Bulletin of the *Transilvania* University of Braşov • Vol 13(62), No. 2 - 2020
Series III: Mathematics, Informatics, Physics, 387-398
https://doi.org/10.31926/but.mif.2020.13.62.2.2

ON A SUBCLASS OF ANALYTIC FUNCTIONS OF FRACTAL POWER WITH NEGATIVE COEFFICIENTS

Zainab E. ABDULNABY[1] **and Rabha W. IBRAHIM**[*2] [12]

Abstract

The purpose of this article is to introduce a new general family of normalized analytic fractal function in the open unit disk. We employ this class to define a fractional differential operator of two fractals. This operator, under some conditions involves the well known Salagean differential operator. Our method is based on the Hadamard product and its generalization of functions with negative coefficients.

2000 *Mathematics Subject Classification:* 30C45, 44C55 .
Key words: Fractional calculus, fractional operator, analytic function, unit disk, subordination and superordination, univalent function.

1 Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. For all $z \in \mathbb{U}$, we define an extension of the fractional Koebe function as follows:

$$f(z) = \frac{z^{\alpha+1}}{(1 - z^{\mu})^{\alpha}}, \tag{1}$$

where $0 \le \alpha \le 1$ and $\mu(\alpha) \ge 1$ such that $\mu(0) = 1$. Let $\mathcal{E}_{\mu}$ be the class of all normalized analytic functions of fractional power f defined in (1) and indicated by

$$f(z) = z + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{n!} z^{\mu n + \alpha}, \tag{2}$$

[1*2] *Corresponding author*
[1]Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq, e-mail: esazainab@yahoo.com
[2*2]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, UAE, e-mail: rabhaibrahim@yahoo.com

For function f in $\mathcal{E}_\mu$ is starlike of order $\lambda(0 \leq \lambda < 1)$ if $\Re\{zf'/f\} > \lambda$ and is convex of order λ if $\Re\{1 + zf''/f'\} > \lambda$, respectively symbolizes by $f \in \mathcal{S}^*_\mu(\lambda)$ and $f \in \mathcal{K}_\mu(\lambda)$ for $|z| < 1$.

For $z \in \mathbb{U}$, let the new differential operator of fractional functional power f be defined by $D^k_{\mu,\alpha} : \mathcal{E}_\mu \to \mathcal{E}_\mu$ and

$$\begin{aligned}
D^0_{\mu,\alpha} f(z) &= f(z) = z + \sum_{n=2}^{\infty} a_n z^{\mu n+\alpha}, \\
D^1_{\mu,\alpha} f(z) &= z(f(z)') = z + \sum_{n=2}^{\infty} (\mu n + \alpha) a_n z^{\mu n+\alpha}, \\
&\vdots \\
D^k_{\mu,\alpha} f(z) &= D^1\left(D^{k-1} f(z)\right) = z + \sum_{n=2}^{\infty} (\mu n + \alpha)^k a_n z^{\mu n+\alpha}.
\end{aligned}$$

Obviously, $D^k_{\mu,\alpha} f(z) \in \mathcal{E}_\mu$ and normalized by

$$D^k_{\mu,\alpha} f(z)\big|_{z=0} = 0 \quad \text{and} \quad (D^k_{\mu,\alpha} f(z))'\big|_{z=0} = 1.$$

Moreover,

$$D^k_{1,0} f(z) = S^k f(z), \quad z \in \mathbb{U}.$$

is the well known Salagean differential operator [4]. The class of fractional analytic functions is defined and studied in [5]-[1].

Now, let $\mathcal{X}_\mu \subset \mathcal{E}_\mu$ the class of all analytic functions of fractional power with negative coefficients defined by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^{\mu n+\alpha}, \quad z \in \mathbb{U}. \tag{3}$$

Let $\mathcal{X}^*_\mu(\lambda)$ and $K_\mu(\lambda)$ be the subclasses of functions f in $\mathcal{X}_\mu$ which are respectively indicates to the class of starlike and convex functions with negative coefficients of order λ in $\mathbb{U}$.

