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SUBCLASSES OF MULTIVALENT NON-BAZILEVIČ FUNCTIONS DEFINED WITH HIGHER ORDER DERIVATIVES

Mohamed K. AOUF[1], Teodor BULBOACĂ[2] and Tamer M. SEOUDY[*,3]

Abstract

By making use of the principle of subordination, we introduce a certain class of multivalent non-Bazilevič functions with higher order. Also, we obtain subordination property, inclusion result, and inequality properties of this class. The results presented here would provide extensions of those given in earlier works.

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1 Introduction

Let $\mathcal{A}(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k,\ z \in \mathbb{U} \quad (p,n \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. We write $\mathcal{A}(p,1) =: \mathcal{A}(p)$ and $\mathcal{A}(1,1) =: \mathcal{A}$.

If f and g are analytic functions in $\mathbb{U}$, we say that f is *subordinate* to g, or g is *superordinate* to f, written symbolically $f \prec g$ or $f(z) \prec g(z)$, if there exists

[1]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, email: mkaouf127@yahoo.com
[2]Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania, email: bulboaca@math.ubbcluj.ro
[3*] *Corresponding author*, Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt, email: tms00@fayoum.edu.eg; Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah, Saudi Arabia, email: tmsaman@uqu.edu.sa

a *Schwarz function* ω, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$ then we have the following equivalence (see [7] and [11]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the first order differential subordination

$$\phi\left(p(z), zp'(z); z\right) \prec h(z), \tag{2}$$

then p is said to be a *solution of the differential subordination* (2). The univalent function q is called a *dominant of the solutions of the differential subordination* (2) if $p(z) \prec q(z)$ for all p satisfying (2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all the dominants q of (2) is called the *best dominant*.

Upon differentiating q-times both sides of (1) we obtain

$$f^{(q)}(z) = \delta(p,q)z^{p-q} + \sum_{k=p+n}^{\infty} \delta(k,q)a_k z^{k-q},\ z \in \mathbb{U},$$

where

$$\delta(p,q) = \frac{p!}{(p-q)!} \quad \left(p \in \mathbb{N},\ q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ p \geq q\right).$$

Several research has investigated higher order derivatives of multivalent functions, see, for example [2], [3], [4] and [9].

Now we introduce the class $\mathcal{N}_p^n(\lambda, \alpha, q; A, B)$, defined as follows:

Definition 1. *A function $f \in \mathcal{A}(p,n)$, with $f^{(q)}(z) \neq 0$ for all $z \in \dot{\mathbb{U}} := \mathbb{U} \setminus \{0\}$, is said to be the class $\mathcal{N}_p^n(\lambda, \alpha, q; A, B)$ if it satisfies the subordination condition*

$$(1+\lambda)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \lambda \frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz},$$

where all the powers are principal values, and throughout the paper, unless otherwise mentioned, the parameters λ, α, p, n, q, A, B are constrained as follows:

$$\lambda \in \mathbb{C},\ 0 < \alpha < 1,\ p, n \in \mathbb{N},\ q \in \mathbb{N}_0,\ p > q,\ \text{and } -1 \leq B < A \leq 1.$$

Furthermore, let denote $\mathcal{N}_p^n(\lambda, \alpha, q; 1-2\sigma, -1) =: \mathcal{N}_p^n(\lambda, \alpha, q; \sigma)$, that is $f \in \mathcal{N}_p^n(\lambda, \alpha, q; \sigma)$ if and only if $f \in \mathcal{A}(p,n)$ satisfies the condition

$$\operatorname{Re}\left\{(1+\lambda)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \lambda \frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}\right\} > \sigma,\ z \in \mathbb{U},$$

with $0 \leq \sigma < 1$.

