space are characterized via a system of partial differential equations. Then, in Section 5 and 6, we shall respectively classify Ricci, curvature and matter collineations on the low-three-dimensional generalized space equipped with Lorentzian left-invariant metric.

2 Preliminaries

Let (M, g) be a connected pseudo-Riemannian and x a point of M. A symmetry at x is an isometry s_x of M, having x as isolated fixed point. When (M, g) is a symmetric space, each point x admits a symmetry s_x reversing geodesics through the point. Hence, s_x is involutive for all x. This property was generalized by A.J. Ledger, who defined a regular $s-$structure as a family $\{s_x : x \in M\}$ of symmetries of (M, g) satisfying

$$s_x \circ s_y = s_z \circ s_x, \;\; z = s_x(y),$$

for all x, y of M. The of an $s-$structure is the least integer $k \geq 2$, such that $(s_x)^k = id_M$ for all x (it may happen that $k = \infty$). A generalized symmetric space is a connected pseudo-Riemannian (M, g) admitting a regular $s-$structure. The order of a generalized symmetric space is the minimum of all integers $k \geq 2$ such that M admits a regular $s-$structure of order k.

Following [13], any proper (that is, non-symmetric) three-dimensional generalized symmetric space (M, g) is of order 4. Moreover, it is given by the space $\mathbb{R}^3(x, y, t)$ with the pseudo-Riemannian metric

$$g = \varepsilon(e^{2t}dx^2 + e^{-2t}dy^2) + \lambda dt^2, \tag{1}$$

where $\varepsilon = \pm 1$ and $\lambda \neq 0$ is a real constant. Depending on the values of ε and λ, these metrics attain any possible signature: $(3, 0)$, $(0, 3)$, $(2, 1)$, $(1, 2)$.

3 Connection and curvature of three-dimensional generalized symmetric spaces

Let (M, g) be a three-dimensional generalized symmetric space which is the space $\mathbb{R}^3(x, y, t)$, and denote by ∇, R and Ric the Levi-Civita connection, the Riemann curvature tensor and the Ricci tensor of M, respectively. We used the coordinates $(x_1, x_2, x_3) = (x, y, t)$ and the corresponding basis of coordinate vector fields basis $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}$ by $\{\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\}$.

The Levi-Civita connection ∇ of (M, g) with respect to the coordinates vector fields $\{\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\}$.

The non-vanishing components of the Levi-Civita connection ∇ of (M, g) are given by

$$\begin{cases} \nabla_{\partial_{x_1}}\partial_{x_1} = -\frac{\varepsilon}{\lambda}e^{2x_3}\partial_{x_3}, \;\; \nabla_{\partial_{x_1}}\partial_{x_3} = \partial_{x_1}, \\ \nabla_{\partial_{x_2}}\partial_{x_2} = \frac{\varepsilon}{\lambda}e^{-2x_3}\partial_{x_3}, \;\; \nabla_{\partial_{x_2}}\partial_{x_3} = -\partial_{x_2}, \\ \nabla_{\partial_{x_3}}\partial_{x_1} = \partial_{x_1}, \qquad \nabla_{\partial_{x_3}}\partial_{x_2} = -\partial_{x_2}. \end{cases} \tag{2}$$

The curvature tensor R taken with the sign convention

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The non-vanishing curvature tensor R components are computed as

$$\left\{ \begin{array}{l} R(\partial_{x_1}, \partial_{x_2})\partial_{x_1} = -\frac{\varepsilon}{\lambda} e^{2x_3} \partial_{x_2}, \;\; R(\partial_{x_1}, \partial_{x_2})\partial_{x_2} = \frac{\varepsilon}{\lambda} e^{-2x_3} \partial_{x_1}, \\ R(\partial_{x_1}, \partial_{x_3})\partial_{x_1} = \frac{\varepsilon}{\lambda} e^{2x_3} \partial_{x_3}, \;\; R(\partial_{x_1}, \partial_{x_3})\partial_{x_3} = -\partial_{x_1}, \\ R(\partial_{x_2}, \partial_{x_3})\partial_{x_2} = \frac{\varepsilon}{\lambda} e^{-2x_3} \partial_{x_3}, \;\; R(\partial_{x_2}, \partial_{x_3})\partial_{x_3} = -\partial_{x_2}. \end{array} \right. \tag{3}$$

