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A NOTE ON C. C. YANG'S QUESTION CORRESPONDING TO LINEAR SHIFT OPERATOR

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Abstract

In this paper, we investigate shared value problems of finite ordered meromorphic functions with the linear shift operators governed by them, which practically provide an answer to Yang's question. We exhibit a number of examples which will justify some assertions in the paper. Based on some examples relevant with the discussion, we also place a question in the penultimate section for future research.

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1 Introduction, Definitions and Results

Throughout the paper by $\mathbb{N}$ and $\mathbb{C}$ we respectively denote the set of all natural numbers and the set of all complex numbers. We put $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Throughout the paper by f and g be mean two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For a meromorphic function f and $a \in \overline{\mathbb{C}}$, each z with $f(z) = a$ will be called an a-point of f.

For some $a \in \overline{\mathbb{C}}$, if $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

The basis of the present paper is the theory of value distribution of meromorphic functions. So we will use standard definitions and notations from [4]. In particular, $N(r, a; f) = N(r; a; f)$ ($\overline{N}(r, a; f) = \overline{N}(r; a; f)$) denotes the counting function (reduced counting function) of a-points of meromorphic functions f, $T(r, f)$ is the Nevanlinna characteristic function of f and $S(r, f)$ is used to denote each functions which is of smaller order than $T(r, f)$ when $r \to \infty$ outside of a possible exceptional set of finite linear measure. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes

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any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure. If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \longrightarrow \infty} \frac{m(r, a; f)}{T(r, f)}$$

is called Nevanlinna deficiency of the value a.

Let c be a non-zero complex constant. Then by $f(z + c)$ we mean the shift operator of f and define its difference operator by $\Delta_c f(z) = f(z + c) - f(z)$ and $\Delta_c^s f(z) = \Delta_c \left(\Delta_c^{(s-1)}(f(z))\right)$, $s \in \mathbb{N}$, $s \geq 2$, all are nothing but a linear combination of different shift operator. It will be reasonable to introduce linear shift operator as follows:

$$L(f) = \sum_{i=1}^{k} a_i f(z + c_i),$$

where $a_i (\neq 0) \in \mathbb{C}$ for $1 \leq i \leq k$ and at least $(k - 1)$ c_i's are non-zero complex numbers. In the paper, we will consider $L(f) \not\equiv \mathbf{0}$, $f(z)$, to make it meaningful.

We start our discussion on a question made by Yang [9].

In 1976, Yang [9] proposed the following problem:

Suppose that $f(z)$ and $g(z)$ are two entire functions such that $f(z)$ and $g(z)$ share 0 CM and and $f'(z)$ and $g'(z)$ share 1 CM. What can be said about the relationship between $f(z)$ and $g(z)$?

Shibazaki [8] answer the question of Yang [9] in the following manner:

Theorem A. *[8] Suppose that $f(z)$ and $g(z)$ are entire functions of finite order such that $f'(z)$ and $g'(z)$ share 1 CM. If $\delta(0; f) > 0$ and 0 is a Picard value of $g(z)$, then either $f(z) \equiv g(z)$ or $f'(z).g'(z) \equiv 1$.*

In 1991, Yi-Yang [10] obtained the following theorem:

Theorem B. *[10] Let $f(z)$ and $g(z)$ be meromorphic functions satisfying $\delta(\infty; f) = \delta(\infty; g) = 1$. If f' and g' share 1 CM and $\delta(0; f) + \delta(0; g) > 1$, then either $f \equiv g$ or $f'.g' \equiv 1$.*

There have been a continuous extensions and refinements of Yang's result. But in case of shift operator no such remarkable progress on the same question was done.

Lately, in 2013, as an attempt to solve Yang's question, Liu-Qi-Yi [7] proved the following result on linear shift operator.

Theorem C. *[7] Let c_j, a_j, b_j $(j = 1, 2, \ldots, k)$ be complex constants and let $f(z)$ and $g(z)$ be two finite order meromorphic functions satisfying $\delta(\infty; f) = \delta(\infty; g) = 1$. Let $L(f) = \sum_{i=1}^{k} a_i f(z + c_i)$ and $L(g) = \sum_{i=1}^{k} b_i g(z + c_i)$. Suppose that $L(f).L(g) \neq 0$. If $L(f)$ and $L(g)$ share 1 CM and $\delta(0; f) + \delta(0; g) > 1$, $L(f) \equiv L(g)$ or $L(f).L(g) \equiv 1$.*

In this paper we have taken into account Yang's problem and using the notion of weighted sharing, we will try to provide a solution for a more general setting namely linear shift operator, which will improve a number of existing results.