Remark that the above defined classes generalize and extend many others defined by different authors, and we emphasize the following well-known special cases:

(i) $\mathcal{N}_p^n(\lambda, \alpha, 0; A, B) =: \mathcal{N}_p^n(\lambda, \alpha; A, B)$, see Aouf and Seoudy [5];

(ii) $\mathcal{N}_p^n(\lambda, \alpha, 0; \sigma) =: \mathcal{N}_p^n(\lambda, \alpha; \sigma)$, see Aouf and Seoudy [5];

(iii) $\mathcal{N}_1^1(\lambda, \alpha, 0; A, B) =: \mathcal{N}(\lambda, \alpha; A, B)$, see Wang el. at [15];

(iv) $\mathcal{N}_1^1(-1, \alpha, 0; \sigma) =: \mathcal{N}(\alpha; \sigma)$, with $0 \le \sigma < 1$, where $\mathcal{N}(\alpha; \sigma)$ is the class of non-Bazilevič functions of order σ which was defined and studied by Tuneski and Darus [14];

(v) $\mathcal{N}_1^1(-1, \alpha, 0; 1, -1) =: \mathcal{N}(\alpha)$, where $\mathcal{N}(\alpha)$ is the class of non-Bazilevič functions which was introduced and studied by Obradović [12].

2 Preliminary results

In order to establish our main results, we need the following lemmas.

Lemma 1. *[8, 11] Let the function h be a convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function k analytic in $\mathbb{U}$ is given by*

$$k(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots,\ z \in \mathbb{U}. \tag{3}$$

If

$$k(z) + \frac{zk'(z)}{\gamma} \prec h(z) \quad (\operatorname{Re}\gamma \ge 0,\ \gamma \ne 0), \tag{4}$$

then

$$k(z) \prec \psi(z) = \frac{\gamma}{n} z^{-\frac{\gamma}{n}} \int_0^z t^{\frac{\gamma}{n}-1} h(t)\, dt \prec h(z),$$

and ψ is the best dominant of (4).

Lemma 2. *[10] Let* F *be convex in $\mathbb{U}$. If $f, g \in \mathcal{A}$, with $f(z) \prec \mathrm{F}(z)$ and $g(z) \prec \mathrm{F}(z)$, then $\gamma f(z) + (1-\gamma) g(z) \prec \mathrm{F}(z)$, where $0 \le \gamma \le 1$.*

Lemma 3. *[13] Let*

$$f(z) = \sum_{k=1}^{\infty} a_k z^k,\ z \in \mathbb{U},$$

be analytic in $\mathbb{U}$ and

$$g(z) = \sum_{k=1}^{\infty} b_k z^k,\ z \in \mathbb{U},$$

be convex (univalent) in $\mathbb{U}$. If $f(z) \prec g(z)$, then

$$|a_k| \le |b_1|,\ k \in \mathbb{N}.$$

Lemma 4. *[16] For real or complex numbers a, b, c with $c \neq 0, -1, -2, \ldots,$ the next relations hold:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b; c; z),\ z \in \mathbb{C} \setminus (1, +\infty)$$

$$\textit{for } \operatorname{Re} c > \operatorname{Re} b > 0;$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right),\ z \in \mathbb{C} \setminus (1, +\infty);$$

$${}_2F_1\left(1, 1; 2; \frac{z}{z+1}\right) = \frac{(1+z)\ln(1+z)}{z},\ z \in \mathbb{C} \setminus (-\infty, -1) \cup \{0\}.$$

Putting $z = 1$ in the above last identity we get

$${}_2F_1\left(1, 1; 2; \frac{1}{2}\right) = 2\ln 2. \tag{5}$$

In the present paper we obtain subordination properties, inclusion results, distortion theorems and inequality properties of the class $\mathcal{N}_p^n(\lambda, \alpha, q; A, B)$. The results presented here would provide generalizations and extensions of those given in many earlier works.

3 Main results

We begin by presenting our first subordination property given by the below theorem.

