Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 1(63), No. 1 - 2021, 241-246
https://doi.org/10.31926/but.mif.2021.1.63.1.17

PRESERVATION OF LIPSCHITZ CONSTANTS BY SOME GENERALIZED BASKAKOV AND SZÁSZ - MIRAKYAN OPERATORS

Maria TALPĂU DIMITRIU[1]

Abstract

In this paper, we consider a Stancu type generalization of Baskakov operators and Szász-Mirakyan operators and we prove the preservation of Lipschitz constants by these operators.

2020 *Mathematics Subject Classification:* 41A36, 26A16.
Key words: Baskakov operators, Szász-Mirakyan operators, Preservation of Lipschitz constants.

1 Introduction

Preservation of the Lipschitz classes by Bernstein operators was showed by Hajek (only for the exponent $\alpha = 1$) [6], Lindvall (using probabilistic methods) [7] and Brown, Elliott and Paget (an elementary proof) [4]. For other operators, elementary or probabilistic proofs were given in [5], [8], [1], [2], [12].

In [11], D. D. Stancu introduced the following generalized Bernstein operators

$$S_{n,r,s}(f,x) = \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \sum_{i=0}^{s} p_{s,i}(x) f\left(\frac{j+ir}{n}\right), \tag{1}$$

$f \in C[0,1]$, $x \in [0,1]$, where $n \in \mathbb{N}$, $r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $rs < n$. Bernstein's operators are obtained for $s = 0$ or $s = 1$, $r = 0$ or $s = 1$, $r = 1$.

In the following we will prove the preservation property of Lipschitz classes by the Stancu type generalizations of Baskakov and Szász-Mirakyan operators.

We denote by $Lip_M\alpha$ the class of Lipschitz continuous functions on $[0,\infty)$ with exponent $\alpha \in (0,1]$ and the Lipschitz constant $M > 0$ i.e. the set of all real valued continuous functions f defined on $[0,\infty)$ that verify the condition

$$|f(x) - f(y)| \leq M \cdot |x-y|^{\alpha}, \, (\forall) x, y \in [0,\infty).$$

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: mdimitriu@unitbv.ro

2 A Stancu type generalization of the Baskakov operators

The Baskakov operators [3] are defined by

$$V_n(f,x)=\sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right),\ x\in[0,\infty)\,,\ n\in\mathbb{N} \tag{2}$$

where

$$v_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

and $f:[0,\infty)\longrightarrow\mathbb{R}$ is such that the above series converges.

We define for $r\in\mathbb{N}_0$, $s\in\mathbb{N}$ and $n\in\mathbb{N}$, $n>rs$, the operator

$$V_{n,r,s}(f,x)=\sum_{i=0}^{\infty} v_{n-rs,i}(x)\sum_{j=0}^{\infty} v_{s,j}(x) f\left(\frac{i+rj}{n}\right),\ x\in[0,\infty)\,, \tag{3}$$

where $f:[0,\infty)\longrightarrow\mathbb{R}$ is such that the iterated series (3) is convergent.

Remark 1. *Baskakov's operators are obtained for $r=0$.*

Remark 2. *For the test function $e_k(x)=x^k$, $k=0,1$, using $V_n(e_0,x)=1$, $V_n(e_1,x)=x$, we obtain $V_{n,r,s}(e_0,x)=1$ and $V_{n,r,s}(e_1,x)=x$.*

Theorem 1. *Let $M>0$ and $\alpha\in(0,1]$. If $f\in Lip_M\alpha$, then f is in the domain of $V_{n,r,s}$ and $V_{n,r,s}(f)\in Lip_M\alpha$.*

Proof. Let $0\le x<y<\infty$. We have the following representations

$$\begin{aligned}
&V_{n,r,s}(f,y)\\
&=\sum_{i=0}^{\infty}\binom{n-rs+i-1}{i}\frac{y^i}{(1+y)^{n-rs+i}}\sum_{j=0}^{\infty}\binom{s+j-1}{j}\frac{y^j}{(1+y)^{s+j}} f\left(\frac{i+rj}{n}\right)\\
&=\sum_{i=0}^{\infty}\sum_{k_1=0}^{i}\binom{n-rs+i-1}{i}\frac{1}{(1+y)^{n-rs+i}}\binom{i}{k_1}x^{k_1}(y-x)^{i-k_1}\cdot\\
&\cdot\sum_{j=0}^{\infty}\sum_{k_2=0}^{j}\binom{s+j-1}{j}\frac{1}{(1+y)^{s+j}}\binom{j}{k_2}x^{k_2}(y-x)^{j-k_2} f\left(\frac{i+rj}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\sum_{i=k_1}^{\infty}\sum_{k_2=0}^{\infty}\sum_{j=k_2}^{\infty}\frac{(n-rs+i-1)!}{(n-rs-1)!k_1!(i-k_1)!}\cdot\frac{(s+j-1)!}{(s-1)!k_2!(j-k_2)!}\cdot\\
&\cdot\frac{x^{k_1+k_2}(y-x)^{i-k_1+j-k_2}}{(1+y)^{n-rs+i+s+j}} f\left(\frac{i+rj}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs+k_1+l_1-1)!}{(n-rs-1)!k_1!l_1!}\cdot\frac{(s+k_2+l_2-1)!}{(s-1)!k_2!l_2!}\cdot\\
&\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{(1+y)^{n-rs+k_1+l_1+s+k_2+l_2}} f\left(\frac{k_1+l_1+r(k_2+l_2)}{n}\right)
\end{aligned}$$