Next we recall the following definition known as weighted sharing of values which has a remarkable influence on the uniqueness theory as far as sharing of values are concerned.

Definition 1. *[5, 6] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2. *[5] For $a \in \overline{\mathbb{C}}$ we denote by $N_2(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$.*

Following theorems are the main results of this paper.

Theorem 1. *Let f and g be two non-constant meromorphic functions of finite order and $L(f)$, $L(g)$ share $(1, 2)$. If*

$$\delta(0; f) + \delta(0; g) + k\delta(\infty; f) + k\delta(\infty; g) + (k-1)min\{\delta(\infty; f), \delta(\infty; g)\} > 3k \tag{1.1}$$

then either $L(f) \equiv L(g)$ or $L(f).L(g) \equiv 1$.

In particular when $k = 1$, $L(f) \equiv L(g)$ implies $f \equiv g$.

Theorem 2. *Let f and g be two non-constant meromorphic functions of finite order and share $(0, \infty)$ and $L(f)$, $L(g)$ share $(1, 2)$. If*

$$2\max\{\delta(0; f), \delta(0; g)\} + k\delta(\infty; f) + k\delta(\infty; g) + (k-1)min\{\delta(\infty; f), \delta(\infty; g)\} > 3k \tag{1.2}$$

then either $L(f) \equiv L(g)$ or $L(f).L(g) \equiv 1$.

In particular for $k = 1$, $L(f) \equiv L(g)$ implies $f \equiv g$.

For $k = 1$ the following example shows that the condition (1.1) is sharp.

Example 1. *Let $f(z) = (e^z + 1)^2$, $g(z) = -\frac{e^z}{2}$. Then for $c = \pi i$, $f(z+c)$, $g(z+c)$ share $(1, \infty)$. Here $\delta(0; f) + \delta(0; g) + \delta(\infty; f) + \delta(\infty; g) = 3$, but neither $f(z+c) \equiv g(z+c)$ nor $f(z+c).g(z+c) \equiv 1$.*

For $k = 1$ following two examples show that the condition (1.2) is sharp.

Example 2. *Let $f(z) = -e^z(e^z+1)$, $g(z) = -\frac{1+e^z}{e^{2z}}$. Then f, g share $(0, \infty)$ and for $c = \pi i$, $f(z+c)$, $g(z+c)$ share $(1, \infty)$. Here $2\max\{\delta(0; f), \delta(0; g)\} + \delta(\infty; f) + \delta(\infty; g) = 3$, but neither $f(z+c) \equiv g(z+c)$ nor $f(z+c).g(z+c) \equiv 1$.*

Example 3. *Let $f(z) = -\frac{e^{2z}}{e^z+1}$, $g(z) = -\frac{1}{e^z(e^z+1)}$. Then f, g share $(0, \infty)$ and for $c = \pi i$, $f(z+c)$, $g(z+c)$ share $(1, \infty)$. Here $2\max\{\delta(0; f), \delta(0; g)\} + \delta(\infty; f) + \delta(\infty; g) = 3$, but neither $f(z+c) \equiv g(z+c)$ nor $f(z+c).g(z+c) \equiv 1$.*

Next examples will show that when (1.1) is satisfied then the conclusion will occur.

Example 4. *Let* $f(z)=\frac{e^z+a}{e^{4mz}}$, $g(z)=\frac{i}{e^{(4m-1)z}}$, $m\in\mathbb{N}$, $a\in\mathbb{C}\setminus\{0\}$. *Here* $\delta(0;f)+\delta(0;g)+2\delta(\infty;f)+2\delta(\infty;g)+\min\{\delta(\infty;f),\delta(\infty;g)\}=6+\frac{4m-1}{4m}>6$. *Choosing* $a_1=1$, $a_2=-1$, $c_1=\pi i$, $c_2=\frac{\pi i}{2}$, *we see that* $L(f)$, $L(g)$ *share* $(1,\infty)$ *and* $L(f)=-\frac{(1+i)}{e^{(4m-1)z}}\neq\Delta_c f$ *and* $L(g)=-\frac{(1+i)}{e^{(4m-1)z}}\neq\Delta_c g$ *for any* $c\in\mathbb{C}\setminus\{0\}$. *Here* $L(f)\equiv L(g)$.