Theorem 1. *If $f \in \mathcal{N}_p^n(\lambda, \alpha, q; A, B)$, with $\operatorname{Re}\lambda > 0$, then*

$$\left(\frac{\delta(p, q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} \prec Q(z) \prec \frac{1+Az}{1+Bz}, \tag{6}$$

where the function Q given by

$$Q(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1, 1; \dfrac{(p-q)\alpha}{\lambda n} + 1; \dfrac{Bz}{1+Bz}\right), & B \neq 0, \\ 1 + \dfrac{(p-q)\alpha}{(p-q)\alpha + \lambda n}Az, & B = 0, \end{cases}$$

is the best dominant of (6). Furthermore,

$$\operatorname{Re}\left(\frac{\delta(p, q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} > \rho,\ z \in \mathbb{U}, \tag{7}$$

where

$$\rho = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\, {}_2F_1\left(1, 1; \dfrac{(p-q)\alpha}{\lambda n} + 1; \dfrac{B}{B-1}\right), & \textit{if} \quad B \neq 0, \\ 1 - \dfrac{(p-q)\alpha}{(p-q)\alpha + \lambda n}A, & \textit{if} \quad B = 0. \end{cases}$$

The estimate (7) is the best possible.

Proof. Let $f \in \mathcal{N}_p^n(\lambda, \alpha, q; A, B)$ and define

$$g(z) = \left(\frac{\delta(p,q) z^{p-q}}{f^{(q)}(z)} \right)^{\alpha}, \ z \in \mathbb{U}. \tag{8}$$

Then, the function g is analytic in $\mathbb{U}$ and has the form (3). Taking the derivatives in the both sides of (8) we get

$$\begin{aligned}(1+\lambda)\left(\frac{\delta(p,q) z^{p-q}}{f^{(q)}(z)} \right)^{\alpha} - \lambda \frac{z f^{(1+q)}(z)}{(p-q) f^{(q)}(z)} \left(\frac{\delta(p,q) z^{p-q}}{f^{(q)}(z)} \right)^{\alpha} \\ = g(z) + \frac{\lambda}{(p-q)\alpha} z g'(z), \ z \in \mathbb{U}.\end{aligned} \tag{9}$$

Since $f \in \mathcal{N}_p^n(\lambda, \alpha, q; A, B)$, from (9) it follows that

$$g(z) + \frac{\lambda}{(p-q)\alpha} z g'(z) \prec \frac{1+Az}{1+Bz},$$

and using Lemma 1 for $\gamma = \dfrac{(p-q)\alpha}{\lambda}$ we deduce that

$$\left(\frac{\delta(p,q) z^{p-q}}{f^{(q)}(z)} \right)^{\alpha} \prec Q(z) = \frac{(p-q)\alpha}{\lambda n} z^{-\frac{(p-q)\alpha}{\lambda n}} \int_0^z t^{\frac{(p-q)\alpha}{\lambda n}-1} \frac{1+At}{1+Bt}\, dt$$
$$= \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1, 1; \dfrac{(p-q)\alpha}{\lambda n} + 1; \dfrac{Bz}{1+Bz}\right), & \text{if} \quad B \neq 0, \\ 1 + \dfrac{(p-q)\alpha}{(p-q)\alpha + \lambda n} A z, & \text{if} \quad B = 0, \end{cases} \tag{10}$$

where we have made a change of variables followed by the use of identities in Lemma 4 with $a = 1$, $b = \dfrac{(p-q)\alpha}{\lambda n}$ and $c = b+1$, and this proves the assertion (6).

Next, in order to prove the assertion (7) it is sufficient to show that

$$\inf \{\operatorname{Re} Q(z) : z \in \mathbb{U}\} = Q(-1).$$

Indeed, for $|z| \le r < 1$ we have

$$\operatorname{Re} \frac{1+Az}{1+Bz} \ge \frac{1-Ar}{1-Br}.$$

Setting

$$G(z,s) = \frac{1+Asz}{1+Bsz}$$

and

$$dv(s) = \frac{(p-q)\alpha}{\lambda n} s^{\frac{(p-q)\alpha}{\lambda n}-1}\, ds, \quad (0 \le s \le 1),$$

which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z)=\int_0^1 G(z,s)\,dv(s),$$

and therefore

$$\operatorname{Re} Q(z) \geq \int_0^1 \frac{1-Asr}{1-Bsr}\,dv(s)=Q(-r),\ |z|\leq r<1.$$

