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# MEROMORPHIC SOLUTIONS OF HIGHER ORDER NON-HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

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**Abstract**

In this paper, we investigate the growth of meromorphic solutions of non-homogeneous linear difference equation

$$A_n(z)f(z+c_n)+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=A_{n+1}(z),$$

where $A_{n+1}(z),\cdots,A_0(z)$ are (entire) or meromorphic functions and $c_j$ $(1,\cdots,n)$ are non-zero distinct complex numbers. Under some conditions on the (lower) order and the (lower) type of the coefficients, we obtain estimates on the lower bound of the order of meromorphic solutions of the above equation. We extend early results due to Luo and Zheng.

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*Key words:* linear difference equation, meromorphic solution, order, type, lower order, lower type.

## 1 Introduction and statement of main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions [6, 9, 17].

In the following, we recall some fundamental definitions which are used later.

**Definition 1.** *([6, 9, 17]) Let $f$ be a meromorphic function. Then, the order $\rho(f)$ of $f$ is defined by*

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r},$$

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*where $T(r,f)$ is the characteristic function of Nevanlinna (see [6, 9, 17]). If $f$ is entire function, then the order of $f$ is defined as*

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r}=\limsup_{r\to+\infty}\frac{\log\log M(r,f)}{\log r},$$

*where* $M(r,f)=\max_{|z|=r}|f(z)|$.

**Definition 2.** *([6, 9, 17]) Let $f$ be a meromorphic function. Then, the lower order $\mu(f)$ of $f$ is defined by*

$$\mu(f)=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r}.$$

*If $f$ is entire function, then the lower order of $f$ is defined as*

$$\mu(f)=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r}=\liminf_{r\to+\infty}\frac{\log\log M(r,f)}{\log r}.$$

**Definition 3.** *([2, 9]) Let $f$ be a meromorphic function of order $(0<\rho(f)<\infty)$. Then, the type $\tau(f)$ of $f$ is defined by*

$$\tau(f)=\limsup_{r\to+\infty}\frac{T(r,f)}{r^{\rho(f)}}.$$

*If $f$ is entire function, then the type $\tau_M(f)$ of $f$ with order $(0<\rho(f)<\infty)$ is defined as*

$$\tau_M(f)=\limsup_{r\to+\infty}\frac{\log M(r,f)}{r^{\rho(f)}}.$$

**Definition 4.** *([4, 10]) The lower type $\underline{\tau}(f)$ of a meromorphic function $f$ with lower order $(0<\mu(f)<\infty)$ is defined by*

$$\underline{\tau}(f)=\liminf_{r\to+\infty}\frac{T(r,f)}{r^{\mu(f)}}.$$

*If $f$ is entire function, then the lower type $\underline{\tau}_M(f)$ of $f$ with lower order $(0<\mu(f)<\infty)$ is defined as*

$$\underline{\tau}_M(f)=\liminf_{r\to+\infty}\frac{\log M(r,f)}{r^{\mu(f)}}.$$

**Definition 5.** *([11]) Let $f$ be a meromorphic function. Then, the exponent of convergence of pole-sequence $\lambda\left(\frac{1}{f}\right)$ of $f$ is defined by*

$$\lambda\left(\frac{1}{f}\right)=\limsup_{r\to+\infty}\frac{\log N(r,f)}{\log r},$$

*where $N(r,f)$ is the integrated counting function of poles of $f$ in $\{z:|z|\le r\}$.*

**Definition 6.** *([9, 17]) For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of $a$ with respect to a meromorphic function $f$ is defined as*

$$\delta\left(a,f\right)=\liminf_{r\to+\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)}=1-\limsup_{r\to+\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{T\left(r,f\right)},\ a\neq\infty,$$

$$\delta\left(\infty,f\right)=\liminf_{r\to+\infty}\frac{m\left(r,f\right)}{T\left(r,f\right)}=1-\limsup_{r\to+\infty}\frac{N\left(r,f\right)}{T\left(r,f\right)}.$$

Recently, many articles focused on complex difference equations [1, 4, 13, 14]. The back-ground for these studies lies in the recent difference counterparts of Nevanlinna theory. The key result here is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [7, 8] and Chiang-Feng [5], independently. Several authors have investigated the properties of meromorphic solutions of complex linear difference equation

$$A_n(z)f(z+c_n)+A_{n-1}(z)f(z+c_{n-1})+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=0 \quad (1)$$

when one coefficient has maximal order or among coefficients having the maximal order, exactly one has its type strictly greater than others and achieved some important results (see e.g. [5, 12, 18, 19]). Very recently [15], Luo and Zheng have studied the growth of meromorphic solutions of (1) when more than one coefficient has maximal lower order and the lower type strictly greater than the type of other coefficients, and obtained the following Theorems 1 and 2.

**Theorem 1.** *([15]) Let $A_j(z)(j=0,...,n)$ be entire functions, and let $k,l \in \{0,1,...,n\}$. If the following three assumptions hold simultaneously:*

(1) $\max\{\mu(A_k),\rho(A_j),j\neq k,l\}=\rho\leq\mu(A_l)<\infty,\mu(A_l)>0$;

(2) $\underline{\tau}_M(A_l)>\underline{\tau}_M(A_k)$, *when* $\mu(A_l)=\mu(A_k)$;

(3) $\max\{\tau_M(A_j):\rho(A_j)=\mu(A_l),j\neq k,l\}=\tau_1<\underline{\tau}_M(A_l)$, *when* $\mu(A_l)=\max\{\rho(A_j),j\neq k,l\}$.

*Then every meromorphic solution $f\not\equiv 0$ of (1) satisfies $\rho(f)\geq\mu(A_l)+1$.*

**Theorem 2.** *([15]) Let $A_j(z)(j=0,...,n)$ be meromorphic functions, and let $k,l\in\{0,1,...,n\}$. If the following four assumptions hold simultaneously:*

(1) $\max\{\mu(A_k),\rho(A_j),j\neq k,l\}=\rho\leq\mu(A_l)<\infty,\mu(A_l)>0$;

(2) $\delta\underline{\tau}(A_l)>\underline{\tau}(A_k)$, *when* $\mu(A_l)=\mu(A_k)$;

(3) $\max\{\tau(A_j):\rho(A_j)=\mu(A_l),j\neq k,l\}=\tau_1<\delta\underline{\tau}(A_l)$, *when* $\mu(A_l)=\max\{\rho(A_j),j\neq k,l\}$;

(4) $\delta\left(\infty,A_l\right)=\delta>0$.