and

$$
\begin{aligned}
&V_{n,r,s}(f,x)\\
&=\sum_{k_1=0}^{\infty}\binom{n-rs+k_1-1}{k_1}\frac{x^{k_1}}{(1+x)^{n-rs+k_1}}\sum_{k_2=0}^{\infty}\binom{s+k_2-1}{k_2}\frac{x^{k_2}}{(1+x)^{s+k_2}}f\left(\frac{k_1+rk_2}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\binom{n-rs+k_1-1}{k_1}\frac{x^{k_1}}{(1+y)^{n-rs+k_1}}\cdot\frac{1}{\left(1-\frac{y-x}{1+y}\right)^{n-rs+k_1}}\cdot\\
&\cdot\sum_{k_2=0}^{\infty}\binom{s+k_2-1}{k_2}\frac{x^{k_2}}{(1+y)^{s+k_2}}\cdot\frac{1}{\left(1-\frac{y-x}{1+y}\right)^{s+k_2}}f\left(\frac{k_1+rk_2}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\binom{n-rs+k_1-1}{k_1}\frac{x^{k_1}}{(1+y)^{n-rs+k_1}}\sum_{l_1=0}^{\infty}\binom{n-rs+k_1+l_1-1}{l_1}\left(\frac{y-x}{1+y}\right)^{l_1}\cdot\\
&\cdot\sum_{k_2=0}^{\infty}\binom{s+k_2-1}{k_2}\frac{x^{k_2}}{(1+y)^{s+k_2}}\sum_{l_2=0}^{\infty}\binom{s+k_2+l_2-1}{l_2}\left(\frac{y-x}{1+y}\right)^{l_2}f\left(\frac{k_1+rk_2}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs+k_1+l_1-1)!}{(n-rs-1)!k_1!l_1!}\cdot\frac{(s+k_2+l_2-1)!}{(s-1)!k_2!l_2!}\cdot\\
&\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{(1+y)^{n-rs+k_1+l_1+s+k_2+l_2}}f\left(\frac{k_1+rk_2}{n}\right)
\end{aligned}
$$

Given that $f\in Lip_M\alpha$, the function $\varphi(x)=x^{\alpha}, x\in[0,\infty)$ is a concave function and $V_{n,r,s}(e_0,x)=1$ with the above representation, we obtain

$$
\begin{aligned}
&|V_{n,r,s}(f,y)-V_{n,r,s}(f,x)|\\
&\le\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs+k_1+l_1-1)!}{(n-rs-1)!k_1!l_1!}\cdot\frac{(s+k_2+l_2-1)!}{(s-1)!k_2!l_2!}\cdot\\
&\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{(1+y)^{n-rs+k_1+l_1+s+k_2+l_2}}\left|f\left(\frac{k_1+l_1+r(k_2+l_2)}{n}\right)-f\left(\frac{k_1+rk_2}{n}\right)\right|\\
&\le\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs+k_1+l_1-1)!}{(n-rs-1)!k_1!l_1!}\cdot\frac{(s+k_2+l_2-1)!}{(s-1)!k_2!l_2!}\cdot\\
&\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{(1+y)^{n-rs+k_1+l_1+s+k_2+l_2}}M\left(\frac{k_1+l_1+r(k_2+l_2)}{n}-\frac{k_1+rk_2}{n}\right)^{\alpha}\\
&\le M\left(\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs+k_1+l_1-1)!}{(n-rs-1)!k_1!l_1!}\cdot\frac{(s+k_2+l_2-1)!}{(s-1)!k_2!l_2!}\cdot\right.\\
&\left.\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{(1+y)^{n-rs+k_1+l_1+s+k_2+l_2}}\left(\frac{k_1+l_1+r(k_2+l_2)}{n}-\frac{k_1+rk_2}{n}\right)\right)^{\alpha}\\
&=M\left(V_{n,r,s}(e_1,y)-V_{n,r,s}(e_1,x)\right)^{\alpha}=M(y-x)^{\alpha}.
\end{aligned}
$$