Example 5. *Let* $f(z)=-\frac{e^{4z}+1}{(1+i)e^{(4m+1)z}}$, $g(z)=-\frac{e^{(4m+1)z}}{(1-i)(e^{4z}+1)}$, $m\geq 4$ *be an integer. Here* $\delta(0;f)+\delta(0;g)+2\delta(\infty;f)+2\delta(\infty;g)+\min\{\delta(\infty;f),\delta(\infty;g)\}=6+\frac{4m-15}{4m+1}>6$. *Choosing* $a_1=1$, $a_2=1$, $c_1=\pi i$, $c_2=\frac{\pi i}{2}$, *we see that* $L(f)$, $L(g)$ *share* $(1,\infty)$ *and* $L(f)=\frac{e^{4z}+1}{e^{(4m+1)z}}\neq\Delta_c f$ *and* $L(g)=\frac{e^{(4m+1)z}}{e^{4z}+1}\neq\Delta_c g$ *for any* $c\in\mathbb{C}\setminus\{0\}$. *Here* $L(f).L(g)\equiv 1$.

Example 6. *Let* $f(z)=-\frac{e^z}{2}$, $g(z)=-\frac{e^z(e^z+1)}{2}$. *Then* $\delta(0;f)+\delta(0;g)+2\delta(\infty;f)+2\delta(\infty;g)+\min\{\delta(\infty;f),\delta(\infty;g)\}>6$. *Here* $\Delta_{\pi i}f=e^z$, *and also* $\Delta_{\pi i}g=e^z$. *So* $\Delta_{\pi i}f$, $\Delta_{\pi i}g$ *share* $(1,\infty)$ *and* $\Delta_{\pi i}f=\Delta_{\pi i}g$.

Example 7. *Let* $f(z)=-\frac{e^z}{2}$, $g(z)=-\frac{e^z+1}{2e^{2z}}$. *Then* $\delta(0;f)+\delta(0;g)+2\delta(\infty;f)+2\delta(\infty;g)+\min\{\delta(\infty;f),\delta(\infty;g)\}>6$. *Here* $\Delta_{\pi i}f=e^z$, $\Delta_{\pi i}g=\frac{1}{e^z}$. *Therefore* $\Delta_{\pi i}f$, $\Delta_{\pi i}g$ *share* $(1,\infty)$ *and* $\Delta_{\pi i}f.\Delta_{\pi i}g\equiv 1$.

Example 8. *Let* $k=2p$, *where* $p\in\mathbb{N}$. *Also let* $f(z)=e^{2mz}(e^z+a)$, $g(z)=-e^{2mz}(e^z-a)$, $m\in\mathbb{N}$, $a\in\mathbb{C}\setminus\{0\}$. *Then*$\delta(0;f)+\delta(0;g)+k\delta(\infty;f)+k\delta(\infty;g)+(k-1)min\{\delta(\infty;f),\delta(\infty;g)\}=3k+\frac{2m-1}{2m+1}>3k$. *Choosing* $a_i=1$, $1\leq i\leq k$; $c_1=c_3=\ldots=c_{k-1}=2\pi i$; $c_2=c_4=\ldots=c_k=\pi i$; $L(f)$, $L(g)$ *share* $(1,\infty)$ *and* $L(f)=L(g)=2ape^{2mz}$.