Now, we define a new class $\mathcal{C}_{\mu,k}(\gamma, \beta)$ of functions $f \in \mathcal{X}_\mu$, which satisfy the following condition

$$\mathcal{C}_{\mu,k}(\gamma,\beta) = \left\{ f : f \in \mathcal{X}_\mu \quad \text{and} \quad \Re\{\frac{(D^k f(z))' + z(D^k f(z))''}{(D^k f(z))' + \gamma z(D^k f(z))''}\} > \beta \right\} \tag{4}$$
$$(0 \leq \beta < 1, 0 \leq \gamma < 1, k \in \mathbb{N} \cup \{0\}, z \in \mathbb{U}).$$

The aim of the present paper is to establish new results concerning the quasi-Owa-Hadamard product of $f \in \mathcal{E}_\mu$ in $\mathcal{C}_{\mu,k}(\gamma, \beta)$.

2 Coefficient inequalities

First, we suppose that $0 \leq \gamma < 1$, $0 \leq \beta < 1$ and $k \in \mathbb{N} \cup \{0\}$ with the fractal numbers α and $\mu(\alpha)$.

Theorem 1. Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^{\mu n+\alpha}$, then $f \in \mathcal{C}_{\mu,k}(\gamma,\beta)$ if and only if

$$\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}\left\{\mu n+\alpha-\beta[1+\gamma(\mu n+\alpha)]\right\}a_n \leq 1-\beta. \tag{5}$$

Proof. Assume that Eq.(5) is true and let $|z| = 1$. Then we get

$$\begin{aligned}
&\left|\frac{(D^k_{\mu,\alpha}f(z))' + z(D^k_{\mu,\alpha}f(z))''}{(D^k_{\mu,\alpha}f(z))' + \gamma z(D^k_{\mu,\alpha}f(z))''} - 1\right| \\
&\qquad = \left|\frac{\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}(\mu n+\alpha-1)(1-\gamma)a_n z^{\mu n+\alpha}}{1-\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}[1+\gamma(\mu n+\alpha-1)]a_n z^{\mu n+\alpha}}\right| \\
&\qquad \leq \frac{\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}(\mu n+\alpha-1)(1-\gamma)a_n}{1-\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}[1+\gamma(\mu n+\alpha-1)]a_n} \leq 1-\beta.
\end{aligned}$$

By this, the values of $\frac{(D^k_{\mu,\alpha}f(z))' + z(D^k_{\mu,\alpha}f(z))''}{(D^k_{\mu,\alpha}f(z))' + \gamma z(D^k_{\mu,\alpha}f(z))''}$ in a circle c entered at $w = 1$ and with radius is $1-\beta$. Therefore $f(z)$ in class $\mathcal{C}_{\mu,k}(\gamma,\beta)$. Conversely, let $f(z) \in \mathcal{C}_{\mu,k}(\gamma,\beta)$, then

$$\begin{aligned}
&\Re\{\frac{(D^k_{\mu,\alpha}f(z))' + z(D^k_{\mu,\alpha}f(z))''}{(D^k_{\mu,\alpha}f(z))' + \gamma z(D^k_{\mu,\alpha}f(z))''}\} \\
&\qquad = \Re\{\frac{1-\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+2}a_n z^{\mu n+\alpha}}{1-\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}[1+\gamma(\mu n+\alpha-1)]a_n z^{\mu n+\alpha}}\} > \beta.
\end{aligned} \tag{6}$$

Select values of z such that the imaginary part is zero, then $\frac{(D^k_{\mu,\alpha}f(z))' + z(D^k_{\mu,\alpha}f(z))''}{(D^k_{\mu,\alpha}f(z))' + \gamma z(D^k_{\mu,\alpha}f(z))''}$ is real and let $z \to -1$, we have

$$1-\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+2}a_n > \beta\{1-\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}[1+\gamma(\mu n+\alpha-1)]a_n\}. \tag{7}$$

which equivalents to (5). This implies that the function $f(z) \in \mathcal{C}_{\mu,k}(\gamma,\beta)$. $\square$

Corollary 1. For $f(z) \in \mathcal{C}_{\mu,k}(\gamma,\beta)$, we obtain

$$a_n \leq \frac{1-\beta}{(\mu n+\alpha)^{k+1}\{\mu n+\alpha-\beta[1+\gamma(\mu n+\alpha-1)]\}}, \quad n = \{2,3,\cdots\} \tag{8}$$

then, the sharpness is satisfied from (8) as follows

$$F(z) = z - \frac{1-\beta}{(\mu n+\alpha)^{k+1}\{\mu n+\alpha-\beta[1+\gamma(\mu n+\alpha-1)]\}} z^{\mu n+\alpha}.$$

2.1 Quasi-Owa-Hadamared product

The quasi-Hadamard product is a modification of the Hadamard product, which has been proposed by Owa [2]. In this section, we utilize some generalized results due to Owa and Srivastava [3] for the quasi-Hadamard product of a new class of univalent functions.
Let $f_j \in \mathcal{X}_\mu$ be functions given by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^{\mu n+\alpha}, \quad j = 1, 2,\ z \in \mathbb{U}. \tag{9}$$

then the Hadamard product of two functions $f_j \in \mathcal{X}_\mu$ for $j = 1, 2$ is defined by [?]