So $V_{n,r,s}f\in Lip_M\alpha$. □

3 A Stancu type generalization of the Szász-Mirakyan operators

The Szász-Mirakyan operators [9], [10] are defined by

$$M_n(f,x)=\sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n}\right),\ x\in[0,\infty)\,,\ n\in\mathbb{N}, \tag{4}$$

where

$$m_{n,k}(x)=\frac{n^k}{k!}\cdot\frac{x^k}{e^{nx}}$$

and $f:[0,\infty)\longrightarrow\mathbb{R}$ is such that the above series converges.

We define for $r\in\mathbb{N}_0$, $s\in\mathbb{N}$ and $n\in\mathbb{N}$, $n>rs$, the operator

$$M_{n,r,s}(f,x)=\sum_{i=0}^{\infty} m_{n-rs,i}(x)\sum_{j=0}^{\infty} m_{s,j}(x) f\left(\frac{i+rj}{n}\right),\ x\in[0,\infty)\,, \tag{5}$$

where $f:[0,\infty)\longrightarrow\mathbb{R}$ is such that the iterated series (5) is convergent.

Remark 3. *Szász-Mirakyan operators are obtained for* $r=0$.

Remark 4. *Using* $M_n(e_0,x)=1$, $M_n(e_1,x)=x$ *we obtain* $M_{n,r,s}(e_0,x)=1$ *and* $M_{n,r,s}(e_1,x)=x$.

Theorem 2. *Let* $M>0$ *and* $\alpha\in(0,1]$. *If* $f\in Lip_M\alpha$, *then* f *is in the domain of* $M_{n,r,s}$ *and* $M_{n,r,s}(f)\in Lip_M\alpha$.

Proof. Let $0\le x<y<\infty$. We have the following representations

$$\begin{aligned}
&M_{n,r,s}(f,y)\\
&=\sum_{i=0}^{\infty}\frac{(n-rs)^i}{i!}\cdot\frac{y^i}{e^{(n-rs)y}}\sum_{j=0}^{\infty}\frac{s^j}{j!}\cdot\frac{y^j}{e^{sy}} f\left(\frac{i+rj}{n}\right)\\
&=\sum_{i=0}^{\infty}\sum_{k_1=0}^{i}\frac{(n-rs)^i}{i!}\cdot\frac{1}{e^{(n-rs)y}}\binom{i}{k_1}x^{k_1}(y-x)^{i-k_1}\cdot\\
&\cdot\sum_{j=0}^{\infty}\sum_{k_2=0}^{j}\frac{s^j}{j!}\cdot\frac{1}{e^{sy}}\binom{j}{k_2}x^{k_2}(y-x)^{j-k_2} f\left(\frac{i+rj}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\sum_{i=k_1}^{\infty}\sum_{k_2=0}^{\infty}\sum_{j=k_2}^{\infty}\frac{(n-rs)^i s^j}{k_1!(i-k_1)!k_2!(j-k_2)!}\cdot\frac{x^{k_1+k_2}(y-x)^{i-k_1+j-k_2}}{e^{(n-rs+s)y}} f\left(\frac{i+rj}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs)^{k_1+l_1}s^{k_2+l_2}}{k_1!l_1!k_2!l_2!}\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{e^{(n-rs+s)y}} f\left(\frac{k_1+l_1+r(k_2+l_2)}{n}\right)
\end{aligned}$$

and

$$\begin{aligned}
&M_{n,r,s}(f,x)\\
&=\sum_{k_1=0}^{\infty}\frac{(n-rs)^{k_1}}{k_1!}\cdot\frac{x^{k_1}}{e^{(n-rs)x}}\sum_{k_2=0}^{\infty}\frac{s^{k_2}}{k_2!}\cdot\frac{x^{k_2}}{e^{sx}}f\left(\frac{k_1+rk_2}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(n-rs)^{k_1}s^{k_2}}{k_1!k_2!}\cdot\frac{x^{k_1+k_2}}{e^{(n-rs+s)y}}e^{(n-rs)(y-x)}e^{s(y-x)}f\left(\frac{k_1+rk_2}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(n-rs)^{k_1}s^{k_2}}{k_1!k_2!}\cdot\frac{x^{k_1+k_2}}{e^{(n-rs+s)y}}\cdot\\
&\cdot\sum_{l_1=0}^{\infty}\frac{(n-rs)^{l_1}(y-x)^{l_1}}{l_1!}\sum_{l_2=0}^{\infty}\frac{s^{l_2}(y-x)^{l_2}}{l_2!}f\left(\frac{k_1+rk_2}{n}\right)\\
&=\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs)^{k_1+l_1}s^{k_2+l_2}}{k_1!l_1!k_2!l_2!}\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{e^{(n-rs+s)y}}f\left(\frac{k_1+rk_2}{n}\right)
\end{aligned}$$