Letting $r\to 1^-$ in the above inequality we obtain the assertion (7). Finally, the estimate (7) is the best possible because the function Q is the best dominant of (6). □

Using the elementary inequality $\operatorname{Re}\left(w^{\frac{1}{m}}\right)\geq(\operatorname{Re} w)^{\frac{1}{m}}$ for $\operatorname{Re} w>0$ and $m\geq 1$ in Theorem 1 we obtain the next result:

Corollary 1. *If $f\in\mathcal{N}_p^n(\lambda,\alpha,q;A,B)$, with $\operatorname{Re}\lambda>0$ and $B\neq 0$, then*

$$\operatorname{Re}\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}>\left[\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\frac{(p-q)\alpha}{\lambda n}+1;\frac{B}{B-1}\right)\right]^{\frac{1}{\alpha}},\quad z\in\mathbb{U},$$

and the estimate is the best possible.

Corollary 2. *If $f\in\mathcal{N}_p^n(\lambda,\alpha,q;A^*,B)$, with $\operatorname{Re}\lambda>0$, $-1\leq B<A^*\leq 1$, and $B\neq 0$, where A^* is given by*

$$A^*=\frac{B\,{}_2F_1\left(1,1;\dfrac{(p-q)\alpha}{\lambda n}+1;\dfrac{Bz}{1+Bz}\right)}{{}_2F_1\left(1,1;\dfrac{(p-q)\,\alpha}{\lambda n}+1;\dfrac{Bz}{1+Bz}\right)+B-1},\tag{11}$$

then

$$\operatorname{Re}\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}>0,\ z\in\mathbb{U},$$

and the result is sharp.

Proof. In view of Corollary 1, if $f\in\mathcal{N}_p^n(\lambda,\alpha,q;A^*,B)$ that is

$$(1+\lambda)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}-\lambda\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}\prec\frac{1+A^*z}{1+Bz},$$

then

$$\operatorname{Re}\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}>\left[\frac{A^*}{B}+\left(1-\frac{A^*}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\frac{(p-q)\alpha}{\lambda n}+1;\frac{B}{B-1}\right)\right]^{\frac{1}{\alpha}},\quad z\in\mathbb{U}.\tag{12}$$

Substituting the value of A^* given by (11) in the right hand side of the inequality (12) we obtain

$$\operatorname{Re} \frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)} > 0, \ z \in \mathbb{U},$$

which proves our result. $\square$

For the special case $A = 1 - 2\sigma$, with $0 \le \sigma < 1$ and $B = -1$, Theorem 1 reduces to the following result:

Corollary 3. *If $f \in \mathcal{N}_p^n(\lambda, \alpha, q; \sigma)$, with $\operatorname{Re} \lambda > 0$ and $0 \le \sigma < 1$, then*

$$\operatorname{Re} \left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)} \right)^{\alpha} > \sigma + (1-\sigma) \left[{}_2F_1 \left(1, 1; \frac{(p-q)\alpha}{\lambda n} + 1; \frac{1}{2} \right) - 1 \right], \ z \in \mathbb{U},$$

and the result is sharp.

Putting $q = 0$ and $p = 1$ in Corollary 3 we get:

Corollary 4. *If $f \in \mathcal{N}_1^n(\lambda, \alpha; \sigma)$, with $\operatorname{Re} \lambda > 0$ and $0 \le \sigma < 1$, then*

$$\operatorname{Re} \left(\frac{z}{f(z)} \right)^{\alpha} > \sigma + (1-\sigma) \left[{}_2F_1 \left(1, 1; \frac{\alpha}{\lambda n} + 1; \frac{1}{2} \right) - 1 \right], \ z \in \mathbb{U},$$

and the result is sharp.

Remark 1. *(i) Our result of Corollary 4 with $n = 1$ is an improvement of the result obtained by Wang et al. [15, Corollary 1];*

(ii) Our result of Corollary 4 is an improvement of the result obtained by Alamoushi and Darus [1, Corollary 10] for $\beta = 0$.