*Then every meromorphic solution* $f \not\equiv 0$ *of* (1) *satisfies* $\rho(f) \geq \mu(A_l)$.

The purpose of this paper is to extend the results of Theorems 1 and 2 for the complex non-homogeneous linear difference equation

$$A_n(z)f(z+c_n)+A_{n-1}(z)f(z+c_{n-1})+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z) = A_{n+1}(z). \tag{2}$$

We mainly obtain the following two results.

**Theorem 3.** *Let* $A_j(z)(j = 0, ..., n+1)$ *be entire functions, and let* $k, l \in \{0, 1, ..., n+1\}$. *If the following three assumptions hold simultaneously:*

(1) $\max\{\mu(A_k), \rho(A_j), j \neq k, l\} = \rho \leq \mu(A_l) < \infty$, $\mu(A_l) > 0$;

(2) $\underline{\tau}_M(A_l) > \underline{\tau}_M(A_k)$, *when* $\mu(A_l) = \mu(A_k)$;

(3) $\max\{\tau_M(A_j) : \rho(A_j) = \mu(A_l),\ j \neq k, l\} = \tau_1 < \underline{\tau}_M(A_l)$, *when* $\mu(A_l) = \max\{\rho(A_j),\ j \neq k, l\}$.

*Then every meromorphic solution* $f$ *of* (2) *satisfies* $\rho(f) \geq \mu(A_l)$ *if* $A_{n+1} \not\equiv 0$. *Furthermore, if* $A_{n+1} \equiv 0$, *then every meromorphic solution* $f \not\equiv 0$ *of* (2) *satisfies* $\rho(f) \geq \mu(A_l) + 1$.

**Theorem 4.** *Let* $A_j(z)(j = 0, ..., n+1)$ *be meromorphic functions, and let* $k, l \in \{0, 1, ..., n+1\}$. *If the following five assumptions hold simultaneously:*

(1) $\max\{\mu(A_k), \rho(A_j),\ j \neq k, l\} = \rho \leq \mu(A_l) < \infty, \mu(A_l) > 0$;

(2) $\underline{\tau}(A_l) > \underline{\tau}(A_k)$, *when* $\mu(A_l) = \mu(A_k)$;

(3)
$$\sum_{\rho(A_j)=\mu(A_l)>0,\ j\neq k,l} \tau(A_j) < \underline{\tau}(A_l) < +\infty$$
*when* $\mu(A_l) = \max\{\rho(A_j), j \neq k, l\}$;

(4)
$$\sum_{\rho(A_j)=\mu(A_l)>0,\ j\neq k,l} \tau(A_j) + \underline{\tau}(A_k) < \underline{\tau}(A_l) < +\infty$$
*when* $\mu(A_l) = \mu(A_k) = \max\{\rho(A_j), j \neq k, l\}$;

(5) $\lambda\left(\frac{1}{A_l}\right) < \mu(A_l) < \infty$.

*Then every meromorphic solution* $f$ *of* (2) *satisfies* $\rho(f) \geq \mu(A_l)$ *if* $A_{n+1} \not\equiv 0$. *Furthermore, if* $A_{n+1} \equiv 0$, *then every meromorphic solution* $f \not\equiv 0$ *of* (2) *satisfies* $\rho(f) \geq \mu(A_l) + 1$.

## 2 Some Auxiliary Lemmas

The proofs of our results depend mainly on the following lemmas.

**Lemma 1.** *([5]) Let $f$ be a meromorphic function of finite order $\rho$, and let $c_1, c_2 (c_1 \neq c_2)$ be two arbitrary complex numbers. Let $\varepsilon > 0$ be given, then there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_1$, we have*

$$\exp\{-r^{\rho-1+\varepsilon}\} \leq \left|\frac{f(z+c_1)}{f(z+c_2)}\right| \leq \exp\{r^{\rho-1+\varepsilon}\}.$$

**Lemma 2.** *([6]) Let $f$ be a meromorphic function, $c$ be a non-zero complex constant. Then we have that for $r \longrightarrow +\infty$*

$$(1+o(1))T(r-|c|, f(z)) \leq T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f(z)).$$

**Lemma 3.** *([3]) Let $f$ be a meromorphic function of finite order $\rho$. Then for any given $\varepsilon > 0$, there exists a set $E_2 \subset (1, +\infty)$ having finite linear measure and finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0,1] \cup E_2$ and sufficiently large $r$, we have*

$$\exp\{-r^{\rho+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\rho+\varepsilon}\}.$$

**Lemma 4.** *([4, 10]) Let $f$ be an entire function with $\mu(f) < \infty$. Then for any given $\varepsilon (> 0)$, there exists a subset $E_3 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_3$, we have*

$$\mu(f) = \lim_{\substack{r \to +\infty \\ r \in E_3}} \frac{\log\log M(r,f)}{\log r}$$

*and*

$$M(r,f) < \exp\{r^{\mu(f)+\varepsilon}\}.$$

**Lemma 5.** *([4, 16]) Let $f$ be an entire function with $0 < \mu(f) < \infty$. Then for any given $\varepsilon (> 0)$, there exists a subset $E_4 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_4$, we have*

$$\underline{\tau}_M(f) = \lim_{\substack{r \to +\infty \\ r \in E_4}} \frac{\log M(r,f)}{\log r}$$

*and*

$$M(r,f) < \exp\{(\underline{\tau}_M(f) + \varepsilon) r^{\mu(f)}\}.$$

**Lemma 6.** *([5]) Let $f$ be a meromorphic function of finite order $\rho(f) < \infty$, and let $c_1, c_2$ two complex numbers distinct. Then for each $\varepsilon > 0$ we have*

$$m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = O(r^{\rho(f)-1+\varepsilon}).$$