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} \big(\prod_{j=1}^{2} a_{n,j}\big) z^{\mu n+\alpha}.$$

and for $j = 1, ..., p$, we obtain

$$(f_1 * f_2 * \cdots * f_p)(z) = z - \sum_{n=2}^{\infty} \big(\prod_{j=1}^{p} a_{n,j}\big) z^{\mu n+\alpha}$$

Theorem 2. *If the functions $f_j(z)$ belong to $\mathcal{C}_{\mu,k}(\gamma, \beta_j), j = 1, 2, \cdots, p$, then $(f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{C}_{\mu,k}(\gamma, \eta)$, and*

$$\begin{aligned} &\eta(\alpha, k, \gamma, \beta_j) \\ \leq\ & 1 - \frac{(1-\gamma)\prod_{j=1}^{p}(1-\beta_j)}{(2\mu+\alpha)^{(p-1)(k+1)}\prod_{j=1}^{p}[2\mu+\alpha-\beta_j(1+\gamma)] - (1+\gamma)\prod_{j=1}^{p}(1-\beta_j)}. \end{aligned} \tag{10}$$

The result is sharp for the function

$$f_j(z) = z - \frac{1-\beta_j}{(2\mu+\alpha)^{k+1}[2\mu+\alpha-\beta_j(1+\gamma)]} z^{2\mu+\alpha}, (j = 1, 2, \cdots, p, |z| < 1). \tag{11}$$

Proof. For $p = 1$, we obtain that $\eta = \beta_1$. For $p = 2$, then (5) yields

$$\sum_{n=2}^{\infty} (\mu n+\alpha)^{k+1} \frac{\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}{1-\beta_j} a_{n,j} \leq 1, \quad j = 1, 2. \tag{12}$$

By using Cauchy-Schwarz inequality,we obtain

$$\sum_{n=2}^{\infty} (\mu n+\alpha)^{k+1} \sqrt{\prod_{j=1}^{p=2} \left(\frac{\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}{1-\beta_j} a_{n,j} \right)} \leq 1, \quad j = 1, 2. \tag{13}$$

To prove $p = 2$, we have to find the largest η such that

$$\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}\frac{\{\mu n+\alpha-\eta[1+\gamma(\mu n+\alpha-1)]\}}{1-\eta}a_{n,1}a_{n,2}\leq 1 \tag{14}$$

or, its equal to

$$\frac{\{\mu n+\alpha-\eta[1+\gamma(\mu n+\alpha-1)]\}}{1-\eta}\sqrt{a_{n,1}a_{n,2}} \leq \sqrt{\prod_{j=1}^{p=2}\left(\frac{\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}{1-\beta_j}\right)}, \tag{15}$$

and that is

$$\sqrt{a_{n,1}a_{n,2}}\leq \frac{1-\eta}{\{\mu n+\alpha-\eta[1+\gamma(\mu n+\alpha-1)]\}}\sqrt{\prod_{j=1}^{p=2}\left(\frac{\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}{1-\beta_j}\right)}, \tag{16}$$

Further from (13), we need to find the largest η as follow

$$\frac{\{\mu n+\alpha-\eta[1+\gamma(\mu n=\alpha-1)]\}}{1-\eta} \leq (\mu n+\alpha)^{k+1}\prod_{j=1}^{p=2}\left(\frac{\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}{1-\beta_j}\right), \tag{17}$$

which is equal to

$$\begin{aligned}\eta &\leq \frac{(\mu n+\alpha)^{k+1}\prod_{j=1}^{p=2}\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\} -(\mu n+\alpha)\prod_{j=1}^{p=2}(1-\beta_j)}{(\mu n+\alpha)^{k+1}\prod_{j=1}^{p=2}\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\} -[1+\gamma(\mu n+\alpha-1)]\prod_{j=1}^{p=2}(1-\beta_j)}\\ &= 1-\frac{(\mu n+\alpha-1)(1-\gamma)\prod_{j=1}^{p=2}(1-\beta_j)}{(\mu n+\alpha)^{k+1}\prod_{j=1}^{p=2}\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\} -[1+\gamma(\mu n+\alpha-1)]\prod_{j=1}^{p=2}(1-\beta_j)}.\end{aligned} \tag{18}$$

