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CAPUTO-FABRIZIO FRACTIONAL DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

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Abstract

This paper deals with some existence and uniqueness of solutions for a class of functional Caputo-Fabrizio fractional differential equations. Some applications are made of a generalization of the classical Darbo fixed point theorem for Fréchet spaces associate with the concept of measure of noncompactness. The last section illustrates our results with some examples.

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1 Introduction

There has been a significant development in the area of the theory of fractional calculus and fractional differential equations [31]. For some fundamental results in this subject, we refer the reader to the monographs [3, 6, 7, 29, 23, 36], and the papers [4, 12]. These fractional differential equations involves Riemann-Liouville, Caputo, Hadamard and Hilfer fractional differential operators.

In recent times, a new fractional differential operator having a kernel with exponential decay has been introduced by Caputo and Fabrizio [15]. This approach of fractional derivative is known as the Caputo-Fabrizio operator which has attracted many research scholars due to the fact that it has a non-singular kernel. Several mathematicians were recently busy in development of Caputo-Fabrizio fractional differential equations, see; [11, 16, 20, 21, 22, 24, 30, 33], and the references therein.

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Recently, several researchers obtained other results by application of the technique of measure of noncompactness; see [9, 10, 32], and the references therein. In [1, 2, 5], Abbas *et al.* cosidered several classes of fractional differential equations in Fréchet spaces. Motivated by the above papers, in this article we discuss the existence of solutions for the following Caputo-Fabrizio fractional differential equation

$$({}^{CF}D_0^r u)(t) = f(t, u(t));\ t \in \mathbb{R}_+ := [0, \infty), \tag{1}$$

with the following initial condition

$$u(0) = u_0 \in E, \tag{2}$$

where $T > 0$, $(E, \|\cdot\|)$ is a (real or complex) Banach space, $r \in (0,1)$, $f : \mathbb{R}_+ \times E \to E$ is a given function, and ${}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0,1)$.

Next, we discuss the existence of solutions for the fractional differential equation (1), with the following nonlocal condition

$$u(0) + Q(u) = u_0, \tag{3}$$

where $u_0 \in E$, $Q : C(\mathbb{R}_+, E) \to E$ is a given function. Nonlocal problems are used to represent mathematical models for evolution of various phenomena, such as nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution and nonlocal combustion, see; [14, 17, 25, 34, 35], and the references therein.

This paper initiates the existence of solutions for functional differential equations involving the Caputo-Fabrizio fractional derivative in Fréchet spaces.

2 Preliminaries

Let C be the Banach space of all continuous functions v from $I := [0, T];\ T > 0$ into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t \in I} \|v(t)\|.$$

By $L^1(I)$, we denote the space of Bochner-integrable functions $v : I \to E$ with the norm

$$\|v\|_1 = \int_0^T \|v(t)\| dt.$$

Let $X := C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t \in [0,n]} \|v(t)\|;\ n \in \mathbb{N},$$

and the distance

$$d(u, v) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|u - v\|_n}{1 + \|u - v\|_n};\ u, v \in X.$$

Definition 1. *A nonempty subset $B \subset X$ is said to be bounded if*

$$\sup_{v \in B} \|v\|_n < \infty; \ for \ n \in \mathbb{N}.$$

We recall the following definition of the notion of a sequence of measures of noncompactness [18, 19].

Definition 2. *Let $\mathcal{M}_F$ be the family of all nonempty and bounded subsets of a Fréchet space F. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$ where $\mu_n : \mathcal{M}_F \to [0, \infty)$ is said to be a family of measures of noncompactness in the real Fréchet space F if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_F$:*

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is: $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact,

(b) $\mu_n(B_1) \le \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$,

(c) $\mu_n(ConvB) = \mu_n(B)$ for $n \in \mathbb{N}$,

(d) If $\{B_i\}_{i=1,\cdots}$ is a sequence of closed sets from $\mathcal{M}_F$ such that $B_{i+1} \subset B_i$; $i = 1, \cdots$ and if $\lim_{i\to\infty} \mu_n(B_i) = 0$, for each $n \in \mathbb{N}$, then the intersection set $B_\infty := \Cap_{i=1}^{\infty} B_i$ is nonempty.