**Lemma 7.** *([4, 18]) Let $f$ be a meromorphic function with $\mu(f) < \infty$. Then for any given $\varepsilon(> 0)$, there exists a subset $E_5 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_5$, we have*

$$T(r, f) < r^{\mu(f)+\varepsilon}.$$

**Lemma 8.** *([4, 15]) Let $f$ be a meromorphic function with $0 < \mu(f) < \infty$. Then for any given $\varepsilon(> 0)$, there exists a subset $E_6 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_6$, we have*

$$T(r, f) < (\underline{\tau}(f) + \varepsilon) r^{\mu(f)}.$$

# 3 Proof of Theorem 3

*Proof.* If $f$ has infinite order, then the result holds. Now we suppose that $\rho(f) < \infty$. We divide (2) by $f(z + c_l)$ to get

$$-A_l(z) = \sum_{j=1, j\neq k,l}^{n} A_j(z) \frac{f(z + c_j)}{f(z + c_l)} + A_k(z) \frac{f(z + c_k)}{f(z + c_l)} + A_0(z) \frac{f(z)}{f(z + c_l)} - \frac{A_{n+1}(z)}{f(z + c_l)}. \tag{3}$$

Therefore

$$|A_l(z)| \leq \sum_{j=1, j\neq k,l}^{n} |A_j(z)| \left| \frac{f(z + c_j)}{f(z + c_l)} \right| + |A_k(z)| \left| \frac{f(z + c_k)}{f(z + c_l)} \right| + |A_0(z)| \left| \frac{f(z)}{f(z + c_l)} \right| + \left| \frac{A_{n+1}(z)}{f(z + c_l)} \right|. \tag{4}$$

It follows by Lemma 1 that for any $\varepsilon(> 0)$, there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left| \frac{f(z)}{f(z + c_l)} \right| \leq \exp\{r^{\rho-1+\varepsilon}\}, \quad \left| \frac{f(z + c_j)}{f(z + c_l)} \right| \leq \exp\{r^{\rho-1+\varepsilon}\}, \quad j \neq l. \tag{5}$$

By Lemma 2, we have

$$\rho(f(z + c_l)) = \rho\left(\frac{1}{f(z + c_l)}\right) = \rho(f).$$

Then by Lemma 3, for the above $\varepsilon$, there exists a subset $E_2 \subset (1, +\infty)$, having finite logarithmic measure such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E_2$ and sufficiently large $r$, we have

$$\left| \frac{1}{f(z + c_l)} \right| \leq \exp\{r^{\rho(f)+\varepsilon}\}. \tag{6}$$

In the following, we divide the proof into four cases:

(i) We suppose that $\rho < \mu(A_l)$.
By the definition of $\rho(A_j)$ for the above $\varepsilon$ and sufficiently large $r$, we have

$$|A_j(z)| \leq \exp\{r^{\rho(A_j)+\varepsilon}\} \leq \exp\{r^{\rho+\varepsilon}\}, \ j \neq k, l. \tag{7}$$

By the definition of $\mu(A_l)$, for sufficiently small $\varepsilon > 0$ and sufficiently large $r$, we have

$$|A_l(z)| \geq \exp\{r^{\mu(A_l)-\varepsilon}\}. \tag{8}$$

By the definition of $\mu(A_k)$ and Lemma 4, for any given $\varepsilon(> 0)$, there exists a subset $E_3 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_3$, we have

$$|A_k(z)| \leq \exp\{r^{\mu(A_k)+\varepsilon}\}. \tag{9}$$

By substituting (5)-(9) into (4), for all $z$ satisfying $|z| = r \in E_3 \setminus ([0, 1] \cup E_1 \cup E_2)$, we obtain

$$\exp\{r^{\mu(A_l)-\varepsilon}\} \leq (n-1)\exp\{r^{\rho+\varepsilon}\}\exp\{r^{\rho(f)-1+\varepsilon}\} + \exp\{r^{\mu(A_k)+\varepsilon}\}\exp\{r^{\rho(f)-1+\varepsilon}\}$$
$$+ \exp\{r^{\rho+\varepsilon}\}\exp\{r^{\rho(f)+\varepsilon}\}. \tag{10}$$

Now, we may choose sufficiently small $\varepsilon$ satisfying $0 < 3\varepsilon < \mu(A_l) - \rho$, we deduce from (10) that for $|z| = r \in E_3 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \to +\infty$

$$\exp\{r^{\mu(A_l)-2\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\},$$

that is, $\mu(A_l) \leq \rho(f) + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.
(ii) We suppose that $\max\{\rho(A_j), j \neq k, l\} = \alpha < \mu(A_k) = \mu(A_l)$, $\underline{\tau}_M(A_l) > \underline{\tau}_M(A_k)$.
By the definition of $\rho(A_j)$ for any given $\varepsilon(> 0)$ and sufficiently large $r$, we have

$$|A_j(z)| \leq \exp\{r^{\rho(A_j)+\varepsilon}\} \leq \exp\{r^{\alpha+\varepsilon}\}, \ j \neq k, l. \tag{11}$$

By the definition of $\underline{\tau}_M(A_l)$, for sufficiently small $\varepsilon > 0$ and sufficiently large $r$, we have

$$|A_l(z)| \geq \exp\{(\underline{\tau}_M(A_l) - \varepsilon)\, r^{\mu(A_l)}\}. \tag{12}$$

By the definition of $\underline{\tau}_M(A_k)$ and Lemma 5, for any given $\varepsilon(> 0)$, there exists a subset $E_4 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_4$, we have

$$|A_k(z)| \leq \exp\{(\underline{\tau}_M(A_k) + \varepsilon)\, r^{\mu(A_k)}\} = \exp\{(\underline{\tau}_M(A_k) + \varepsilon)\, r^{\mu(A_l)}\}. \tag{13}$$

By substituting (5), (6) and (11)-(13) into (4), for $r \in |z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$, we have

$$\exp\{(\underline{\tau}_M(A_l) - \varepsilon)\, r^{\mu(A_l)}\} \leq (n-1)\exp\{r^{\alpha+\varepsilon}\}\exp\left\{r^{\rho(f)-1+\varepsilon}\right\}$$
$$+ \exp\{(\underline{\tau}_M(A_k) + \varepsilon)\, r^{\mu(A_l)}\}\exp\left\{r^{\rho(f)-1+\varepsilon}\right\} + \exp\{r^{\alpha+\varepsilon}\}\exp\left\{r^{\rho(f)+\varepsilon}\right\}. \tag{14}$$