Putting $q = j - 1$, with $1 \le j \le p$ in Corollary 3 we obtain the next result:

Corollary 5. *If $f \in \mathcal{A}(p,n)$ satisfies the condition*

$$\operatorname{Re} \left[(1+\lambda) \left(\frac{\delta(p,j-1)z^{p-j+1}}{f^{(j-1)}(z)} \right)^{\alpha} - \frac{\lambda z f^{(j)}(z)}{(p-j+1) f^{(j-1)}(z)} \left(\frac{\delta(p,j-1)z^{p-j+1}}{f^{(j-1)}(z)} \right)^{\alpha} \right]$$

$> \sigma$, $z \in \mathbb{U}$, with $\operatorname{Re} \lambda > 0$ and $0 \le \sigma < 1$, then

$$\operatorname{Re} \left(\frac{\delta(p,j-1)z^{p-j+1}}{f^{(j-1)}(z)} \right)^{\alpha} > \sigma + (1-\sigma) \left[{}_2F_1 \left(1, 1; \frac{(p-j+1)\alpha}{\lambda n} + 1; \frac{1}{2} \right) - 1 \right] \ z \in \mathbb{U},$$

and the result is sharp.

Putting $p = n = 1$ and $\lambda = \alpha = \dfrac{1}{2}$ in Corollary 4, and using the relation (5) we have:

Corollary 6. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(3\sqrt{\frac{z}{f(z)}}-\frac{zf'(z)}{f(z)}\sqrt{\frac{z}{f(z)}}\right)>2\sigma,\ z\in\mathbb{U},$$

with $0\le\sigma<1$, then

$$\operatorname{Re}\sqrt{\frac{z}{f(z)}}>\sigma+(1-\sigma)\left(2\ln 2-1\right),\ z\in\mathbb{U},$$

and the result is sharp.

Theorem 2. *If $f\in\mathcal{N}_p^n(\alpha,q;\eta)$, with $\lambda>0$ and $0\le\eta<1$, then $f\in\mathcal{N}_p^n(\lambda,\alpha,q;\eta)$ for $|z|<R$, where*

$$R=\left(\frac{\sqrt{(p-q)^2\alpha^2+n^2\lambda^2}-n\lambda}{(p-q)\alpha}\right)^{\frac{1}{n}}. \tag{13}$$

The bound R is the best possible.

Proof. For $f\in\mathcal{N}_p^n(\alpha,q;\eta)$, let define the function u by

$$\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}=\eta+(1-\eta)u(z),\ z\in\mathbb{U}. \tag{14}$$

Then, the function u is of the form (3), and has a positive real part in $\mathbb{U}$. Differentiating (14) we have

$$\frac{1}{1-\eta}\left[(1+\lambda)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}-\lambda\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}-\eta\right]$$
$$=u(z)+\frac{\lambda}{(p-q)\alpha}zu'(z),\ z\in\mathbb{U}, \tag{15}$$

and using the following well-known estimate (see [6, Theorem 1])

$$\frac{|zu'(z)|}{\operatorname{Re}u(z)}\le\frac{2nr^n}{1-r^{2n}},\ |z|=r<1,$$

from (15) we deduce that

$$\frac{1}{1-\eta}\left[(1+\lambda)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}-\lambda\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}-\eta\right]$$
$$\ge\left(1-\frac{2\lambda nr^n}{(p-q)\alpha\left(1-r^{2n}\right)}\right)\operatorname{Re}u(z),\ |z|=r<1. \tag{16}$$

It is easily seen that the right-hand side of (16) is positive provided that $r<R$, where R is given by (13).