Some Properties:

(e) We call the family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ to be homogeneous if $\mu_n(\lambda B) = |\lambda|\mu_n(B)$; for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.

(f) If the family $\{\mu_n\}_{n\in\mathbb{N}}$ satisfied the condition $\mu_n(B_1 \cup B_2) \le \mu_n(B_1) + \mu_n(B_2)$, for $n \in \mathbb{N}$, it is called subadditive.

(g) It is sublinear if both conditions (e) and (f) hold.

(h) We say that the family of measures $\{\mu_n\}_{n\in\mathbb{N}}$ has the maximum property if

$$\mu_n(B_1 \cup B_2) = \max\{\mu_n(B_1), \mu_n(B_2)\},$$

(i) The family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

Example 1. *[18], [27] For $B \in \mathcal{M}_X$, $x \in B$, $n \in \mathbb{N}$ and $\epsilon > 0$, let us denote by $\omega^n(x, \epsilon)$ the modulus of continuity of the function x on the interval $[0, n]$; that is,*

$$\omega^n(x, \epsilon) = \sup\{\|x(t) - x(s)\| : t, s \in [0, n], |t - s| \le \epsilon\}.$$

Further, let us put

$$\omega^n(B, \epsilon) = \sup\{\omega^n(x, \epsilon) : x \in B\},$$

$$\omega_0^n(B) = \lim_{\epsilon\to 0^+} \omega^n(B, \epsilon),$$

$$\bar{\alpha}^n(B) = \sup_{t\in[0,n]} \alpha(B(t)) := \sup_{t\in[0,n]} \alpha(\{x(t) : x \in B\}),$$

and

$$\beta_n(B) = \omega_0^n(B) + \bar{\alpha}^n(B).$$

The family of mappings $\{\beta_n\}_{n\in\mathbb{N}}$ *where* $\beta_n : \mathcal{M}_X \to [0,\infty)$, *satisfies the conditions (a)-(d) fom Definition 2.*

Lemma 1. *[13] If* Y *is a bounded subset of a Banach space* F, *then for each* $\epsilon > 0$, *there is a sequence* $\{y_k\}_{k=1}^{\infty} \subset Y$ *such that*

$$\mu(Y) \le 2\mu(\{y_k\}_{k=1}^{\infty}) + \epsilon,$$

where μ *is the Kuratowskii measure of noncompactness.*

Lemma 2. *[26] If* $\{u_k\}_{k=1}^{\infty} \subset L^1([0,n])$ *is uniformly integrable, then* $\mu_n(\{u_k\}_{k=1}^{\infty})$ *is measurable for* $n \in \mathbb{N}^*$, *and*

$$\mu\left(\left\{\int_1^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \le 2\int_1^t \mu(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t \in [0,n]$.

Definition 3. *Let* Ω *be a nonempty subset of a Fréchet space* F, *and let* $A : \Omega \to F$ *be a continuous operator which transforms bounded subsets of onto bounded ones. One says that* A *satisfies the Darbo condition with constants* $(k_n)_{n\in\mathbb{N}}$ *with respect to a family of measures of noncompactness* $\{\mu_n\}_{n\in\mathbb{N}}$, *if*

$$\mu_n(A(B)) \le k_n\mu_n(B)$$

for each bounded set $B \subset \Omega$ *and* $n \in \mathbb{N}$.
If $k_n < 1;\ n \in \mathbb{N}$ *then* A *is called a contraction with respect to* $\{\mu_n\}_{n\in\mathbb{N}}$.