Now, let suppose that

$$\Phi(n) = 1-\frac{(\mu n+\alpha-1)(1-\gamma)\prod_{j=1}^{p=2}(1-\beta_j)}{(\mu n+\alpha)^{k+1}\prod_{j=1}^{p=2}\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\} -[1+\gamma(\mu n+\alpha-1)]\prod_{j=1}^{p=2}(1-\beta_j)}. \tag{19}$$

Since $\Phi'(n) \geq 0$ for $(n \geq 2)$. This yields

$$\begin{aligned} \eta &\leq \Phi(2) \\ &= 1 - \frac{(2\mu+\alpha-1)(1-\gamma)\prod_{j=1}^{p=2}(1-\beta_j)}{\begin{array}{c}(2\mu+\alpha)^{k+1}\prod_{j=1}^{p=2}\{2\mu+\alpha-\beta_j[1+\gamma(2\mu+\alpha-1)]\} \\ -[1+\gamma(2\mu+\alpha-1)]\prod_{j=1}^{p=2}(1-\beta_j)\end{array}}. \end{aligned} \tag{20}$$

Hence, the prove is true for $p = 2$. Now, let assume that, the result is true for $p > 0$, then

$$(f_1 * f_2 * \cdots * f_p * f_{p+1})(z) \in \mathcal{C}_{\mu,k}(\gamma, \xi),$$

where

$$\xi = 1 - \frac{(1-\gamma)(1-\eta)(1-\beta_{p+1})}{\begin{array}{c}(2\mu+\alpha)^{p(k+1)}[2\mu+\alpha-\eta(1+\gamma)][2\mu+\alpha-\beta_{p+1}(1+\gamma)] \\ -(1+\gamma)(1-\eta)(1-\beta_{p+1})\end{array}}. \tag{21}$$

where η is given by (32). It follows from (21) that

$$\xi \leq 1 - \frac{(1-\gamma)\prod_{j=1}^{p+1}(1-\beta_j)}{\begin{array}{c}(2\mu+\alpha)^{p(k+1)}\prod_{j=1}^{p+1}[2\mu+\alpha-\beta_j(1+\gamma)] \\ -(1+\gamma)\prod_{j=1}^{p+1}(1-\beta_j)\end{array}}. \tag{22}$$

Hence, the result is true for $p+1$. lastly, taking into account the function f_j given by (11), we have

$$\begin{aligned} (f_1 * f_2 * \cdots * f_p)(z) &= z - \{\prod_{j=1}^{p} \frac{1-\beta_j}{(2\mu+\alpha)^{k+1}[2\mu+\alpha-\beta_j(1+\gamma)]}\} z^{2\mu+\alpha} \\ &= z - \mathbb{A}_2 z^{2\mu+\alpha}, \end{aligned}$$

where

$$\mathbb{A}_2 = \prod_{j=1}^{p}\left(\frac{1-\beta_j}{(2\mu+\alpha)^{k+1}[2\mu+\alpha-\beta_j(1+\gamma)]}\right).$$

Therefore,

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(\mu n+\alpha)^{k+1}\{\mu n+\alpha-\eta[1+\gamma(\mu n+\alpha-1)]\}}{1-\eta}\mathbb{A}_2 \\ &= \frac{(2\mu+\alpha)^{k+1}\{2\mu+\alpha-\eta[1+\gamma]\}}{1-\eta} \times \prod_{j=1}^{p}\left(\frac{1-\beta_j}{(2\mu+\alpha)^{k+1}[2\mu+\alpha-\beta_j(1+\gamma)]}\right) \\ &= 1 \end{aligned} \tag{23}$$

Thus, the sharpness for $f_p(z)$ defined by (11). □

In next results, we set some sharpness properties for $\beta_j = \beta\,(j = 1, 2, \cdots, p)$ in Theorem 2 as follows:

Corollary 2. *For $j = 1, 2, \cdots, p$, let $f_j(z) \in \mathcal{C}_{\mu,k}(\gamma, \beta)$ then $(f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{C}_{\mu,k}(\gamma, \eta_1)$ where*

$$\eta_1 \leq 1 - \frac{(1-\gamma)(1-\beta)^p}{(2\mu+\alpha)^{(p-1)(k+1)}\left[2\mu+\alpha-\beta(1+\gamma)\right]^p - (1+\gamma)(1-\beta)^p}$$

and the sharpness property is satisfied for the following function

$$f_j(z) = z - \frac{1-\beta}{(2\mu+\alpha)^{k+1}[2\mu+\alpha-\beta(1+\gamma)]} z^{2\mu+\alpha}, \quad |z| < 1.$$

If $k = 0$ in above Theorem 2, then we obtain

Corollary 3. *For $j = 1, 2, \cdots, p$, let $f_j(z) \in \mathcal{C}_{\mu,k}(\gamma, \beta_j)$ then $(f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{C}_{\mu,k}(\gamma, \eta_2)$ where*

$$\eta_2 \leq 1 - \frac{(1-\gamma)\prod_{j=1}^{p}(1-\beta_j)}{(2\mu+\alpha)^{p-1}\prod_{j=1}^{p}\left[2\mu+\alpha-\beta_j(1+\gamma)\right] - (1+\gamma)\prod_{j=1}^{p}(1-\beta_j)}.$$

the sharpness property is satisfied for the following function

$$f_j(z) = z - \frac{1-\beta_j}{(2\mu+\alpha)[2\mu+\alpha-\beta_j(1+\gamma)]} z^{2\mu+\alpha}, \quad j = 1, 2, \cdots, p, |z| < 1. \quad (24)$$

If $k = 0$ and $\beta_j = \beta$ in above Theorem 2, then we get the following results

Corollary 4. *If $f_j(z), (j = 1, 2, \cdots, p) \in \mathcal{C}_{\mu,k}(\gamma, \beta)$, then $(f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{C}_{\mu,k}(\gamma, \eta_3)$, where*

$$\eta_3 \leq 1 - \frac{(1-\gamma)(1-\beta)^p}{(2\mu+\alpha)^{p-1}\left[2\mu+\alpha-\beta(1+\gamma)\right]^p - (1+\gamma)(1-\beta)^p}.$$

The result is sharp for the function

$$f_j(z) = z - \frac{1-\beta}{(2\mu+\alpha)[2\mu+\alpha-\beta(1+\gamma)]} z^{2\mu+\alpha}, \quad j = 1, 2, \cdots, p, |z| < 1. \quad (25)$$

If $\gamma = 0$ in above Theorem 2, then we get

Corollary 5. *For $j = 1, 2, \cdots, p$ let $f_j(z)(\in \mathcal{C}_{\mu,k}(\beta_j)$, then $(f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{C}_k(\eta_4)$, where*

$$\eta_4 \leq 1 - \frac{\prod_{j=1}^{p}(1-\beta_j)}{(2\mu+\alpha)^{(p-1)(k+1)}\prod_{j=1}^{p}(2\mu+\alpha-\beta_j) - \prod_{j=1}^{p}(1-\beta_j)}.$$

The sharpness result is satisfied for the function

$$f_j(z) = z - \frac{1-\beta_j}{(2\mu+\alpha)(2\mu+\alpha-\beta_j)} z^{2\mu+\alpha}, \quad j = 1, 2, \cdots, p, |z| < 1. \quad (26)$$

If $k = 0, p = 2$ and $\beta_j = \beta$ in above Theorem 2, then we have

Corollary 6. *For $j = 1, 2, \cdots, p$ let $f_j(z) \in \mathcal{C}_\mu(\gamma, \beta)$ then $(f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{C}_\mu(\gamma, \eta_5)$, where*

$$\eta_5 \leq 1 - \frac{(1-\gamma)(1-\beta)^2}{(2\mu+\alpha)\left[2\mu+\alpha-\beta(1+\gamma)\right]^2-(1+\gamma)(1-\beta)^2}.$$

The sharpness property for $f_j(z)$, $(j = 1, 2)$ given by (26).