In order to show that the bound R is the best possible, we consider the function $f \in \mathcal{A}(p,n)$ defined by

$$\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} = \eta + (1-\eta)\frac{1+z^n}{1-z^n},\ z \in \mathbb{U}.$$

Since

$$\frac{1}{1-\eta}\left[(1+\lambda)\left(\frac{\delta(p,q)\,z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \lambda\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \eta\right]$$
$$= \frac{(p-q)\alpha\ \left(1-z^{2n}\right)+2\lambda nz^n}{(p-q)\alpha\left(1-z^n\right)^2} = 0$$

for $z = R\exp\left(\frac{\pi i}{n}\right)$, we conclude that the bound is the best possible. □

Taking $q = 0$ in Theorem 2 we get the following special case:

Corollary 7. *If $f \in \mathcal{A}(p,n)$, with $\lambda > 0$ and $0 \le \eta < 1$, satisfy the inequality*

$$\operatorname{Re}\left(\frac{z^p}{f(z)}\right)^{\alpha} > \eta,\ z \in \mathbb{U},$$

then

$$\operatorname{Re}\left[(1+\lambda)\left(\frac{z^p}{f(z)}\right)^{\alpha} - \frac{\lambda zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\alpha}\right] > \eta,\ z \in \mathbb{U},$$

for $|z| < R^*$, where

$$R^* = \left(\frac{\sqrt{p^2\alpha^2+n^2\lambda^2}-n\lambda}{p\alpha}\right)^{\frac{1}{n}}.$$

The bound R^* is the best possible.

Theorem 3. *If $\lambda_2 \ge \lambda_1 \ge 0$ and $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, then*

$$\mathcal{N}_p^n\left(\lambda_2,\alpha,q;A_2,B_2\right) \subset \mathcal{N}_p^n\left(\lambda_1,\alpha,q;A_1,B_1\right). \tag{17}$$

Proof. If we let $f \in \mathcal{N}_p^n\left(\lambda_2,\alpha,q;A_2,B_2\right)$, then we have

$$(1+\lambda_2)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \lambda_2\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} \prec \frac{1+A_2z}{1+B_2z}.$$

Since $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, we easily find that

$$(1+\lambda_2)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \lambda_2\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}$$
$$\prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}, \tag{18}$$

that is $f \in \mathcal{N}_p^n(\lambda_2, \alpha, q; A_1, B_1)$. Thus, the assertion of Theorem 3 holds for $\lambda_2 = \lambda_1 \geq 0$.

Suppose that $\lambda_2 > \lambda_1 \geq 0$. From (18) and Theorem 1, it follows that $f \in \mathcal{N}_p^n(0, \alpha, q; A_1, B_1)$, that is

$$\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} \prec \frac{1+A_1 z}{1+B_1 z}. \tag{19}$$

At the same time, we have

$$\begin{aligned}(1+\lambda_1)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \lambda_1\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} \\ = \left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} \\ +\frac{\lambda_1}{\lambda_2}\left[(1+\lambda_2)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \lambda_2\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha}\right], \ z \in \mathbb{U}.\end{aligned} \tag{20}$$

Moreover, since $0 \leq \dfrac{\lambda_1}{\lambda_2} < 1$ and the function $\dfrac{1+A_1 z}{1+B_1 z}$, with $-1 \leq B_1 < A_1 \leq 1$ is convex in $\mathbb{U}$, combining (18), (19), (20) and Lemma 2 we deduce that

$$(1+\lambda_1)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \lambda_1\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} \prec \frac{1+A_1 z}{1+B_1 z},$$

that is, $f \in \mathcal{N}_p^n(\lambda_1, \alpha, q; A_1, B_1)$, and the conclusion (17) of our theorem is proved. $\square$

Theorem 4. *If $f \in \mathcal{N}_p^n(\lambda, \alpha, q; A, B)$, with $\lambda > 0$ and $-1 \leq B < A \leq 1$, then*

$$\begin{aligned}\frac{(p-q)\alpha}{\lambda n}\int_0^1 \frac{1-Au}{1-Bu}\, u^{\frac{(p-q)\alpha}{\lambda n}-1}\,du < \operatorname{Re}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} \\ < \frac{(p-q)\alpha}{\lambda n}\int_0^1 \frac{1+Au}{1+Bu}\, u^{\frac{(p-q)\alpha}{\lambda n}-1}\,du, \ z \in \mathbb{U}.\end{aligned} \tag{21}$$