Given that $f\in Lip_M\alpha$, the function $\varphi(x)=x^{\alpha}, x\in[0,\infty)$ is a concave function and $M_{n,r,s}(e_0,x)=1$ with the above representation, we obtain

$$\begin{aligned}
&|M_{n,r,s}(f,y)-M_{n,r,s}(f,x)|\\
&\le\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs)^{k_1+l_1}s^{k_2+l_2}}{k_1!l_1!k_2!l_2!}\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{e^{(n-rs+s)y}}\cdot\\
&\cdot\left|f\left(\frac{k_1+l_1+r(k_2+l_2)}{n}\right)-f\left(\frac{k_1+rk_2}{n}\right)\right|\\
&\le\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs)^{k_1+l_1}s^{k_2+l_2}}{k_1!l_1!k_2!l_2!}\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{e^{(n-rs+s)y}}\cdot\\
&\cdot M\left(\frac{k_1+l_1+r(k_2+l_2)}{n}-\frac{k_1+rk_2}{n}\right)^{\alpha}\\
&\le M\Bigg(\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_2=0}^{\infty}\frac{(n-rs)^{k_1+l_1}s^{k_2+l_2}}{k_1!l_1!k_2!l_2!}\cdot\frac{x^{k_1+k_2}(y-x)^{l_1+l_2}}{e^{(n-rs+s)y}}\cdot\\
&\cdot\left(\frac{k_1+l_1+r(k_2+l_2)}{n}-\frac{k_1+rk_2}{n}\right)\Bigg)^{\alpha}\\
&=M\left(M_{n,r,s}(e_1,y)-M_{n,r,s}(e_1,x)\right)^{\alpha}=M(y-x)^{\alpha}.
\end{aligned}$$

So $M_{n,r,s}f\in Lip_M\alpha$. □

References

[1] Adell, J.A. and de la Cal, J., *Using stochastic processes for studying Bernstein-type operators*, Rend. Circ. Mat. Palermo (2) Suppl. **33** (1993), 125-141.

[2] Adell, J.A. and de la Cal, J., *Preservation of moduli of continuity by Bernstein-type operators*, in Approximation, Probability and Related Fields (G. A. Anastassiou and S. T. Rachev Eds.), Plenum Press, New York, (1994), 1-18.

[3] Baskakov, V., *An instance of a sequence of linear positive operators in the space of continuous functions*, Doklady Akademii Nauk SSSR **113** (1957), 249-251.

[4] Brown, B.M., Elliot, D. and Paget, D.F., *Lipschitz constants for the Bernstein polynomials of a Lipschitz continuous function*, J. Approx. Theory, **49** (1987), 196-199.

[5] Della Vechia, B., *On the preservation of Lipschitz constants for some linear operators*, Bol. Un. Mat. Ital. B, **3(7)** (1989), 125–136.

[6] Hajek, D., *Uniform polynomial approximation*, Amer. Math. Monthly **72** (1965), 681.

[7] Lindvall, T., *Bernstein polynomials and the law of large numbers*, Math. Sci. **7** (1982), 127-139.

[8] Khan, M.K. and Peters, M.A., *Lipschitz constants for some approximation operators of a Lipschitz continuous function*, Journal of Approximation Theory, **59** (1989), 307-315.

[9] Mirakjan, G. M., *Approximation des fonctions continues au moyen de polynomes de la forme* $e^{-nx}\sum_{k=0}^{m_n} C_{k,n}x^k$ [*Approximation of continuous functions with the aid of polynomials of* $e^{-nx}\sum_{k=0}^{m_n} C_{k,n}x^k$], Comptes Rendus de l'Académie des Sciences de lÚRSS **31** (1941), 201–205. (in French)

[10] Szász, O., *Generalizations of S. Bernstein's polynomials to the infinite interval*, J. Res. Natl. Bur. Stand. **45** (1950), no. 3, 239–245.

[11] Stancu, D.D., *A note on a multiparameter Bernstein-type approximating operator*, Mathematica (Cluj) **26(49)** (1984), no. 2, 153–157.

[12] Trif, T., *An elementary proof of the preservation of Lipschitz constants by the Meyer-König and Zeller operators*, JIPAM. Journal of Inequalities in Pure and Applied Mathematics, **4**, (2003), no. 5, Article 90.