Example 9. *Let* $k=2p$, *where* $p\in\mathbb{N}$. *Also let* $f(z)=\frac{e^{2mz}(e^z+a)}{k}$, $g(z)=\frac{(e^z-a)}{ke^{2(m+1)z}}$, $m\in\mathbb{N}$, $a\in\mathbb{C}\setminus\{0\}$. *Then* $\delta(0;f)+\delta(0;g)+k\delta(\infty;f)+k\delta(\infty;g)+(k-1)min\{\delta(\infty;f),\delta(\infty;g)\}=3k+\frac{4m^2+2m-1}{(2m+1)(2m+2)}>3k$. *Choosing* $a_1=a_3=\ldots=a_{k-1}=1$; $a_2=a_4=\ldots=a_k=-1$; $c_1=c_3=\ldots=c_{k-1}=2\pi i$; $c_2=c_4=\ldots=c_k=\pi i$; $L(f)=e^{(2m+1)z}$, $L(g)=\frac{1}{e^{(2m+1)z}}$ *share* $(1,\infty)$ *and* $L(f).L(g)=1$.

Following examples satisfy Theorem 2.

Example 10. *Let* $f(z)=\frac{e^{4mz}}{e^z+1}$, $g(z)=-\frac{e^{(4m+1)z}}{e^z+1}$, $m\geq 2$ *be an integer. Then* f *and* g *share* $(0,\infty)$ *and* $2\max\{\delta(0;f),\delta(0;g)\}+2\delta(\infty;f)+2\delta(\infty;g)+\min\{\delta(\infty;f),\delta(\infty;g)\}=6+\frac{16m^2-16m-3}{4m(4m+1)}>6$. *Choosing* $a_1=1$, $a_2=-1$, $c_1=\pi i$, $c_2=\frac{\pi i}{2}$, *we see that* $L(f)$, $L(g)$ *share* $(1,\infty)$ *and* $L(f)=\frac{(1+i)e^{(4m+1)z}}{(1-e^z)(ie^z+1)}\neq\Delta_c f$ *and* $L(g)=\frac{(1+i)e^{(4m+1)z}}{(1-e^z)(ie^z+1)}\neq\Delta_c g$ *for any* $c\in\mathbb{C}\setminus\{0\}$. *Here* $L(f)\equiv L(g)$.

Example 11. *Let* $f(z)=\frac{1}{e^{pz}(e^z+1)}$, $g(z)=-\frac{1}{e^{(p+1)z}(e^z+1)}$, $p=4m+3$, $m\in\mathbb{N}$. *Then* f *and* g *share* $(0,\infty)$ *and* $2\max\{\delta(0;f),\delta(0;g)\}+2\delta(\infty;f)+2\delta(\infty;g)+\min\{\delta(\infty;f),\delta(\infty;g)\}=2+\frac{2p}{p+1}+\frac{2(p+1)}{p+2}+\frac{p}{p+1}=6+\frac{p^2-2p-6}{(p+1)(p+2)}>6$. *Here* $\Delta_{\frac{\pi i}{2}}f$, $\Delta_{\frac{\pi i}{2}}g$ *share* $(1,\infty)$ *and* $\Delta_{\frac{\pi i}{2}}f=\frac{(i-1)}{e^{pz}(e^z+1)(ie^z+1)}=\Delta_{\frac{\pi i}{2}}g$.

Following two examples show that respectively in Theorems 1 and 2 the sharing of the value 1 can not be omitted.

Example 12. *Let* $f(z) = e^{mz}(e^z + 1)$, $g(z) = e^{mz}(e^z - 1)$, $m \geq 2$ *be an integer and choose* $c = \pi i$. *Clearly* f, g *does not share* $(0, \infty)$ *but they fail to share the value* 1. *Here though* $\delta(0; f) + \delta(0; g) + \delta(\infty; f) + \delta(\infty; g) > 3$ *but neither* $f(z + c) \equiv g(z + c)$ *nor* $f(z + c).g(z + c) \equiv 1$.

Example 13. *Let* $f(z) = \frac{e^{6z}}{e^z+1}$, $g(z) = \frac{1}{e^{5z}(e^z+1)}$. *Clearly* f, g *share* $(0, \infty)$. *Choosing* $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}$, $c_1 = 2\pi i$, $c_2 = \pi i$, *we have* $L(f) = \frac{e^{6z}}{1-e^{2z}}$, $L(g) = \frac{1}{e^{4z}(e^{2z}-1)}$. *Clearly* $L(f)$, $L(g)$ *fail to share the value* 1. *Here though* $2\max\{\delta(0; f), \delta(0; g)\} + 2\delta(\infty; f) + 2\delta(\infty; g) + \min\{\delta(\infty; f), \delta(\infty; g)\} > 6$ *but neither* $L(f) \equiv L(g)$ *nor* $L(f).L(g) \equiv 1$.