Now, we may choose sufficiently small $\varepsilon$, $0 < 2\varepsilon < \min\{\mu(A_l) - \alpha, \underline{\tau}_M(A_l) - \underline{\tau}_M(A_k)\}$, then from (14) for $r \in E_4 \setminus ([0,1] \cup E_1 \cup E_2)$ sufficiently large, we obtain

$$\exp\{(\underline{\tau}_M(A_l) - \underline{\tau}_M(A_k) - 2\varepsilon) r^{\mu(A_l)-\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\},$$

that is, $\mu(A_l) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.
(iii) We suppose that $\mu(A_k) < \max\{\rho(A_j), j \neq k, l\} = \mu(A_l)$ and $\max\{\tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} = \tau_1 < \underline{\tau}_M(A_l)$. By the definitions of $\rho(A_j)$ and $\tau(A_j)$, for any given $\varepsilon > 0$ and sufficiently large $r$, we have

$$|A_j(z)| \leq \begin{cases} \exp\{r^{\rho(A_j)+\varepsilon}\} \leq \exp\{r^{\mu(A_l)-\varepsilon}\}, \text{ if } \rho(A_j) < \mu(A_l),\ j \neq k, l, \\ \exp\{(\tau_1 + \varepsilon) r^{\mu(A_l)}\}, \text{ if } \rho(A_j) = \mu(A_l),\ j \neq k, l. \end{cases} \tag{15}$$

Then, by substituting (5), (6), (9), (12) and (15) into (4), for all $z$ satisfying $|z| = r \in E_3 \setminus ([0,1] \cup E_1 \cup E_2)$ sufficiently large, we have

$$\begin{aligned} \exp\{(\underline{\tau}_M(A_l) - \varepsilon) r^{\mu(A_l)}\} &\leq O\left(\exp\{(\tau_1 + \varepsilon) r^{\mu(A_l)}\} \exp\left\{r^{\rho(f)-1+\varepsilon}\right\}\right) \\ &+ O\left(\exp\{r^{\mu(A_l)-\varepsilon}\} \exp\left\{r^{\rho(f)-1+\varepsilon}\right\}\right) \\ + \exp\{r^{\mu(A_k)+\varepsilon}\} \exp\left\{r^{\rho(f)-1+\varepsilon}\right\} &+ \exp\{(\tau_1 + \varepsilon) r^{\mu(A_l)}\} \exp\left\{r^{\rho(f)+\varepsilon}\right\}. \end{aligned} \tag{16}$$

Now, we may choose sufficiently small $\varepsilon$ satisfying

$$0 < 2\varepsilon < \min\{\mu(A_l) - \mu(A_k), \underline{\tau}_M(A_l) - \tau_1\},$$

then from (16) for sufficiently large $r \in E_3 \setminus ([0,1] \cup E_1 \cup E_2)$, we get

$$\exp\{(\underline{\tau}_M(A_l) - \tau_1 - 2\varepsilon) r^{\mu(A_l)-\varepsilon}\} \leq \exp\left\{r^{\rho(f)+\varepsilon}\right\},$$

that is, $\mu(A_l) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.
(iv) We suppose that $\max\{\rho(A_j), j \neq k, l\} = \mu(A_k) = \mu(A_l)$ and

$$\max\{\underline{\tau}_M(A_k), \tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} = \tau_2 < \underline{\tau}_M(A_l).$$

Then, by substituting (5), (6), (12), (13) and (15) into (4), for all $z$ satisfying $|z| = r \in E_4 \setminus ([0,1] \cup E_1 \cup E_2)$ sufficiently large, we obtain

$$\begin{aligned} \exp\{(\underline{\tau}_M(A_l) - \varepsilon) r^{\mu(A_l)}\} &\leq O\left(\exp\{(\tau_2 + \varepsilon) r^{\mu(A_l)}\} \exp\left\{r^{\rho(f)-1+\varepsilon}\right\}\right) \\ &+ O\left(\exp\{r^{\mu(A_l)-\varepsilon}\} \exp\left\{r^{\rho(f)-1+\varepsilon}\right\}\right) \\ + \exp\{(\underline{\tau}_M(A_k) + \varepsilon) r^{\mu(A_l)}\} \exp\left\{r^{\rho(f)-1+\varepsilon}\right\} &+ \exp\{(\tau_2 + \varepsilon) r^{\mu(A_l)}\} \exp\left\{r^{\rho(f)+\varepsilon}\right\}. \end{aligned} \tag{17}$$

Now, we may choose sufficiently small $\varepsilon$ satisfying

$$0 < 2\varepsilon < \underline{\tau}_M(A_l) - \tau_2,$$

then from (17) for sufficiently large $|z| = r \in E_4 \setminus ([0,1] \cup E_1 \cup E_2)$, we get

$$\exp\{(\underline{\tau}_M(A_l) - \tau_2 - 2\varepsilon) r^{\mu(A_l)-\varepsilon}\} \leq \exp\left\{r^{\rho(f)+\varepsilon}\right\},$$

that is, $\mu(A_l) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.
Further, if $A_{n+1} \equiv 0$, then by Theorem 1, every meromorphic solution $f \not\equiv 0$ of (2) satisfies $\rho(f) \geq \mu(A_l) + 1$. □

## 4 Proof of Theorem 4

*Proof.* If $f$ has infinite order, then the result holds. Now, we suppose that $\rho(f) < \infty$. It follows by (2) and Lemma 2 that

$$T(r, A_l(z)) = m(r, A_l(z)) + N(r, A_l(z)) \leq \sum_{j=0, j\neq k,l}^{n+1} m(r, A_j(z)) + m(r, A_k(z))$$

$$+ \sum_{j=1, j\neq l}^{n} m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + m\left(r, \frac{1}{f(z+c_l)}\right)$$

$$+N(r, A_l(z)) + O(1) \leq \sum_{j=0, j\neq k,l}^{n+1} T(r, A_j(z)) + T(r, A_k(z)) + N(r, A_l(z))$$

$$+T\left(r, \frac{1}{f(z+c_l)}\right) + \sum_{j=1, j\neq l}^{n} m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1)$$

$$\leq \sum_{j=0, j\neq k,l}^{n+1} T(r, A_j(z)) + T(r, A_k(z)) + N(r, A_l(z))$$

$$+2T(r + |c_l|, f(z)) + \sum_{j=1, j\neq l}^{n} m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1)$$

$$\leq \sum_{j=0, j\neq k,l}^{n+1} T(r, A_j(z)) + T(r, A_k(z)) + N(r, A_l(z))$$

$$+2T(2r, f) + \sum_{j=1, j\neq l}^{n} m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1). \quad (18)$$