The Ricci curvature Ric is defined by

$$Ric(X,Y) = trace\{Z \to R(Z,X)Y\}. \tag{4}$$

The components $\{Ric_{ij}\}$ of Ric are defined by

$$Ric(\partial_{x_i}, \partial_{x_j}) = Ric_{ij} = \sum_{k=1}^{3} g(\partial_{x_k}, \partial_{x_k}) g\left(R(\partial_{x_k}, \partial_{x_i})\partial_{x_j}, \partial_{x_k}\right). \tag{5}$$

The non-vanishing components $\{Ric_{ij}\}$ are computed as

$$Ric_{\partial_{x_3}, \partial_{x_3}} = -2. \tag{6}$$

The scalar curvature τ of (M,g) is constant and we have

$$\tau = trRic = \sum_{i=1}^{3} g(\partial_{x_i}, \partial_{x_i}) Ric(\partial_{x_i}, \partial_{x_i}) = -\frac{2}{\lambda}. \tag{7}$$

4 Affine, homothetic and Killing vector fields

We first classify affine, homothetic and Killing vector fields of the three-dimensional generalized space. The classifications we obtain are summarized in the following theorem.

Theorem 1. *Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3}$ be an arbitrary vector field on the three-dimensional generalized space.*

1. *X is a Killing vector field if and only if*

$$\left\{ \begin{array}{l} f_1 = -c_1 x_1 + c_2, \\ f_2 = c_1 x_2 + c_3, \\ f_3 = c_1, \quad c_i \in \mathbb{R}. \end{array} \right.$$

2. *There are no homothetic non-Killing vector fields.*

3. *X is a affine vector field if and only if*

$$\left\{ \begin{array}{l} f_1 = -c_1 x_1 + c_2, \\ f_2 = c_1 x_2 + c_3, \\ f_3 = c_1, \; c_i \in \mathbb{R}. \end{array} \right.$$

Proof. Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3}$ denote an arbitrary vector field on the three-dimensional generalized space, for some arbitrary smooth functions f_1, f_2, f_3 on the three-dimensional generalized space. Starting from (1), a direct calculation yields the following description of the Lie derivative of the metric tensor g :

$$\begin{cases} (\mathcal{L}_X g)(\partial_{x_1}, \partial_{x_1}) = 2\varepsilon e^{2x_3}(\partial_{x_1} f_1 + f_3), \\ (\mathcal{L}_X g)(\partial_{x_1}, \partial_{x_2}) = \varepsilon(e^{-2x_3}\partial_{x_1} f_2 + e^{2x_3}\partial_{x_2} f_1), \\ (\mathcal{L}_X g)(\partial_{x_1}, \partial_{x_3}) = \varepsilon e^{2x_3}\partial_{x_3} f_1 + \lambda\partial_{x_1} f_3, \\ (\mathcal{L}_X g)(\partial_{x_2}, \partial_{x_2}) = 2\varepsilon e^{-2x_3}(\partial_{x_2} f_2 - f_3), \\ (\mathcal{L}_X g)(\partial_{x_2}, \partial_{x_3}) = \varepsilon e^{-2x_3}\partial_{x_3} f_2 + \lambda\partial_{x_2} f_3, \\ (\mathcal{L}_X g)(\partial_{x_3}, \partial_{x_3}) = 2\lambda\partial_{x_3} f_3. \end{cases} \tag{8}$$

In order to determine the homothetic and Killing vector fields, we then must solve the system of PDEs obtained require that $\mathcal{L}_X g = \eta g$, for some real constant η. The solutions of the corresponding system of PDEs give us the homothetic and Killing (in the case $\eta = 0$) vector fields of the three-dimensional generalized symmetric space,

$$\begin{cases} \partial_{x_1} f_1 + f_3 = \frac{\eta}{2}, \\ e^{-2x_3}\partial_{x_1} f_2 + e^{2x_3}\partial_{x_2} f_1 = 0, \\ \varepsilon e^{2x_3}\partial_{x_3} f_1 + \lambda\partial_{x_1} f_3 = 0, \\ \partial_{x_2} f_2 - f_3 = \frac{\eta}{2}, \\ \varepsilon e^{-2x_3}\partial_{x_3} f_2 + \lambda\partial_{x_2} f_3 = 0, \\ \partial_{x_3} f_3 = \frac{\eta}{2}. \end{cases} \tag{9}$$