The next example shows that for $k = 1$, under the same situation as in Theorem 1 sharing of $(1, 2)$ can not be relaxed to $(1, 0)$.

Example 14. *Let* $f(z) = e^z$, $g(z) = \frac{2e^z-1}{e^{2z}}$. *Then for* $c = \pi i$, $f(z + c)$, $g(z + c)$ *share* $(1, 0)$. *Also* $\delta(0; f) + \delta(0; g) + \delta(\infty; f) + \delta(\infty; g) > 3$. *But neither* $f(z + c) \equiv g(z + c)$ *nor* $f(z + c).g(z + c) \equiv 1$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[6] Let* F *and* G *be two non constant meromorphic functions such that they share* $(1, 2)$. *Then one of following cases holds:*

(i) $T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + S(r, F) + S(r, G)$

and the same inequality holds for $T(r, G)$.
(ii) $F \equiv G$.
(iii) $F.G \equiv 1$.

Lemma 2. *[1, 2]Let* f *be a meromorphic function of finite order and let* c *be a non-zero complex number. Then, we have*

$$m\left(r, \frac{f(z + c)}{f(z)}\right) = S(r, f).$$

The following basic inequalities, by [[3], lemma 8.3] are frequently used in value distribution for differences.

Lemma 3. *[3] Let* f *be a non-constant meromorphic function of finite order and* $c \in \mathbb{C}$. *Then*

$$N\left(r, \frac{1}{f(z + c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r, f),$$

$$N\left(r, f(z+c)\right) \leq N\left(r, f(z)\right) + S(r,f),$$

$$\overline{N}\left(r, \frac{1}{f(z+c)}\right) \leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r,f)$$

and

$$\overline{N}\left(r, f(z+c)\right) \leq \overline{N}\left(r, f(z)\right) + S(r,f).$$

Henceforth unless otherwise stated for two non-constant meromorphic functions f and g we denote by $F = L(f)$ and $G = L(g)$.

Lemma 4. *Let f be a non-constant meromorphic functions of finite order and let $L(f)$ $(\not\equiv 0)$ be defined as in previous. Then*

$$N\left(r, \frac{1}{L(f)}\right) \leq T(r, L(f)) - T(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f)$$

and

$$N\left(r, \frac{1}{L(f)}\right) \leq N\left(r, \frac{1}{f}\right) + (k-1)N(r, f(z)) + S(r,f).$$

Proof. Using Lemma 2,

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{L(f)}{f}\right) + m\left(r, \frac{1}{L(f)}\right) + O(1) \leq m\left(r, \frac{1}{L(f)}\right) + S(r,f).$$

In view of above, using the First Fundamental Theorem, we have

$$\begin{aligned} N\left(r, \frac{1}{L(f)}\right) &\leq T(r, L(f)) - m\left(r, \frac{1}{L(f)}\right) + S(r,f) \qquad (2.1)\\ &\leq T(r, L(f)) - m\left(r, \frac{1}{f}\right) + S(r,f) \\ &\leq T(r, L(f)) - T(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned}$$

Now,

$$\begin{aligned} T(r, L(f)) &\leq m\left(r, \frac{L(f)}{f}\right) + m(r,f) + N(r, L(f)) \\ &\leq m(r,f) + kN(r,f) + S(r,f) \\ &\leq T(r,f) + (k-1)N(r,f) + S(r,f). \end{aligned}$$

From (2.1)

$$N\left(r, \frac{1}{L(f)}\right) \leq N\left(r, \frac{1}{f}\right) + (k-1)N(r, f(z)) + S(r,f).$$

□

Lemma 5. *Let f and g be two non-constant meromorphic functions of finite order. Then $S(r, L(f))$ can be replaced by $S(r, f)$ and $S(r, L(g))$ can be replaced by $S(r, g)$.*

Proof.