By Lemma 6, for any given $\varepsilon(> 0)$, we have

$$m\left(r, \frac{f(z)}{f(z+c_l)}\right) = O(r^{\rho(f)-1+\varepsilon}), \quad m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) = O(r^{\rho(f)-1+\varepsilon}), \quad j \neq l. \quad (19)$$

By the definition of $\lambda(\frac{1}{A_l})$, for the above $\varepsilon$ and sufficiently large $r$, we have

$$N(r, A_l) \leq r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon}. \quad (20)$$

(i) We suppose that $\rho < \mu(A_l)$.
By the definition of $\rho(A_j)$ and $\rho(f)$ for the above $\varepsilon$ and sufficiently large $r$, we have

$$T(r, A_j) \leq r^{\rho(A_j)+\varepsilon} \leq r^{\rho+\varepsilon}, \ j \neq k, l, \quad (21)$$

$$T(r, f) \leq r^{\rho(f)+\varepsilon}. \quad (22)$$

By the definition of $\mu(A_l)$, for sufficiently small $\varepsilon > 0$ and sufficiently large $r$, we have
$$T(r, A_l) \geq r^{\mu(A_l)-\varepsilon}. \tag{23}$$
By the definition of $\mu(A_k)$ and Lemma 7, for any given $\varepsilon(> 0)$, there exists a subset $E_5 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_5$, we have
$$T(r, A_k) \leq r^{\mu(A_k)+\varepsilon}. \tag{24}$$
By substituting (19)-(24) into (18) for sufficiently large $r \in E_5$, we obtain
$$r^{\mu(A_l)-\varepsilon} \leq nr^{\rho+\varepsilon} + r^{\mu(A_k)+\varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + 2\,(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \tag{25}$$
Now, we may choose sufficiently small $\varepsilon$ satisfying
$$0 < 3\varepsilon < \min\left\{\mu(A_l) - \rho, \mu(A_l) - \lambda\left(\frac{1}{A_l}\right)\right\},$$
we deduce from (25) that for $|z| = r \in E_5$, $r \to +\infty$
$$r^{\mu(A_l)-2\varepsilon} \leq r^{\rho(f)+\varepsilon},$$
that is, $\mu(A_l) \leq \rho(f) + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.
Further, if $A_{n+1} \equiv 0$, then by substituting (19)-(21), (23) and (24) into (18) for sufficiently large $r \in E_5$, we obtain
$$r^{\mu(A_l)-\varepsilon} \leq (n-1)\, r^{\rho+\varepsilon} + r^{\mu(A_k)+\varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \tag{26}$$
Now, we may choose sufficiently small $\varepsilon$ satisfying
$$0 < 3\varepsilon < \min\left\{\mu(A_l) - \rho, \mu(A_l) - \lambda\left(\frac{1}{A_l}\right)\right\},$$
we deduce from (26) that for $|z| = r \in E_5$, $r \to +\infty$
$$r^{\mu(A_l)-2\varepsilon} \leq r^{\rho(f)-1+\varepsilon},$$
that is, $\mu(A_l) \leq \rho(f) - 1 + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l) + 1$.
(ii) We suppose that $\max\{\rho(A_j), j \neq k, l\} = \alpha < \mu(A_k) = \mu(A_l)$, $\underline{\tau}(A_l) > \underline{\tau}(A_k)$.
By the definition of $\rho(A_j)$, for any given $\varepsilon(> 0)$ and sufficiently large $r$, we have
$$T(r, A_j) \leq r^{\rho(A_j)+\varepsilon} \leq r^{\alpha+\varepsilon},\ j \neq k, l. \tag{27}$$
By the definition of $\underline{\tau}(A_l)$, for sufficiently small $\varepsilon > 0$ and sufficiently large $r$, we have
$$T(r, A_l) \geq (\underline{\tau}(A_l) - \varepsilon)\, r^{\mu(A_l)}. \tag{28}$$
By the definition of $\underline{\tau}(A_k)$ and Lemma 8, for any given $\varepsilon(> 0)$, there exists a subset $E_6 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_6$, we have
$$T(r, A_k) \leq (\underline{\tau}(A_k) + \varepsilon)\, r^{\mu(A_k)} = (\underline{\tau}(A_k) + \varepsilon)\, r^{\mu(A_l)}. \tag{29}$$

By substituting (19), (20), (22), (27)-(29) into (18), for sufficiently large $r \in E_6$, we obtain

$$(\underline{\tau}(A_l) - \varepsilon)\, r^{\mu(A_l)} \le n r^{\alpha+\varepsilon} + (\underline{\tau}(A_k) + \varepsilon)\, r^{\mu(A_l)}$$

$$+ r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + 2\,(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \tag{30}$$

Now, we may choose sufficiently small $\varepsilon$ satisfying

$$0 < 2\varepsilon < \min\left\{\mu(A_l) - \alpha, \underline{\tau}(A_l) - \underline{\tau}(A_k), \mu(A_l) - \lambda\left(\frac{1}{A_l}\right)\right\},$$

then from (30) for $r \in E_6$ sufficiently large, we get

$$(\underline{\tau}(A_l) - \underline{\tau}(A_k) - 2\varepsilon) r^{\mu(A_l)-\varepsilon} \le r^{\rho(f)+\varepsilon},$$

that is, $\mu(A_l) \le \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_l)$.
Further, if $A_{n+1} \equiv 0$, then by substituting (19), (20), (27)-(29) into (18) for sufficiently large $r \in E_6$, we obtain

$$(\underline{\tau}(A_l) - \varepsilon)\, r^{\mu(A_l)} \le (n-1)\, r^{\alpha+\varepsilon} + (\underline{\tau}(A_k) + \varepsilon)\, r^{\mu(A_l)}$$

$$+ r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \tag{31}$$

Now, we may choose sufficiently small $\varepsilon$ satisfying

$$0 < 2\varepsilon < \min\left\{\mu(A_l) - \alpha, \underline{\tau}(A_l) - \underline{\tau}(A_k), \mu(A_l) - \lambda\left(\frac{1}{A_l}\right)\right\},$$

we deduce from (31) that for $|z| = r \in E_6$, $r \to +\infty$

$$(\underline{\tau}(A_l) - \underline{\tau}(A_k) - 2\varepsilon) r^{\mu(A_l)-\varepsilon} \le r^{\rho(f)-1+\varepsilon},$$

that is, $\mu(A_l) \le \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_l) + 1$.
(iii) We suppose that $\mu(A_k) < \max\{\rho(A_j), j \ne k, l\} = \mu(A_l)$ and

$$\sum_{\rho(A_j)=\mu(A_l)>0,\, j\ne k,l} \tau(A_j) < \underline{\tau}(A_l) < +\infty.$$