Definition 4. *[15, 24] The Caputo-Fabrizio fractional integral of order* $0 < r < 1$ *for a function* $h \in L^1(I)$ *is defined by*

$${}^{CF}I^r h(\tau) = \frac{2(1-r)}{M(r)(2-r)}h(\tau) + \frac{2r}{M(r)(2-r)}\int_0^\tau h(x)dx, \quad \tau \ge 0$$

where $M(r)$ is normalization constant depending on r.

Definition 5. *[15, 24] The Caputo-Fabrizio fractional derivative for a function* $h \in C^1(I)$ *of order* $0 < r < 1$, *is defined by*

$${}^{CF}D^r h(\tau) = \frac{(2-r)M(r)}{2(1-r)}\int_0^\tau \exp\left(-\frac{r}{1-r}(\tau - x)\right)h'(x)dx;\ \tau \in I.$$

Note that $({}^{CF}D^r)(h) = 0$ *if and only if* h *is a constant function.*

Lemma 3. *Let $h \in L^1(I)$. A function $u \in \mathcal{C}$ is a solution of problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t); \quad t \in I \\ u(0) = u_0, \end{cases} \tag{4}$$

if and only if u satisfies the following integral equation

$$u(t) = C + a_r h(t) + b_r \int_0^t h(s)ds, \tag{5}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \; b_r = \frac{2r}{(2-r)M(r)}, \; C = u_0 - a_r h(0).$$

Proof. Suppose that u satisfies (4). From Proposition 1 in [24]; the equation

$$({}^{CF}D_0^r u)(t) = h(t)$$

implies that

$$u(t) - u(0) = a_r(h(t) - h(0)) + b_r \int_0^t h(s)ds.$$

Thus from the initial condition $u(0) = u_0$, we obtain

$$u(t) = u_0 - a_r h(0) + a_r h(t) + b_r \int_0^t h(s)ds.$$

Hence we get (5).
Conversely, if u satisfies (5), then $({}^{CF}D_0^r u)(t) = h(t)$; for $t \in I$, and $u(0) = u_0$. □

In the sequel we will make use of the following generalization of the classical Darbo fixed point theorem for Fréchet spaces.

Theorem 1. *[18, 19] Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space F and let $V : \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

3 Existence Results

Now, we shall prove the main results concerning the existence of solutions of our problems.

Let us introduce the following hypotheses.

(H_1) The function $t \mapsto f(t,u)$ is measurable on $\mathbb{R}_+$ for each $u \in E$, and the function $u \mapsto f(t,u)$ is continuous on E for a.e. $t \in \mathbb{R}_+$.

(H_2) There exists a continuous function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,u)\| \le p(t)(1 + \|u\|); \; \textit{for a.e. } t \in \mathbb{R}_+, \textit{ and each } u \in E.$$

(H_3) For each bounded set $B \subset E$ and for each $t \in \mathbb{R}_+$, we have

$$\mu(f(t,B)) \leq p(t)\mu(B),$$

where μ is a measure of noncompactness on the Banach space E.

(H_4) The function $Q : C(\mathbb{R}_+, E) \to E$ is continuous, and there exists a constant $q^* > 0$, such that

$$\|Q(u)\| \leq q^*(1 + \|u\|_\infty);\ for\ each\ u \in C(\mathbb{R}_+, E).$$

Moreover, for each bounded set $B_1 \subset X$, we have

$$\mu(Q(B_1)) \leq q^* \sup_{t \in I_n} \mu(B_1(t)),$$

where $B_1(t) = \{u(t) : u \in B_1\};\ t \in I_n;\ n \in \mathbb{N}$.

For $n \in \mathbb{N}$, let

$$p_n^* = \sup_{t \in [0,n]} p(t),$$

and define on $X := C(\mathbb{R}_+, E)$ the family of measure of noncompactness by

$$\mu_n(D) = \omega_0^n(D) + \sup_{t \in [0,n]} \mu(D(t)),$$

where $D(t) = \{v(t) \in E : v \in D\};\ t \in [0,n]$.

3.1 The Initial Value Problem

In this section, we are concerned with the existence results of the problem (1)-(2).