$$\begin{aligned} T(r, L(f)) &= T\left(r, \sum_{j=1}^{k} a_j f(z+c_j)\right) \\ &\leq \sum_{j=1}^{k} T(r, f(z+c_j)) \\ &= \sum_{j=1}^{k} m(r, f(z+c_j)) + \sum_{j=1}^{k} N(r, f(z+c_j)) \\ &\leq \sum_{j=1}^{k} m\left(r, \frac{f(z+c_j)}{f(z)}\right) + km(r, f) + \sum_{j=1}^{k} N(r, f(z+c_j)). \end{aligned}$$

Using Lemmas 2 and 3 we get

$$T(r, L(f)) \leq km(r, f) + kN(r, f) + S(r, f) = kT(r, f) + S(r, f).$$

So $S(r, L(f))$ can be replaced by $S(r, f)$. Similarly $S(r, L(g))$ can be replaced by $S(r, g)$. □

3 Proofs of the theorems

Proof of Theorem 1. Let us assume that (i) of Lemma 1 holds. Now using Lemmas 2, 3, 4 and 5 we get

$$\begin{aligned} & T(r, f) \\ \leq\ & T(r, L(f)) - N\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{f}\right) + S(r, f) \\ \leq\ & N_2(r, 0; L(f)) + N_2(r, 0; L(g)) + N_2(r, \infty; L(f)) + N_2(r, \infty; L(g)) - N\left(r, \frac{1}{L(f)}\right) \\ & + N\left(r, \frac{1}{f}\right) + S(r, f) + S(r, g) \\ \leq\ & N_2(r, 0; L(f)) - N(r, 0; L(f)) + N_2(r, 0; L(g)) + N(r, \infty; L(f)) + N(r, \infty; L(g)) \\ & + N(r, 0; f) + S(r, f) + S(r, g) \\ \leq\ & N(r, 0; L(g)) + N(r, \infty; L(f)) + N(r, \infty; L(g)) + N(r, 0; f) + S(r, f) + S(r, g) \\ \leq\ & N(r, 0; g) + (k-1)N(r, \infty; g) + kN(r, \infty; f) + kN(r, \infty; g) + N(r, 0; f) + S(r, f) \\ & + S(r, g). \end{aligned}$$

i.e.,

$$\begin{aligned} T(r, f) \leq\ & N(r, 0; f) + N(r, 0; g) + kN(r, \infty; f) + kN(r, \infty; g) + (k-1)N(r, \infty; g) \\ & + S(r, f) + S(r, g). \end{aligned} \tag{3.1}$$

Similarly we can obtain

$$\begin{aligned} T(r,g) \;\leq\;& N(r,0;f)+N(r,0;g)+kN(r,\infty;f)+kN(r,\infty;g)+(k-1)N(r,\infty;f) \\ & +S(r,f)+S(r,g). \end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we have for $\varepsilon > 0$,

$$\begin{aligned} & T(r) \\ \leq\;& \left(1-\delta(0;f)+\frac{\varepsilon}{5}\right)T(r)+\left(1-\delta(0;g)+\frac{\varepsilon}{5}\right)T(r)+k\left(1-\delta(\infty;f)+\frac{\varepsilon}{5k}\right)T(r) \\ & +k\left(1-\delta(\infty;g)+\frac{\varepsilon}{5k}\right)T(r)+(k-1)\left(1-min\{\delta(\infty;f),\delta(\infty;g)\}+\frac{\varepsilon}{5(k-1)}\right)T(r) \\ & +S(r). \end{aligned}$$

i.e.,

$$\delta(0;f)+\delta(0;g)+k\delta(\infty;f)+k\delta(\infty;g)+(k-1)min\{\delta(\infty;f),\delta(\infty;g)\}-\varepsilon \leq 3k.$$

Since $\varepsilon > 0$ is arbitrary, so we have

$$\delta(0;f)+\delta(0;g)+k\delta(\infty;f)+k\delta(\infty;g)+(k-1)min\{\delta(\infty;f),\delta(\infty;g)\} \leq 3k,$$

which contradicts (1.1). Thus from Lemma 1, we can conclude that either $L(f) \equiv L(g)$ or $L(f).L(g) \equiv 1$. □