Then, there exists a set $J \subseteq \{0, 1, \ldots, n+1\} \setminus \{k, l\}$ such that for $j \in J$, we have $\rho(A_j) = \mu(A_l)$ with $\sum\limits_{j \in J} \tau(A_j) = \tau_1 < \underline{\tau}(A_l)$ and for $j \in \{0, 1, \ldots, n + 1\} \setminus (J \cup \{k, l\})$, we have $\rho(A_j) < \mu(A_l)$. Hence, for any given $\varepsilon > 0$ and sufficiently large $r$, we have

$$T(r, A_j) \le (\tau(A_j) + \varepsilon)\, r^{\mu(A_l)},\ j \in J \tag{32}$$

and

$$T(r, A_j) \le r^{\mu(A_l)-\varepsilon},\ j \in \{0, 1, \ldots, n+1\} \setminus (J \cup \{k, l\}). \tag{33}$$

Then, by substituting (19), (20), (22), (24), (28), (32) and (33) into (18), for all $z$ satisfying $|z| = r \in E_5$ sufficiently large, we obtain

$$(\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)} \le \sum_{j\in J} (\tau(A_j) + \varepsilon) r^{\mu(A_l)} + \sum_{j\in\{0,1,\dots,n+1\}\setminus(J\cup\{k,l\})} r^{\mu(A_l)-\varepsilon} + r^{\mu(A_k)+\varepsilon}$$

$$+ r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) \le (\tau_1 + n\varepsilon) r^{\mu(A_l)} + O\left(r^{\mu(A_l)-\varepsilon}\right) + r^{\mu(A_k)+\varepsilon}$$

$$+ r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \tag{34}$$

Now, we may choose sufficiently small $\varepsilon$ satisfying

$$0 < \varepsilon < \min\left\{ \frac{\mu(A_l) - \mu(A_k)}{2}, \frac{\underline{\tau}(A_l) - \tau_1}{n+1}, \frac{\mu(A_l) - \lambda\left(\frac{1}{A_l}\right)}{2} \right\},$$

then from (34) for sufficiently large $|z| = r \in E_5$, we get

$$(\underline{\tau}(A_l) - \tau_1 - (n+1)\varepsilon) r^{\mu(A_l)-\varepsilon} \le r^{\rho(f)+\varepsilon},$$

that is, $\mu(A_l) \le \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_l)$.
Further, if $A_{n+1} \equiv 0$, then by substituting (19), (20), (24), (28), (32) and (33) into (18) for sufficiently large $r \in E_5$, we obtain

$$(\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)} \le (\tau_1 + (n-1)\varepsilon) r^{\mu(A_l)} + O\left(r^{\mu(A_l)-\varepsilon}\right) + r^{\mu(A_k)+\varepsilon}$$

$$+ r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \tag{35}$$

Now, we may choose sufficiently small $\varepsilon$ satisfying

$$0 < \varepsilon < \min\left\{ \frac{\mu(A_l) - \mu(A_k)}{2}, \frac{\underline{\tau}(A_l) - \tau_1}{n}, \frac{\mu(A_l) - \lambda\left(\frac{1}{A_l}\right)}{2} \right\},$$

we deduce from (35) that for $|z| = r \in E_5$, $r \to +\infty$

$$(\underline{\tau}(A_l) - \tau_1 - n\varepsilon) r^{\mu(A_l)-\varepsilon} \le r^{\rho(f)-1+\varepsilon},$$

that is, $\mu(A_l) \le \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \ge \mu(A_l) + 1$.
(iv) We suppose that $\max\{\rho(A_j), j \ne k, l\} = \mu(A_k) = \mu(A_l)$ and

$$\sum_{\rho(A_j)=\mu(A_l)>0,\, j\ne k,l} \tau(A_j) + \underline{\tau}(A_k) < \underline{\tau}(A_l) < +\infty.$$

Then, by substituting (19), (20), (22), (28), (29), (32) and (33) into (18), for all $z$ satisfying $|z| = r \in E_6$ sufficiently large, we have

$$(\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)} \le \sum_{j\in J} (\tau(A_j) + \varepsilon) r^{\mu(A_l)} + \sum_{j\in\{0,1,\dots,n+1\}\setminus(J\cup\{k,l\})} r^{\mu(A_l)-\varepsilon}$$

$$+\left(\underline{\tau}(A_k)+\varepsilon\right)r^{\mu(A_l)}+r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon}+2\left(2r\right)^{\rho(f)+\varepsilon}+O(r^{\rho(f)-1+\varepsilon})$$
$$\leq\left(\tau_1+\underline{\tau}(A_k)+(n+1)\,\varepsilon\right)r^{\mu(A_l)}+O\left(r^{\mu(A_l)-\varepsilon}\right)$$
$$+r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon}+2\left(2r\right)^{\rho(f)+\varepsilon}+O(r^{\rho(f)-1+\varepsilon}). \tag{36}$$