If $k = 0, p = 2$, $\beta_j = \beta$ and $\alpha = 1$ in above Theorem (2) then we obtain

Corollary 7. *For $j = 1, 2, \cdots, p$ let $f_j(z) \in \mathcal{C}_\mu(\gamma, \beta)$ then $(f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{C}_\mu(\gamma, \eta_6)$ where*

$$\eta_6 \leq 1 - \frac{(1-\gamma)(1-\beta)^2}{(2\mu+1)\left[2\mu+1-\beta(1+\gamma)\right]^2-(1+\gamma)(1-\beta)^2}.$$

The sharpness result is satisfied for the function

$$f_j(z) = z - \frac{1-\beta_j}{(2\mu+1)(2\mu+1-\beta_j)} z^{2\mu+1}, \quad j = 1, 2, \cdots, p, |z| < 1. \tag{27}$$

If $k = 0, p = 2$, $\beta_j = \beta$ and $\alpha = 0$ in above Theorem (2) then we have

Corollary 8. *For $j = 1, 2, \cdots, p$ let $f_j(z) \in \mathcal{C}_\mu(\gamma, \beta)$ then $(f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{C}_\mu(\gamma, \eta_7)$ where*

$$\eta_7 \leq 1 - \frac{(1-\gamma)(1-\beta)^2}{(2\mu)\left[2\mu-\beta(1+\gamma)\right]^2-(1+\gamma)(1-\beta)^2}.$$

The sharpness result is satisfied for the function

$$f_j(z) = z - \frac{1-\beta_j}{(2\mu)(2\mu-\beta_j)} z^{2\mu}, \quad j = 1, 2, \cdots, p, |z| < 1. \tag{28}$$

If $k = 0, p = 2$, $\beta_j = \beta$, $\alpha = 1$ and $\mu = 1/2$ in above Theorem (2) then we obtain

Corollary 9. *For $j = 1, 2, \cdots, p$ let $f_j(z) \in \mathcal{C}(\gamma, \beta)$ then $(f_1 * f_2 * \cdots * f_p)(z) \in \mathcal{C}(\gamma, \eta_8)$ where*

$$\eta_8 \leq 1 - \frac{(1-\gamma)(1-\beta)^2}{(2)\left[2-\beta(1+\gamma)\right]^2-(1+\gamma)(1-\beta)^2}.$$

The sharpness property is satisfied for the function

$$f_j(z) = z - \frac{1-\beta_j}{2(2-\beta_j)} z^2, \quad |z| < 1. \tag{29}$$

Theorem 3. *For $j = 1, 2, \cdots, p$ let $f_j(z)$ be a function defined by*

$$(f_1 * f_2 * \cdots * f_p)(z) = z - \sum_{n=2}^{\infty} \left(\prod_{j=1}^{p} a_{n,j} \right) z^{\mu n+\alpha} \quad (j = 1, 2, \cdots, p) \tag{30}$$

be in $\mathcal{C}_{\mu,k}(\gamma, \beta_j)$, then the function $\Omega(z)$ given by

$$\Omega(z) = z - \frac{1}{2} \sum_{n=2}^{\infty} \left(\prod_{j=1}^{p} |a_{n,j}| \right) z^{\mu n+\alpha} \tag{31}$$

is also in $\mathcal{C}_{\mu,k}(\gamma, \beta_j)$.

Proof. In view Theorem 1, we easily see that

$$\frac{1}{2} \sum_{n=2}^{\infty} (\mu n + \alpha)^{k+1} \{\mu n + \alpha - \beta[1 + \gamma(\mu n + \alpha)]\} |a_{n,1} + a_{n,2} + \cdots + a_{n,p}| \leq$$
$$\frac{1}{2} \sum_{n=2}^{\infty} (\mu n + \alpha)^{k+1} \{\mu n + \alpha - \beta[1 + \gamma(\mu n + \alpha)]\}$$
$$\times (|a_{n,1}| + |a_{n,2}| + \cdots + |a_{n,p}|) \leq (1 - \beta_j)$$

which implies that $\Omega(z) \in \mathcal{C}_{\mu,k}(\gamma, \beta_j)$.

□

2.2 Generalizations of Hadamard products

For $f_j(z)$ and $j = 1, 2, \cdots, p$ in (9), we define the generalized quasi-hadamard product by

$$(f_1 \circledast f_2 \circledast \cdots \circledast f_p)(z) := z - \sum_{n=2}^{\infty} \left(\prod_{j=1}^{p} (a_{n,j})^{\frac{1}{m_j}} \right) z^{\mu n+\alpha}, \alpha \geq 1 \tag{32}$$

where

$$\left(\sum_{j=1}^{p} \frac{1}{m_j} = 1; m_j > 1; j = 1, 2, \cdots, p \right).$$