We first integrate the last equation in 9 and we get

$$f_3 = \frac{\eta}{2}x_3 + F_3(x_1, x_2),$$

where F_3 is an arbitrary smooth function.
Next we replace f_3 into the third and fifth equations of 9, they respectively become

$$\varepsilon e^{2x_3}\partial_{x_3} f_1 + \lambda\partial_{x_1} F_3 = 0,$$
$$\varepsilon e^{-2x_3}\partial_{x_3} f_2 + \lambda\partial_{x_2} F_3 = 0,$$

which, integrated, yield

$$\begin{aligned} f_1 &= \tfrac{\lambda}{2\varepsilon}e^{-2x_3}\partial_{x_1} F_3 + F_1(x_1, x_2), \\ f_2 &= \tfrac{\lambda}{2\varepsilon}e^{2x_3}\partial_{x_2} F_3 + F_2(x_1, x_2), \end{aligned} \tag{10}$$

for some smooth functions F_1, F_2. Substituting the above expressions of f_1 and f_2 into the second equation of 9, it now gives

$$e^{-2x_3}\partial_{x_1} F_2 + e^{2x_3}\partial_{x_2} F_1 = 0. \tag{11}$$

Since the above equation 11 must hold for all values of x_3, yields $\partial_{x_2} F_1 = \partial_{x_1} F_2 = 0$, that is,

$$F_1 = F_1(x_1), \quad F_2 = F_2(x_2).$$

System 9 reduces to its first and fourth equations. The first and fourth equations in 9 now reads

$$\begin{array}{c}\frac{\lambda}{2\varepsilon}e^{-2x_3}\partial^2_{x_1}F_3 + F_1'(x_1) + \frac{\eta}{2}x_3 + F_3 = 0,\\ \frac{\lambda}{2\varepsilon}e^{2x_3}\partial^2_{x_2}F_3 + F_2'(x_2) - \frac{\eta}{2}x_3 - F_3 = 0.\end{array} \tag{12}$$

Since the above equation 12 holds for all values of x_3, in particular it implies that $F_3 = -F_1'(x_1) = F_2'(x_2) = c_1$, where c_1 is a real constant. Next we replace F_3 in 12 and 10 we obtain

$$\begin{cases} f_1 = -c_1x_1 + c_2,\\ f_2 = c_1x_2 + c_3,\\ f_3 = c_1,\\ \eta = 0,\end{cases} \tag{13}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. And as $\eta = 0$ then there are no homothetic non-Killing vectors fields in three-dimensional generalized symmetric space.

To determine the affine Killing vector fields, we need to calculate the Lie derivative of the Levi-Civita connection ∇. Staring from (2),we find the following possibly non-vanishing components:

$$\begin{cases} (\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_1}) = (\partial^2_{x_1}f_1 + 2\partial_{x_1}f_3 + \frac{\varepsilon}{\lambda}e^{2x_3}\partial_{x_3}f_1)\partial_{x_1} + (\partial^2_{x_1}f_2 + \frac{\varepsilon}{\lambda}e^{2x_3}\partial_{x_3}f_2)\partial_{x_2}\\ +(\partial^2_{x_1}f_3 + \frac{\varepsilon}{\lambda}e^{2x_3}(\partial_{x_3}f_3 - 2f_3 - 2\partial_{x_1}f_1))\partial_{x_3},\\ (\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_2}) = (\partial_{x_1}\partial_{x_2}f_1 + \partial_{x_2}f_3)\partial_{x_1} + (\partial_{x_1}\partial_{x_2}f_2 - \partial_{x_1}f_3)\partial_{x_2}\\ +(\partial_{x_1}\partial_{x_2}f_3 + \frac{\varepsilon}{\lambda}e^{-2x_3}\partial_{x_1}f_2 - \frac{\varepsilon}{\lambda}e^{2x_3}\partial_{x_2}f_1)\partial_{x_3},\\ (\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_3}) = (\partial_{x_1}\partial_{x_3}f_1 + \partial_{x_3}f_3)\partial_{x_1} + (\partial_{x_1}\partial_{x_3}f_2 - 2\partial_{x_1}f_2)\partial_{x_2}\\ +(\partial_{x_1}\partial_{x_3}f_3 - \partial_{x_1}f_3 - \frac{\varepsilon}{\lambda}e^{2x_3}\partial_{x_3}f_1)\partial_{x_3},\\ (\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_2}) = (\partial^2_{x_2}f_1 - \frac{\varepsilon}{\lambda}e^{-2x_3}\partial_{x_3}f_1)\partial_{x_1} + (\partial^2_{x_2}f_2 - 2\partial_{x_2}f_3 - \frac{\varepsilon}{\lambda}e^{-2x_3}\partial_{x_3}f_2)\partial_{x_2}\\ +(\partial^2_{x_2}f_3 + \frac{\varepsilon}{\lambda}e^{-2x_3}(-2f_3 + 2\partial_{x_2}f_2 - \partial_{x_3}f_3))\partial_{x_3},\\ (\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_3}) = (\partial_{x_2}\partial_{x_3}f_1 + 2\partial_{x_2}f_1)\partial_{x_1} + (\partial_{x_2}\partial_{x_3}f_2 - \partial_{x_3}f_3)\partial_{x_2}\\ +(\partial_{x_2}\partial_{x_3}f_3 + \partial_{x_2}f_3 + \frac{\varepsilon}{\lambda}e^{-2x_3}\partial_{x_3}f_2)\partial_{x_3},\\ (\mathcal{L}_X\nabla)(\partial_{x_3},\partial_{x_3}) = (\partial^2_{x_3}f_1 + 2\partial_{x_3}f_1)\partial_{x_1} + (\partial^2_{x_3}f_2 - 2\partial_{x_3}f_2)\partial_{x_2} + \partial^2_{x_3}f_3\partial_{x_3}.\end{cases} \tag{14}$$