Proof of Theorem 2. Noting that here $N(r,0;f) = N(r,0;g)$, in view of (3.1) and (3.2) we get for $\varepsilon > 0$,

$$\begin{aligned} & T(r) \\ \leq\;& 2\left(1-\max\{\delta(0;f),\delta(0;g)\}+\frac{\varepsilon}{8}\right)T(r)+k\left(1-\delta(\infty;f)+\frac{\varepsilon}{4k}\right)T(r)+k\,(1-\delta(\infty;g) \\ & +\frac{\varepsilon}{4k}\Big)T(r)+(k-1)\left(1-min\{\delta(\infty;f),\delta(\infty;g)\}+\frac{\varepsilon}{4(k-1)}\right)T(r)+S(r). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$2\max\{\delta(0;f),\delta(0;g)\}+k\delta(\infty;f)+k\delta(\infty;g)+(k-1)min\{\delta(\infty;f),\delta(\infty;g)\} \leq 3k,$$

which contradicts (1.2). Hence from Lemma 1 we get either $L(f) \equiv L(g)$ or $L(f).L(g) \equiv 1$. □

4 Remark and an open question

For the case $k \geq 2$, from the following examples we see that, the conclusions of Theorems 1, 2 may occur even if the expression on the left hand side of the inequalities (1.1) and (1.2) is less than or equal to the lower bound of (1.1) and (1.2).

Example 15. *Let* $f(z) = -\frac{1}{e^{3z}(e^z+1)}$, $g(z) = \frac{1}{e^{4z}(e^z+1)}$. *Clearly* f *and* g *share* $(0,\infty)$. *Then for* $a_1 = 1$, $a_2 = -1$; $c_1 = 2\pi i$, $c_2 = \frac{\pi i}{2}$, $L(f) = \frac{(i-1)}{e^{3z}(e^z+1)(ie^z+1)} = L(g)$. *We see that* $L(f)$, $L(g)$ *share* $(1,\infty)$ *and* $L(f) \equiv L(g)$. *But* $5 < 2\max\{\delta(0;f), \delta(0;g)\} + 2\delta(\infty;f) + 2\delta(\infty;g) + \min\{\delta(\infty;f), \delta(\infty;g)\} < 6$.

Example 16. *Let* $f(z) = e^z+1$, $g(z) = \frac{e^z+1}{e^{2z}}$. *Clearly* f *and* g *share* $(0,\infty)$. *Then for* $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{2}$; $c_1 = 2\pi i$, $c_2 = \pi i$, $L(f)(= e^z)$, $L(g)\left(= \frac{1}{e^z}\right)$ *share* $(1,\infty)$ *and* $L(f).L(g) \equiv 1$. *But* $2\max\{\delta(0;f), \delta(0;g)\} + 2\delta(\infty;f) + 2\delta(\infty;g) + \min\{\delta(\infty;f), \delta(\infty;g)\} = 6$.

In view of the above examples it is evident that the conditions (1.1) and (1.2) in theorems 1 and 2 respectively are not sharp and consequently the same could further be weakened. But unfortunately at present we do not have sufficient resources to tackle this problem. Considering this the following question is inevitable:
Can conditions (1.1) and (1.2) in Theorems 1 and 2 be further relaxed for the case $k \geq 2$ to make them sharp?

5 Observation and a conjecture

Example 17. *Let* $f(z) = \frac{3e^z-1}{4e^{2z}(1-e^z)}$, $g(z) = \frac{e^{3z}}{1-e^z}$. *Then for* $a_1 = 1$, $a_2 = 1$; $c_1 = 2\pi i$, $c_2 = \pi i$, $L(f) = \frac{3e^{2z}-1}{2e^{2z}(1-e^{2z})}$, $L(g) = \frac{2e^{4z}}{1-e^{2z}}$. *Clearly* $L(f)$, $L(g)$ *share* $(1,\infty)$ *and* $\delta(0;f)+\delta(0;g)+2\delta(\infty;f)+2\delta(\infty;g)+\min\{\delta(\infty;f), \delta(\infty;g)\} = 5$. *But neither* $L(f) \equiv L(g)$ *nor* $L(f).L(g) \equiv 1$.

In view of the above example we conjecture that the lower bound in (1.1) for $k = 2$ could be reduced up to 5 and in general for $k \geq 3$, it might be $3k - 1$. But again we did not succeed to prove it.

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