Now, we may choose sufficiently small $\varepsilon$ satisfying

$$0<\varepsilon<\min\left\{\frac{\underline{\tau}(A_l)-\tau_1-\underline{\tau}(A_k)}{n+2},\frac{\mu(A_l)-\lambda\left(\frac{1}{A_l}\right)}{2}\right\},$$

then from (36) for sufficiently large $|z|=r\in E_6$, we get

$$\left(\underline{\tau}(A_l)-\tau_1-\underline{\tau}(A_k)-(n+2)\,\varepsilon\right)r^{\mu(A_l)-\varepsilon}\leq r^{\rho(f)+\varepsilon},$$

that is, $\mu(A_l)\leq\rho(f)+2\varepsilon$, since $\varepsilon>0$ is arbitrary, then $\rho(f)\geq\mu(A_l)$.
Further, if $A_{n+1}\equiv 0$, then by substituting (19), (20), (28), (29), (32) and (33) into (18), for sufficiently large $r\in E_6$, we obtain

$$\left(\underline{\tau}(A_l)-\varepsilon\right)r^{\mu(A_l)}\leq\left(\tau_1+\underline{\tau}(A_k)+n\varepsilon\right)r^{\mu(A_l)}+O\left(r^{\mu(A_l)-\varepsilon}\right)$$
$$+r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon}+O(r^{\rho(f)-1+\varepsilon}). \tag{37}$$

Now, we may choose sufficiently small $\varepsilon$ satisfying

$$0<\varepsilon<\min\left\{\frac{\underline{\tau}(A_l)-\tau_1-\underline{\tau}(A_k)}{n+1},\frac{\mu(A_l)-\lambda\left(\frac{1}{A_l}\right)}{2}\right\},$$

we deduce from (37) that for $|z|=r\in E_6$, $r\to+\infty$

$$\left(\underline{\tau}(A_l)-\tau_1-\underline{\tau}(A_k)-(n+1)\,\varepsilon\right)r^{\mu(A_l)-\varepsilon}\leq r^{\rho(f)-1+\varepsilon},$$

that is, $\mu(A_l)\leq\rho(f)-1+2\varepsilon$, since $\varepsilon>0$ is arbitrary, then $\rho(f)\geq\mu(A_l)+1$. □

## 5 Examples

**Example 1.** *Consider the non-homogeneous difference equation with entire coefficients*

$$A_3(z)f(z+\sqrt{\pi})+A_2(z)f(z-\sqrt{\pi})+A_1(z)f(z+\frac{\sqrt{\pi}}{2})+A_0(z)f(z)=A_4(z). \tag{38}$$

***Case 1.*** $\max\{\rho(A_j),\mu(A_k):j\neq l,k\}<\mu(A_l)$. *In* (38), *for*

$$A_0\left(z\right)=1,\quad A_1\left(z\right)=e^{-4\sqrt{\pi}z+3\pi},$$

$$A_2(z) = -2e^{8\sqrt{\pi}z-12\pi}, \quad A_3(z) = e^{-4z^2+z+8\pi}, \quad A_4(z) = e^z,$$

*we have*

$$\max\{\rho(A_j) \ \ (j=0,1,4), \mu(A_2)\} = 1 < \mu(A_3) = 2.$$

*As we see, the conditions of Theorem 3 are verified. The function*

$$f(z) = e^{4(z^2-\pi)-8\sqrt{\pi}z}$$

*is a solution of equation* (38) *which satisfies* $\rho(f) = 2 \geq \mu(A_3) = 2$.

***Case 2.*** $\max\{\rho(A_j) : j \neq k, l\} < \mu(A_k) = \mu(A_l)$, $\underline{\tau}_M(A_l) > \underline{\tau}_M(A_k)$. *In* (38), *for*

$$A_0(z) = e^z, \ A_1(z) = e^{z(1-4\sqrt{\pi})+3\pi}, \ A_2(z) = -2e^{z(1+8\sqrt{\pi})-12\pi} + e^{z^2+16\sqrt{\pi}z},$$

$$A_3(z) = e^{-4z^2+z+8\pi} - e^{z^2+16\pi}, \ A_4(z) = e^z,$$

*we get*

$$\max\{\rho(A_j) \ \ (j=0,1,4)\} = 1 < \mu(A_2) = \mu(A_3) = 2, \quad \underline{\tau}_M(A_2) = 1 < 4 = \underline{\tau}_M(A_3).$$

*Obviously, the conditions of Theorem 3 are satisfied. The function*

$$f(z) = e^{4(z^2-\pi)-8\sqrt{\pi}z}$$

*is a solution of equation* (38) *and satisfies* $\rho(f) = 2 \geq \mu(A_3) = 2$.
***Case 3.*** $\mu(A_k) < \max\{\rho(A_j), j \neq k, l\} = \mu(A_l)$ *and* $\max\{\tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} = \tau_1 < \underline{\tau}_M(A_l)$. *In* (38), *for*

$$A_0(z) = 1, \quad A_1(z) = e^{-4\sqrt{\pi}z+3\pi},$$

$$A_2(z) = -2e^{8\sqrt{\pi}z-12\pi}, \quad A_3(z) = e^{-3z^2+8\pi}, \quad A_4(z) = e^{z^2},$$

*we have*

$$\mu(A_0) = 0 < \max\{\rho(A_j) \ \ (j=1,2,4)\} = 2 = \mu(A_3),$$

$$\tau_M(A_4) = 1 < 3 = \underline{\tau}_M(A_3).$$

*Hence, the conditions of Theorem 3 are verified. The function*

$$f(z) = e^{4(z^2-\pi)-8\sqrt{\pi}z}$$

*is a solution of equation* (38) *satisfying* $\rho(f) = 2 \geq \mu(A_3) = 2$.
***Case 4.*** $\max\{\rho(A_j) : j \neq k, l\} = \mu(A_k) = \mu(A_l)$ *and* $\max\{\underline{\tau}_M(A_k), \tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} < \underline{\tau}_M(A_l)$. *In* (38), *for*

$$A_0(z) = e^{z^2}, \quad A_1(z) = e^{z^2-4\sqrt{\pi}z+3\pi},$$

$$A_2(z) = -2e^{z^2+8\sqrt{\pi}z-12\pi}, \quad A_3(z) = e^{-6z^2}, \quad A_4(z) = e^{-2z^2-8\pi},$$

*we get*

$$\max\{\rho(A_j) \ \ (j=0,1,4)\} = \mu(A_2) = \mu(A_3) = 2,$$

$$\max\{\underline{\tau}_M(A_2), \tau_M(A_j), \ \ (j=0,1,4)\} = 2 < 6 = \underline{\tau}_M(A_3).$$

*We see that the conditions of Theorem 3 are satisfied. The function*

$$f(z) = e^{4(z^2-\pi)-8\sqrt{\pi}z}$$

*is a solution of equation* (38) *which satisfies* $\rho(f) = 2 \geq \mu(A_3) = 2$.