In order to determine the affine vector fields, we then must solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative are equal to zero.

From equation $dx3[(\mathcal{L}_X\nabla)(\partial_{x_3},\partial_{x_3})] = 0$ it fellows that

$$f_3 = A(x_1,x_2)x_3 + \overline{A}(x_1,x_2),$$

where A and $\overline{A}$ are real valuables smooth functions.

We then replace f_3 in equations given by $dx3[(\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_2})] = 0$ and $dx3[(\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_2})] = 0$, we get

$$\left\{ \begin{array}{l} \partial_{x_2} f_2 = -\frac{\lambda}{\varepsilon} e^{2x_3}[\partial^2_{x_2} A x_3 + \partial^2_{x_2}\overline{A}] + Ax_3 + \overline{A} + \frac{1}{2}A, \\ \partial_{x_1} f_1 = -\frac{\lambda}{\varepsilon} e^{-2x_3}[\partial^2_{x_1} A x_3 + \partial^2_{x_1}\overline{A}] - Ax_3 - \overline{A} - \frac{1}{2}A. \end{array} \right. \tag{15}$$

and

$$\left\{ \begin{array}{l} \partial_{x_3}\partial_{x_2} f_2 = -\frac{\lambda}{2\varepsilon} e^{2x_3}[2\partial^2_{x_2} A x_3 + 2\partial^2_{x_2}\overline{A} + \partial^2_{x_2} A] + A, \\ \partial_{x_3}\partial_{x_1} f_1 = \frac{\lambda}{2\varepsilon} e^{-2x_3}[-2\partial^2_{x_1} A x_3 - 2\partial^2_{x_1}\overline{A} + \partial^2_{x_1} A] - A. \end{array} \right.$$

Next we replace $f_3, \partial_{x_2} f_2, \partial_{x_1} f_1, \partial_{x_3}\partial_{x_2} f_2$ and $\partial_{x_3}\partial_{x_1} f_1$ in equations given by $dx2[(\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_3})] = 0$ and $dx1[(\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_3})] = 0$, we get

$$\left\{ \begin{array}{l} \partial^2_{x_2} A = \partial^2_{x_1} A = 0, \\ \partial^2_{x_2}\overline{A} = \partial^2_{x_1}\overline{A} = 0. \end{array} \right. \tag{16}$$

Which together with equations given by $dx1[(\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_2})] = 0$ and $dx2[(\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_2})] = 0$, and using equation 16, we get

$$\left\{ \begin{array}{l} \partial_{x_1} A = 0, \\ \partial_{x_2} A = 0. \end{array} \right. \tag{17}$$

Since from the above equations 16 , 17 and by integrating equations 15, we conclude that

$$\left\{ \begin{array}{l} f_1 = -cx_1x_3 - \frac{1}{2}cx_1 - \frac{1}{2}c_1x_1^2 - c_2x_1x_2 - \frac{1}{2}c_3x_1^2x_2 - c_4x_1 + c_5, \\ f_2 = cx_2x_3 + \frac{1}{2}cx_2 + c_1x_1x_2 + \frac{1}{2}c_2x_2^2 + \frac{1}{2}c_3x_1x_2^2 + c_4x_2 + c_6, \\ f_3 = cx_3 + c_1x_1 + c_2x_2 + c_3x_1x_2 + c_4, \end{array} \right. \tag{18}$$

where $c, c_i \in \mathbb{R}$.