Definition 6. *By a solution of the problem (1)-(2) we mean a continuous function $u \in X$ that satisfies the integral equation*

$$u(t) = c + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds,$$

where $c = u_0 - a_r f(0, u_0)$.

Theorem 2. *Assume that the hypotheses $(H_1) - (H_3)$ hold.*
If

$$\ell_n := p_n^*(2a_r + 4nb_r) < 1;$$

for each $n \in \mathbb{N}^*$, then the problem (1)-(2) has at least one solution.

Proof. Consider the operator $N : X \to X$ defined by:

$$(Nu)(t) = c + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds. \tag{6}$$

Clearly, the fixed points of the operator N are solution of the problem (1)-(2).

For any $n \in \mathbb{N}^*$, we set

$$R_n \geq \frac{\|c\| + p_n^*(a_r + nb_r)}{1 - p_n^*(a_r + nb_r)},$$

and we consider the ball

$$B_{R_n} := B(0, R_n) = \{w \in X : \|w\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}^*$, and each $u \in B_{R_n}$ and $t \in [0, n]$ we have

$$\begin{aligned}
|(Nu)(t)| &\leq \|c\| + a_r\|f(t, u(t))\| + b_r \int_0^t \|f(s, u(s))\| ds \\
&\leq \|c\| + a_r p(t)(1 + \|u(t)\|) + b_r \int_0^t p(s)(1 + \|u(s)\|) ds \\
&\leq \|c\| + a_r p_n^*(1 + R_n) + b_r p_n^*(1 + R_n) \int_0^t ds \\
&\leq \|c\| + p_n^*(a_r + nb_r)(1 + R_n) \\
&\leq R_n.
\end{aligned}$$

Thus

$$\|N(u)\|_n \leq R_n. \tag{7}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 1. The proof will be given in two steps.

Step 1. *$N(B_{R_n})$ is bounded and $N : B_{R_n} \to B_{R_n}$ is continuous.*
Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.
Let $\{u_k\}_{k \in \mathbb{N}}$ be a sequence such that $u_k \to u$ in B_{R_n}. Then, for each $t \in [0, n]$, we have

$$\|(Nu_k)(t) - (Nu)(t)\| \leq a_r\|f(t, u_k(t)) - f(t, u(t))\| + b_r \int_0^t \|f(s, u_k(s)) - f(s, u(s))\| ds.$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(u_k) - N(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

Step 2. *For each bounded equicontinuous subset D of B_{R_n}, $\mu_n(N(D)) \leq \ell_n \mu_n(D)$.*
From Lemmas 1 and 2, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence

$\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$\begin{aligned} \mu((ND)(t)) &= \mu\left(\left\{c + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds;\ u \in D\right\}\right) \\ &\leq 2\mu\left(\{a_r f(t, u_k(t))\}_{k=1}^{\infty}\right) + 2\mu\left(\left\{b_r \int_0^t f(s, u_k(s))ds\right\}_{k=1}^{\infty}\right) + \epsilon \\ &\leq 2a_r\mu\left(\{f(t, u_k(t))\}_{k=1}^{\infty}\right) + 4b_r \int_0^t \mu\left(\{f(s, u_k(s))\}_{k=1}^{\infty}\right) ds + \epsilon \\ &\leq 2a_r p(t)\mu\left(\{u_k(t)\}_{k=1}^{\infty}\right) + 4b_r \int_0^t p(s)\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon \\ &\leq 2a_r p_n^* \mu_n(D) + 4nb_r p_n^* \mu_n(D) + \epsilon \\ &= (2a_r + 4nb_r)p_n^*\ \mu_n(D) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((ND)(t)) \leq p_n^*(2a_r + 4nb_r)\ \mu_n(D).$$

Thus

$$\mu_n(N(D)) \leq p_n^*(2a_r + 4nb_r)\ \mu_n(D).$$

As a consequence of steps 1 and 2 together with Theorem 1, we can conclude that N has at least one fixed point in B_{R_n} which is a solution of problem (1)-(2). □

3.2 The Problem with Nonlocal Condition

Now, we are concerned with the existence results of the problem (1),(3).