Proof. Let $f \in \mathcal{N}_p^n(\lambda, \alpha, q; A, B)$, with $\lambda > 0$ and $-1 \leq B < A \leq 1$, from Theorem 1 it follows that (10) holds, which implies that

$$\begin{aligned}\operatorname{Re}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} &< \sup_{z\in\mathbb{U}} \operatorname{Re}\left[\frac{(p-q)\alpha}{\lambda n}\int_0^1 \frac{1+Azu}{1+Bzu}\, u^{\frac{(p-q)\alpha}{\lambda n}-1}\,du\right] \\ &\leq \frac{(p-q)\alpha}{\lambda n}\int_0^1 \left[\sup_{z\in\mathbf{U}} \operatorname{Re}\left(\frac{1+Azu}{1+Bzu}\, u^{\frac{(p-q)\alpha}{\lambda n}-1}\right)\right]du \\ &= \frac{(p-q)\alpha}{\lambda n}\int_0^1 \frac{1+Au}{1+Bu}\, u^{\frac{(p-q)\alpha}{\lambda n}-1}\,du, \ z \in \mathbb{U},\end{aligned} \tag{22}$$

and

$$\begin{aligned}\operatorname{Re}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} &> \inf_{z\in\mathbb{U}} \operatorname{Re}\left[\frac{(p-q)\alpha}{\lambda n}\int_0^1 \frac{1+Azu}{1+Bzu}\, u^{\frac{(p-q)\alpha}{\lambda n}-1}\, du\right] \\ &\geq \frac{(p-q)\alpha}{\lambda n}\int_0^1 \left[\inf_{z\in\mathbb{U}} \operatorname{Re}\left(\frac{1+Azu}{1+Bzu}\, u^{\frac{(p-q)\alpha}{\lambda n}-1}\right)\right] du \\ &= \frac{(p-q)\alpha}{\lambda n}\int_0^1 \frac{1-Au}{1-Bu}\, u^{\frac{(p-q)\alpha}{\lambda n}-1}\, du \ z\in\mathbb{U}. \end{aligned} \tag{23}$$

Combining (22) and (23) we get our conclusion (21). □

Theorem 5. *If $f \in \mathcal{N}_p^n(\lambda,\alpha,q;A,B)$ is of the form* (1)*, then*

$$|a_{p+n}| \leq \frac{\delta\,(p,q+1)}{\delta\,(p+n,q)}\left|\frac{A-B}{(p-q)\alpha+\lambda n}\right|. \tag{24}$$

Proof. From the assumption, we have

$$\begin{aligned}&(1+\lambda)\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} - \lambda\frac{zf^{(1+q)}(z)}{(p-q)f^{(q)}(z)}\left(\frac{\delta(p,q)z^{p-q}}{f^{(q)}(z)}\right)^{\alpha} \\ &= 1 + \frac{\delta\,(p+n,q)}{\delta(p,q)}\left(-\alpha - \frac{\lambda n}{p-q}\right)a_{p+n}z^n + \cdots \prec \frac{1+Az}{1+Bz},\end{aligned}$$

that is

$$\frac{\delta(p+n,q)}{\delta(p,q)}\left(-\alpha - \frac{\lambda n}{p-q}\right)a_{p+n}z^n + \cdots \prec (A-B)z + \ldots.$$

Now, from Lemma 3 we deduce that

$$\left|\frac{\delta\,(p+n,q)}{\delta(p,q)}\left(-\alpha - \frac{\lambda n}{p-q}\right)a_{p+n}\right| \leq |A-B|,$$

which is equivalent to the inequality (24) asserted in our theorem. □

Remark 2. *(i) Putting $q = 0$ in Theorems 2, 3, 4 and 5, respectively, we obtain the results of Aouf and Seoudy [5, Theorems 5, 6, 7 and 8], respectively.*

(ii) Taking $q = 0$ and $p = n = 1$ in Theorems 3, 4 and 5, respectively, we get the results of Wang et. al. [15, Theorems 2, 3 and 9], respectively.

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