Replacing f_1 into equations $dx1[(\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_2})] = 0$ and $dx1[(\mathcal{L}_X\nabla)(\partial_{x_2},\partial_{x_3})] = 0$, we get

$$c = c_2 = c_3 = 0.$$

And similarly replacing f_2 into equations $dx2[(\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_1})] = 0$ and $dx2[(\mathcal{L}_X\nabla)(\partial_{x_1},\partial_{x_3})] = 0$, we get

$$c = c_1 = c_3 = 0.$$

Thus the final solution of PDEs system obtained requiring that all the coefficients in the above Lie derivative of the Levi-Civita connection ∇ are equal to zero are given by

$$\left\{ \begin{array}{l} f_1 = -c_4x_1 + c_5, \\ f_2 = c_4x_2 + c_6, \\ f_3 = c_4, \quad c_i \in \mathbb{R}. \end{array} \right. \tag{19}$$

$\square$

5 Ricci and curvature collineations

In this section we give a full classification of Ricci and curvature collineations vector fields of the three-dimensional generalized space. The classifications we obtain are summarized in the following theorem.

Theorem 2. *Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3}$ be an arbitrary vector field on the three-dimensional generalized space.*

1. *X is a Ricci collineation if and only if*

$$X = f_1\partial_{x_1} + f_2\partial_{x_2} + c\partial_{x_3},$$

where $c \in \mathbb{R}$, and f_1, f_2 are any smooth functions on the three-dimensional generalized space.

2. *X is a curvature collineation vector field if and only if*

$$\begin{cases} f_1 = -c_1x_1 + c_2, \\ f_2 = c_1x_2 + c_3, \\ f_3 = c_1, \end{cases}$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Proof. Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3}$ denote an arbitrary vector field on the three-dimensional generalized space, for some arbitrary smooth functions f_1, f_2, f_3 on the three-dimensional generalized space. Starting from (6), a direct calculation yields the following description of the Lie derivative of the Ricci tensor Ric in the direction of X given by:

$$\begin{cases} (\mathcal{L}_X Ric)(\partial_{x_1}, \partial_{x_1}) = 0, \\ (\mathcal{L}_X Ric)(\partial_{x_1}, \partial_{x_2}) = 0, \\ (\mathcal{L}_X Ric)(\partial_{x_1}, \partial_{x_3}) = -2\partial_{x_1}f_3, \\ (\mathcal{L}_X Ric)(\partial_{x_2}, \partial_{x_2}) = 0, \\ (\mathcal{L}_X Ric)(\partial_{x_2}, \partial_{x_3}) = -2\partial_{x_2}f_3, \\ (\mathcal{L}_X Ric)(\partial_{x_3}, \partial_{x_3}) = -4\partial_{x_3}f_3. \end{cases} \tag{20}$$

Ricci collineations are then calculated by solving the system of PDEs obtained by requiring that all the above coefficients of $\mathcal{L}_X Ric$ vanish.
From equations given by $(\mathcal{L}_X Ric)(\partial_{x_1}, \partial_{x_3}) = 0$, $(\mathcal{L}_X Ric)(\partial_{x_2}, \partial_{x_3}) = 0$ and $(\mathcal{L}_X Ric)(\partial_{x_3}, \partial_{x_3}) = 0$ we get that

$$f_3 = c,$$

where $c \in \mathbb{R}$ and f_1, f_2 are any smooth functions on the three-dimensional generalized space.