Definition 7. *By a solution of the problem (1),(3) we mean a continuous function $u \in X$ that satisfies the integral equation*

$$u(t) = c - Q(u) + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds,$$

where $c = u_0 - a_r f(0, u_0)$.

Now, we shall prove the following theorem concerning the existence of solutions of problem (1),(3).

Theorem 3. *Assume that the hypotheses $(H_1) - (H_4)$ hold. If*

$$\lambda_n := 2q^* + p_n^*(2a_r + 4nb_r) < 1,$$

for each $n \in \mathbb{N}^*$, then the problem (1),(3) has at least one solution.

Proof. Consider the operator $N : X \to X$ defined by:

$$(Gu)(t) = c - Q(u) + a_r f(t, u(t)) + b_r \int_0^t f(s, u(s))ds. \tag{8}$$

Clearly, the fixed points of the operator G are solution of the problem (1),(3).

For any $n \in \mathbb{N}^*$, we set

$$\rho_n \geq \frac{\|c\| + q^* + p_n^*(a_r + nb_r)}{1 - q^* - p_n^*(a_r + nb_r)},$$

and we consider the ball

$$B_{\rho_n} := B(0, \rho_n) = \{w \in X : \|w\|_n \leq \rho_n\}.$$

For any $n \in \mathbb{N}^*$, and each $u \in B_{\rho_n}$ and $t \in [0, n]$ we have

$$\begin{aligned}
\|(Gu)(t)\| &\leq \|c\| + \|Q(u)\| + a_r\|f(t, u(t))\| + b_r \int_0^t \|f(s, u(s))\|ds \\
&\leq \|c\| + q^*(1+\|u\|_\infty)+a_r p(t)(1+\|u(t)\|)+b_r \int_0^t p(s)(1+\|u(s)\|)ds \\
&\leq \|c\| + q^*(1 + \rho_n) + a_r p_n^*(1 + \rho_n) + b_r p_n^*(1 + \rho_n) \int_0^t ds \\
&\leq \|c\| + q^*(1 + \rho_n) + p_n^*(a_r + nb_r)(1 + \rho_n) \\
&\leq \rho_n.
\end{aligned}$$

Thus

$$\|G(u)\|_n \leq \rho_n. \tag{9}$$

This proves that G transforms the ball B_{R_n} into itself. As in the proof of Theorem 2, we can show that the operator $G : B_{\rho_n} \to B_{\rho_n}$ satisfies all the assumptions of Theorem 1. Indeed; $G(B_{\rho_n})$ is bounded, and we can easily prove that $G : B_{\rho_n} \to B_{\rho_n}$ is continuous. Next, from Lemmas 1 and 2, for any $D \subset B_{\rho_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0,n]$, we have

$$\begin{aligned}
\mu((GD)(t)) &= \mu\left(\left\{c - Q(u) + a_r f(t,u(t)) + b_r \int_0^t f(s,u(s))ds;\ u \in D\right\}\right) \\
&\leq 2\mu\left(\{Q(u) + a_r f(t,u_k(t))\}_{k=1}^{\infty}\right) \\
&\quad +2\mu\left(\left\{b_r \int_0^t f(s,u_k(s))ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 2\mu\left(\{Q(u_k)\}_{k=1}^{\infty}\right) + 2a_r\mu\left(\{f(t,u_k(t))\}_{k=1}^{\infty}\right) \\
&\quad +4b_r \int_0^t \mu\left(\{f(s,u_k(s))\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 2q^* mu\left(\{u_k(t)\}_{k=1}^{\infty}\right) + 2a_r p(t)\mu\left(\{u_k(t)\}_{k=1}^{\infty}\right) \\
&\quad +4b_r \int_0^t p(s)\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon \\
&\leq 2q^*\mu_n(D) + 2a_r p_n^*\mu_n(D) + 4nb_r p_n^*\mu_n(D) + \epsilon \\
&= [2q^* + p_n^*(2a_r + 4nb_r)]\mu_n(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((GD)(t)) \leq [2q^* + p_n^*(2a_r + 4nb_r)]\mu_n(D).$$