**Example 2.** *Consider the non-homogeneous difference equation with meromorphic coefficients*

$$A_2(z)f(z+i\pi^2) + A_1(z)f(z+i\frac{\pi^2}{2}) + A_0(z)f(z) = A_3(z). \tag{39}$$

***Case 1.*** $\max\{\rho(A_j), \mu(A_k) : j \neq l, k\} < \mu(A_l)$. *In* (39), *for*

$$A_0(z) = e^{24\pi i z^2}, \quad A_1(z) = \frac{e^{-\frac{8}{\pi}z^3 - 12\pi i z^2}}{\cos(\frac{iz}{\pi})},$$

$$A_2(z) = e^{24\pi^3 z + 8\pi^5 i}, \quad A_3(z) = -e^{-6\pi^3 z - i\pi^5} \sec^2(\frac{iz}{\pi}),$$

*we have*

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 3,$$

$$\max\{\rho(A_j), \mu(A_0) : j = 2,3\} = 2 < \mu(A_1) = 3.$$

*It's clear that the conditions of Theorem 4 are satisfied. The function*

$$f(z) = \frac{e^{\frac{8}{\pi}z^3}}{\sin(\frac{iz}{\pi})}$$

*is a solution of equation* (39) *satisfying* $\rho(f) = 3 \geq \mu(A_3) = 3$.

***Case 2.*** $\max\{\rho(A_j) : j \neq k, l\} = \alpha < \mu(A_k) = \mu(A_l)$, $\underline{\tau}_M(A_l) > \underline{\tau}_M(A_k)$. *In* (39), *for*

$$A_0(z) = -e^{z^3 + 12\pi i z^2 - 6\pi^3 z} + e^{24\pi i z^2}, \ A_1(z) = e^{-\frac{8}{\pi}z^3 + 6\pi^3 z} - \left(\cot(\frac{iz}{\pi})\right) e^{z^3 + i\pi^5},$$

$$A_2(z) = e^{24\pi^3 z + 8\pi^5 i}, \ A_3(z) = -e^{12\pi i z^2 - i\pi^5} \sec(\frac{iz}{\pi}),$$

*we have*

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 3, \quad \max\{\rho(A_j) : j = 2,3\} = 2 < \mu(A_0) = \mu(A_1) = 3,$$

$$\underline{\tau}(A_1) = \frac{8}{\pi^2} > \frac{1}{\pi} = \underline{\tau}(A_0).$$

*Obviously, the conditions of Theorem 4 are satisfied. The function*

$$f(z) = \frac{e^{\frac{8}{\pi}z^3}}{\sin(\frac{iz}{\pi})}$$

*is a solution of equation* (39) *and* $f$ *satisfies* $\rho(f) = 3 \geq \mu(A_1) = 3$.
***Case 3.*** $\mu(A_k) < \max\{\rho(A_j), j \neq k, l\} = \mu(A_l)$ *and*

$$\sum_{\rho(A_j)=\mu(A_l)>0,\, j\neq k,l} \tau(A_j) < \underline{\tau}(A_l) < +\infty.$$

*In* (39), *for*

$$A_0(z) = e^{24\pi i z^2}, \quad A_1(z) = \frac{e^{-\frac{12}{\pi}z^3 - 12\pi i z^2 + 6\pi^3 z}}{\cos(\frac{iz}{\pi})},$$

$$A_2(z) = e^{24\pi^3 z + 8\pi^5 i}, \quad A_3(z) = -e^{-\frac{4}{\pi}z^3 - \pi^5 i}\sec^2(\frac{iz}{\pi}),$$

*we get*

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 3, \quad \mu(A_2) = 1 < \max\{\rho(A_j) : j = 0,3\} = \mu(A_1) = 3,$$

$$\sum_{\rho(A_j)=\mu(A_l)>0,\, j\neq k,l} \tau(A_j) = \tau(A_3) = \frac{4}{\pi^2} < \underline{\tau}(A_1) = \frac{12}{\pi^2}.$$

*Obviously, the conditions of Theorem 4 are satisfied. The function*

$$f(z) = \frac{e^{\frac{8}{\pi}z^3}}{\sin(\frac{iz}{\pi})}$$

*is a solution of equation* (39) *which satisfies* $\rho(f) = 3 \geq \mu(A_1) = 3$.
***Case 4.*** $\max\{\rho(A_j), j \neq k, l\} = \mu(A_k) = \mu(A_l)$ *and*

$$\sum_{\rho(A_j)=\mu(A_l)>0,\, j\neq k,l} \tau(A_j) + \underline{\tau}(A_k) < \underline{\tau}(A_l) < +\infty.$$

*In* (39), *for*

$$A_0(z) = e^{\frac{iz^3}{\pi} + 24\pi i z^2}, \quad A_1(z) = \frac{e^{-\frac{12}{\pi}z^3 - 12\pi i z^2 + 6\pi^3 z}}{\cos(\frac{iz}{\pi})},$$

$$A_2(z) = e^{\frac{iz^3}{\pi} + 24\pi^3 z + 8\pi^5 i}, \quad A_3(z) = -e^{-\frac{4}{\pi}z^3 - \pi^5 i}\sec^2(\frac{iz}{\pi}),$$

*we have*

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 3, \quad \max\{\rho(A_j) : j = 2,3\} = \mu(A_0) = \mu(A_1) = 3,$$

$$\sum_{\rho(A_j)=\mu(A_l)>0,\, j\neq k,l} \tau(A_j)+\underline{\tau}(A_k)=\tau(A_2)+\tau(A_3)+\underline{\tau}\,(A_0)$$
$$=\frac{1}{\pi^2}+\frac{4}{\pi^2}+\frac{1}{\pi^2}=\frac{6}{\pi^2}<\underline{\tau}(A_1)=\frac{12}{\pi^2}.$$

*Obviously, the conditions of Theorem 4 are satisfied. The function*

$$f\,(z)=\frac{e^{\frac{8}{\pi}z^3}}{\sin(\frac{iz}{\pi})}$$

*is a solution of equation* (39) *and* $f$ *satisfies* $\rho(f)=3\geq\mu(A_1)=3.$

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