To determine the curvature collineations, we need to calculate the Lie derivative of the curvature tensor R in the direction of X. Staring from (3),we find the following possibly non-vanishing components:

$$\begin{cases} (\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_2}, \partial_{x_1}) = \frac{\varepsilon}{\lambda}[e^{2x_3}\partial_{x_2} f_1 + e^{-2x_3}\partial_{x_1} f_2]\partial_{x_1} - \frac{\varepsilon}{\lambda} e^{2x_3}[2f_3 + 2\partial_{x_1} f_1]\partial_{x_2} \\ +2\frac{\varepsilon}{\lambda} e^{2x_3}\partial_{x_2} f_3 \partial_{x_3}, \\ (\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_2}, \partial_{x_2}) = 2\frac{\varepsilon}{\lambda} e^{-2x_3}[-f_3 + \partial_{x_2} f_2]\partial_{x_1} - \frac{\varepsilon}{\lambda}[e^{2x_3}\partial_{x_2} f_1 + e^{-2x_3}\partial_{x_1} f_2]\partial_{x_2} \\ -2\frac{\varepsilon}{\lambda} e^{-2x_3}\partial_{x_1} f_3 \partial_{x_3}, \\ (\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) = [\frac{\varepsilon}{\lambda} e^{-2x_3}\partial_{x_3} f_2 - \partial_{x_2} f_3]\partial_{x_1} - [\partial_{x_1} f_3 + \frac{\varepsilon}{\lambda} e^{2x_3}\partial_{x_3} f_1]\partial_{x_2}, \\ (\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_3}, \partial_{x_1}) = -[\frac{\varepsilon}{\lambda} e^{2x_3}\partial_{x_3} f_1 + \partial_{x_1} f_3]\partial_{x_1} - 2\frac{\varepsilon}{\lambda} e^{2x_3}\partial_{x_3} f_2 \partial_{x_2} \\ +2\frac{\varepsilon}{\lambda} e^{2x_3}[f_3 + \partial_{x_1} f_1]\partial_{x_3}, \\ (\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_3}, \partial_{x_2}) = [\frac{\varepsilon}{\lambda} e^{-2x_3}\partial_{x_3} f_2 - \partial_{x_2} f_3]\partial_{x_1} + \frac{\varepsilon}{\lambda}[e^{2x_3}\partial_{x_2} f_1 + e^{-2x_3}\partial_{x_1} f_2]\partial_{x_3}, \\ (\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_3}, \partial_{x_3}) = -2\partial_{x_3} f_3 \partial_{x_1} + [\frac{\varepsilon}{\lambda} e^{2x_3}\partial_{x_3} f_1 + \partial_{x_1} f_3]\partial_{x_3}, \\ (\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_3}, \partial_{x_1}) = [\frac{\varepsilon}{\lambda} e^{2x_3}\partial_{x_3} f_1 - \partial_{x_1} f_3]\partial_{x_2} + \frac{\varepsilon}{\lambda}[e^{2x_3}\partial_{x_2} f_1 - e^{-2x_3}\partial_{x_1} f_2]\partial_{x_3}, \\ (\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_3}, \partial_{x_2}) = -2\frac{\varepsilon}{\lambda} e^{-2x_3}\partial_{x_3} f_1 \partial_{x_1} - [\frac{\varepsilon}{\lambda} e^{-2x_3}\partial_{x_3} f_2 + \partial_{x_2} f_3]\partial_{x_2} \\ +2\frac{\varepsilon}{\lambda} e^{-2x_3}[-f_3 + \partial_{x_2} f_2]\partial_{x_3}, \\ (\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_3}, \partial_{x_3}) = -2\partial_{x_3} f_3 \partial_{x_2} + [\frac{\varepsilon}{\lambda} e^{-2x_3}\partial_{x_3} f_2 + \partial_{x_2} f_3]\partial_{x_3}, \end{cases} \tag{21}$$

In order to determine the curvature collineation vector fields, we then must solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the curvature tensor in the direction of X are equal to zero. Which together with equations

$$\begin{aligned} &dx3[(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_2}, \partial_{x_2})] = 0, && dx3[(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_2}, \partial_{x_1})] = 0, \\ &dx3[(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_3}, \partial_{x_1})] = 0, && dx2[(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_3}, \partial_{x_1})] = 0, \\ &dx1[(\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_3}, \partial_{x_2})] = 0, && dx3[(\mathcal{L}_X R)(\partial_{x_2}, \partial_{x_3}, \partial_{x_2})] = 0 \\ &dx1[(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_3}, \partial_{x_3})] = 0, \end{aligned}$$

and we obtain after integration

$$\begin{cases} f_1 = -c_1 x_1 + B(x_2), \\ f_2 = c_1 x_2 + \overline{B}(x_1), \\ f_3 = c_1, \end{cases} \tag{22}$$

where $c_1 \in \mathbb{R}$ and $B, \overline{B}$ are smooth functions.