Theorem 4. *For $j = 1, 2, \cdots, p$, if $f_j(z) \in \mathcal{C}_{\mu,k}(\gamma, \beta_j)$ then*

$$(f1 \circledast f_2 \circledast \cdots \circledast f_p) \in \mathcal{C}_{\mu,k}(\gamma, \delta),$$

where

$$\omega(\alpha, k, \gamma, \beta_j)$$
$$\leq \left\{ \frac{(2\mu + \alpha)(1 - \gamma) \prod_{j=1}^{2} \left((1 - \beta_j)\right)^{\frac{1}{m_j}}}{\begin{array}{c}(2\mu + \alpha)^{(p-1)(k+1)} \prod_{j=1}^{2} \left(\{2\mu + \alpha - \beta_j[1 + \gamma(2\mu + \alpha - 1)]\}\right)^{\frac{1}{m_j}} \\ -[1 + \gamma(2\mu + \alpha - 1)]) \prod_{j=1}^{2} \left((1 - \beta_j)\right)^{\frac{1}{m_j}}\end{array}} \right\} \tag{33}$$

and for $j = 1, 2, \cdots, p$ the sharpness for functions f_j defined by

$$f_j(z) = z - \{\frac{(2\mu+\alpha)(1-\gamma)(1-\beta_j)}{(2\mu+\alpha)^{k+1}[2\mu+\alpha-\beta_j(1+\gamma(2\mu+\alpha-1))}\}.$$

Proof. For $f_1(z) \in \mathcal{C}_{\mu,k}(\gamma, \beta_1)$ and $f_2(z) \in \mathcal{C}_{\mu,k}(\gamma, \beta_2)]$,

$$\sum_{n=2}^{\infty} \frac{(\mu n+\alpha)^{k+1}\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}{1-\beta_j} a_{n,j} \leq 1$$

where $a_{n,j} \geq 0$, for $(j = 1, 2)$ and $n \geq 2$, we have

$$\prod_{j=1}^{2}\left(\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}\left\{\left(\frac{\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}{1-\beta_j}\right)^{\frac{1}{m_j}}(a_{n,j})^{\frac{1}{m_j}}\right\}^{m_j}\right)^{\frac{1}{m_j}} \leq 1. \tag{34}$$

Then, by using the Holder inequality and (34), we obtain

$$\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}\left\{\prod_{j=1}^{2}\left(\frac{\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}{1-\beta_j}\right)^{\frac{1}{m_j}}(a_{n,j})^{\frac{1}{m_j}}\right\} \leq 1,$$

Or satisfies

$$\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}\prod_{j=1}^{2}(a_{n,j})^{\frac{1}{m_j}} \leq \prod_{j=1}^{2}\left(\frac{1-\beta_j}{\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}\right)^{\frac{1}{m_j}}. \tag{35}$$

Now we need to find the largest ρ such that

$$\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}\left(\frac{\{\mu n+\alpha-\rho[1+\gamma(\mu n+\alpha-1)]\}}{(1-\rho)}\right)\left(\prod_{j=1}^{2}(a_{n,j})^{\frac{1}{m_j}}\right) \leq 1$$

from (35), we see that

$$\begin{aligned}&\sum_{n=2}^{\infty}(\mu n+\alpha)^{k+1}\left(\frac{\{\mu n+\alpha-\rho[1+\gamma(\mu n+\alpha-1)]\}}{(1-\rho)}\right)\left(\prod_{j=1}^{2}|a_{n,j}|^{\frac{1}{m_j}}\right)\\ &\quad\leq \sum_{n=2}^{\infty}\left(\frac{(\mu n+\alpha)^{k+1}\{\mu n+\alpha-\rho[1+\gamma(\mu n+\alpha-1)]\}}{(1-\rho)}\right)\\ &\qquad\times\prod_{j=1}^{2}\left(\frac{1-\beta_j}{(\mu n+\alpha)^{k+1}\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}\right)^{\frac{1}{m_j}} \leq 1,\end{aligned}$$

and that is

$$\left(\frac{\{\mu n+\alpha-\rho[1+\gamma(\mu n+\alpha-1)]\}}{(1-\rho)}\right)\leq (\mu n+\alpha)^{k+1}\prod_{j=1}^{2}\left(\frac{\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}}{(1-\beta_j)}\right)^{\frac{1}{m_j}}$$