Thus

$$\mu_n(G(D)) \leq [2q^* + p_n^*(2a_r + 4nb_r)]\mu_n(D).$$

Hence, from Theorem 1, we can conclude that G has at least one fixed point in B_{ρ_n} which is a solution of problem(1),(3). □

4 Examples

Let

$$l^1 = \left\{u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{k=1}^{\infty} |u_k| < \infty\right\}$$

be the Banach space with the norm

$$\|u\| = \sum_{k=1}^{\infty} |u_k|,$$

and $C(\mathbb{R}_+, l^1)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into l^1, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\|;\ n \in \mathbb{N}.$$

Example 1. Consider the following problem of Caputo-Fabrizio fractional differential equations

$$\begin{cases} ({}^{CF}D_0^{\frac{1}{2}}u_k)(t) = f_k(t, u(t));\ t \in \mathbb{R}_+, \\ u(0) = (1, 2^{-1}, 2^{-2}, \dots, 2^{-n}, \cdots);\ t \in \mathbb{R}_+,\ k = 1, 2, \cdots, \end{cases} \tag{10}$$

where

$$\begin{cases} f_k(t,u) = \dfrac{(2^{-k} + u_k(t)) \sin t}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})(1+\sqrt{t})};\ t \in (0, +\infty),\ u \in l^1, \\ f_k(0, u) = 0; \qquad\qquad u \in l^1, \end{cases}$$

for each $t \in [0, n];\ n \in \mathbb{N}$, with

$$f = (f_1, f_2, \dots, f_k, \dots),\ and\ u = (u_1, u_2, \dots, u_k, \dots).$$

The hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{|\sin t|}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})(1+\sqrt{t})};\ t \in (0, +\infty), \\ p(0) = 0. \end{cases}$$

So; for any $n \in \mathbb{N}$, we have $p_n^* = \frac{1}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})}$, and

$$\ell_n := p_n^*(2a_r + 4nb_r) = \frac{1}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})}(2a_{\frac{1}{2}} + 4nb_{\frac{1}{2}}) = \frac{1}{32} < 1.$$

Simple computations show that all conditions of Theorem 2 are satisfied. Consequently, the problem (10) has at least one solution defined on $\mathbb{R}_+$.

Example 2. Consider now the following problem of Caputo-Fabrizio fractional differential equations

$$\begin{cases} ({}^{CF}D_0^{\frac{1}{2}}u_k)(t) = f_k(t, u(t));\ t \in \mathbb{R}_+, \\ u(0) + Q(u) = (1, 2^{-1}, 2^{-2}, \dots, 2^{-n}, \cdots);\ t \in \mathbb{R}_+,\ k = 1, 2, \cdots, \end{cases} \tag{11}$$

where $Q = (Q_1, Q_2, \dots, Q_k, \dots),\ Q : C(\mathbb{R}_+, l^1) \to l^1$, and

$$Q_k(u) = \frac{2^{-k} + u_k}{64};\ k = 1, 2, \cdots.$$

In addition to hypotheses $(H_1)-(H_3)$, the hypothesis (H_4) is satisfies with $q^* = \frac{1}{64}$. Also we have

$$\lambda_n := 2q_n^* + p_n^*(2a_r + 4nb_r) = \frac{1}{32} + \frac{1}{64(a_{\frac{1}{2}} + 2nb_{\frac{1}{2}})}(2a_{\frac{1}{2}} + 4nb_{\frac{1}{2}}) = \frac{1}{16} < 1.$$

Simple computations show that all conditions of Theorem 3 are satisfied. Consequently, the problem (11) has at least one solution defined on $\mathbb{R}_+$.

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