Next, we replace f_1 and f_2 in equation $dx1[(\mathcal{L}_X R)(\partial_{x_1}, \partial_{x_2}, \partial_{x_1})] = 0$, we get

$$e^{2x_3} B'(x_2) + e^{-2x_3}\overline{B}'(x_1) = 0. \tag{23}$$

Since the above equation (23) holds for all values of x_3, in particular implies that

$$\begin{cases} B(x_2) = c_2, \\ \overline{B}(x_1) = c_3, c_2, c_3 \in \mathbb{R}. \end{cases} \tag{24}$$

The final solution of the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the curvature tensor in the direction of X are equal to zero are given by

$$\begin{cases} f_1 = -c_1 x_1 + c_2, \\ f_2 = c_1 x_2 + c_3, \\ f_3 = c_1, \end{cases}$$

where $c_1, c_2, c_3 \in \mathbb{R}$. □

6 Matter collineations

In this section we classify matter collineation vector fields of the three-dimensional generalized space. The classifications we obtain are summarized in the following theorem.

Theorem 3. *Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3}$ be an arbitrary vector field on the three-dimensional generalized space.*

X is a matter collineation vector field if and only if

$$\begin{cases} f_1 = -c_1x_1 + c_2, \\ f_2 = c_1x_2 + c_3, \\ f_3 = c_1, \end{cases}$$

Where $c_1, c_2, c_3 \in \mathbb{R}$.

Proof. Let $X = f_1\partial_{x_1} + f_2\partial_{x_2} + f_3\partial_{x_3}$ denote an arbitrary vector field on the three-dimensional generalized space, for some arbitrary smooth functions f_1, f_2, f_3 on the three-dimensional generalized space. Starting from equations (1),(6) and (7), a direct calculation yields in the three-dimensional generalized space, with respect to the basis $\{\partial_{x_i}\}_{i\in\{1,2,3\}}$ the tensor $T = Ric - \frac{\tau}{2}g$ is described by:

$$T = \begin{pmatrix} \frac{\varepsilon}{\lambda}e^{2x_3} & 0 & 0 \\ 0 & \frac{\varepsilon}{\lambda}e^{-2x_3} & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{25}$$

When we compute the Lie derivative of T with respect to X and we find:

$$\begin{cases} (\mathcal{L}_XT)(\partial_{x_1}, \partial_{x_1}) = 2\frac{\varepsilon}{\lambda}e^{2x_3}[f_3 + \partial_{x_1}f_1], \\ (\mathcal{L}_XT)(\partial_{x_1}, \partial_{x_2}) = \frac{\varepsilon}{\lambda}[e^{2x_3}\partial_{x_2}f_1 + e^{-2x_3}\partial_{x_1}f_2], \\ (\mathcal{L}_XT)(\partial_{x_1}, \partial_{x_3}) = \frac{\varepsilon}{\lambda}e^{2x_3}\partial_{x_3}f_1 - \partial_{x_1}f_3, \\ (\mathcal{L}_XT)(\partial_{x_2}, \partial_{x_2}) = 2\frac{\varepsilon}{\lambda}e^{-2x_3}[-f_3 + \partial_{x_2}f_2], \\ (\mathcal{L}_XT)(\partial_{x_2}, \partial_{x_3}) = \frac{\varepsilon}{\lambda}e^{-2x_3}\partial_{x_3}f_2 - \partial_{x_2}f_3, \\ (\mathcal{L}_XT)(\partial_{x_3}, \partial_{x_3}) = -2\partial_{x_3}f_3. \end{cases} \tag{26}$$

To determine matter collineations we solve the system of PDEs obtained requiring that all the coefficients in the above Lie derivative of the tensor field T in the direction of X are equal to zero (i.e. $\mathcal{L}_XT = 0$), we get that all solutions coincide with Killing vector fields of the three-dimensional generalized space.

□

Acknowledgements. The author would like to thank the Referees for all helpful comments and suggestions that have improved the quality of our initial manuscript. The author was supported by The National Agency Scientific Research (DGRSDT).

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