which yields

$$\rho\leq\{\frac{(\mu n+\alpha)^{k+1}\prod_{j=1}^{2}\left(\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}\right)^{\frac{1}{m_j}}-(\mu n+\alpha)\prod_{j=1}^{2}\left((1-\beta_j)\right)^{\frac{1}{m_j}}}{(\mu n+\alpha)^{k+1}\prod_{j=1}^{2}\left(\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}\right)^{\frac{1}{m_j}}-[1+\gamma(\mu n+\alpha-1)]\prod_{j=1}^{2}\left((1-\beta_j)\right)^{\frac{1}{m_j}}}\} \tag{36}$$

$$\rho=1-\{\frac{(\mu n+\alpha)(1-\gamma)\prod_{j=1}^{2}\left(1-\beta_j\right)^{\frac{1}{m_j}}}{(\mu n+\alpha)^{k+1}\prod_{j=1}^{2}\left(\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}\right)^{\frac{1}{m_j}}-[1+\gamma(\mu n+\alpha-1)])\prod_{j=1}^{2}\left(1-\beta_j\right)^{\frac{1}{m_j}}}\} \tag{37}$$

let $\Theta(n)$

$$\Theta(n)=1-\{\frac{(\mu n+\alpha)(1-\gamma)\prod_{j=1}^{2}\left(1-\beta_j\right)^{\frac{1}{m_j}}}{(\mu n+\alpha)^{k+1}\prod_{j=1}^{2}\left(\{\mu n+\alpha-\beta_j[1+\gamma(\mu n+\alpha-1)]\}\right)^{\frac{1}{m_j}}-[1+\gamma(\mu n+\alpha-1)])\prod_{j=1}^{2}\left(1-\beta_j\right)^{\frac{1}{m_j}}}\}$$

but for $\Theta(n)\geq 0$ for $(n\leq 2)$. This yields

$$\rho\leq\Theta(2)=1-\{\frac{(2\mu+\alpha)(1-\gamma)\prod_{j=1}^{2}\left((1-\beta_j)\right)^{\frac{1}{m_j}}}{(2\mu+\alpha)^{k+1}\prod_{j=1}^{2}\left(\{2\mu+\alpha-\beta_j[1+\gamma(2\mu+\alpha-1)]\}\right)^{\frac{1}{m_j}}-[1+\gamma(2\mu+\alpha-1)])\prod_{j=1}^{2}\left((1-\beta_j)\right)^{\frac{1}{m_j}}}\} \tag{38}$$

Thus the assertion Theorem 2 holds true when $p=2$. Then, clearly that,

$$(f_1\circledast\cdots\circledast f_{p+1})\in\mathcal{C}_{\mu,k,}(\gamma,\omega)$$

with

$$\omega=1-\{\frac{(2\mu+\alpha)(1-\gamma)\left((1-\beta_j)\right)^{\frac{1}{m_{j+1}}}}{(2\mu+\alpha)^{k+1}\left(\{2\mu+\alpha-\beta_{j+1}[1+\gamma(2\mu+\alpha-1)]\}\right)^{\frac{1}{m_{j+1}}}-[1+\gamma(2\mu+\alpha-1)])\left((1-\beta_{j+1})\right)^{\frac{1}{m_{j+1}}}}\}$$

then we conclude that

$$(f_1 \circledast \ldots \circledast f_p) \in \mathcal{C}_{\mu,k,}(\gamma, \omega, \delta)$$

$\square$

References

[1] Abdulnaby, Zainab E., Rabha W. Ibrahim, and Adem Kılıçman. *On boundedness and compactness of a generalized Srivastava-Owa fractional derivative operator*, Journal of King Saud University-Science **30** (2018), 153-157.

[2] Owa, S. *On the classes of univalent functions with negative coefficients.* Math. Japon **27** (1982), 409-416.

[3] Owa, S., and Srivastava, H. M., *Some generalized convolution properties associated with certain subclasses of analytic functions.* Journal of Inequalities in Pure and Applied Mathematics, **3** (2002), 1-13.

[4] Salagean, G. S. *Subclasses of univalent functions*, In Complex Analysis-Fifth Romanian-Finnish Seminar (1983), 362-372.

[5] Srivastava, H. M., Darus, M. and Ibrahim, R. W., *Classes of analytic functions with fractional powers defined by means of a certain linear operator.* Integral Transforms and Special Functions **22** (2011), 17–28.

[6] Srivastava, H. M., Kılıçmanc, A., Abdulnabyc, Z. E. and Ibrahime, R. W. *Generalized convolution properties based on the modified Mittag-Leffler function.* Journal of Nonlinear Sciences and Applications **10** (2017), 4284-4294.