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POINTWISE $\mathcal{PR}$- PSEUDO SLANT SUBMANIFOLDS OF PARA-KÄHLER MANIFOLDS

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Abstract

We first introduce a natural definition of the pointwise $\mathcal{PR}$-pseudo-slant submanifolds M in para-Kähler manifolds $\overline{M}$ and then investigate the existence of M by presenting some numerical examples. Finally, we derive some necessary and sufficient conditions for the integrability and foliation of the distributions involved with the definition of such submanifolds in $\overline{M}$.

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1 Introduction

The geometry of the submanifolds in different spaces has always been a topic of great interest in differential geometry. In particular, the slant submanifolds of complex, contact (pseudo)-Riemannian manifolds has a long and fascinating history and currently been extensively studied because of its numerous application to mathematics and physics (refer, [6, 7]). B.-Y. Chen in [11] introduced the concept of slant submanifolds of an almost Hermitian manifold that comprises both totally real and holomorphic submanifolds. A. Bejancu also introduced a notion of CR submanifolds in [4] as the generalizations of totally real and holomorphic submanifolds. N. Papaghiuc [14] originated the notion of semi-slant submanifolds that encompasses the classes of CR submanifolds and slant submanifolds. F. Etayo [12] introduced the theory of pointwise slant submanifolds of almost Hermitian manifolds with the name quasi-slant, as an additional extension to slant submanifolds. Furthermore, A. Carriazo in [5] initiated the study of pseudo-slant submanifolds and derived totally real, holomorphic and CR submanifolds as special cases. In [9], B.-Y. Chen and O. J. Garay studied pointwise slant submanifolds

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of Kaehler manifolds and demonstrated some fundamental results of those submanifolds. Since then several differential geometers have studied and classified the pointwise slant geometry in Riemann settings (see, [15, 16, 21, 22]). These geometric setting may found inappropriate, specifically in the general relativity theory where the metric may not be always positive definite i.e. Riemannian. Hence, B.-Y. Chen and M. I. Munteanu in [8] presented the new class of submanifolds called $\mathcal{PR}$-submanifolds by considering the semi-Riemannian non-degenerate metric structure in para-Kaehler manifolds. Analogous to that S.K. Srivastava and the author continued the study for paracosymplectic manifolds [20]. Recently Alegre-Carriazo [1, 2] and Srivastava-Sharma [17, 18] carried out the notion for slant, semi-slant, pseudo slant submanifolds in para geometries particularly in para-Hermitian and paracosymplectic manifolds, and presented analogies and differences between structure admitted semi-Riemannian and Riemannian metrics.

In the present work, we study the geometry of pointwise $\mathcal{PR}$-pseudo-slant submanifolds in para-Kähler manifolds that naturally englobe slant, $\mathcal{PR}$ and pseudo-slant submanifolds. The organization of the article is as follows. In Sect. 2, we recall some basic information about para-Kähler manifold formulas for submanifolds. Sect. 3 includes some definition of pointwise slant submanifold and characterization results for such submanifold. In Sect. 4 we first explain the construction of pointwise $\mathcal{PR}$-pseudo slant submanifolds along with some numerical examples and then derived the conditions of integrability and totally geodesic foliation for the distributions allied to the characterization of a pointwise $\mathcal{PR}$-pseudo-slant submanifold in para-Kähler manifolds.

2 Preliminaries

A smooth manifold $\overline{M}$ of dimension $2m$ is said to have an almost product structure if $\mathcal{P}^2 = I$, where $\mathcal{P}$ is a $(1,1)$ tensor field and I the identity transformation on $\overline{M}$. For this, the pair $(\overline{M}, \mathcal{P})$ is called almost product manifold. An almost paracomplex manifold is an almost product manifold such that the two eigenbundles $T^{\pm}\overline{M}$ corresponding to the eigenvalues ± 1 of $\mathcal{P}$ have the equal dimension. An almost para-Hermitian manifold $(\overline{M}, \mathcal{P}, \overline{g})$ is a smooth manifold associated with an almost product structure $\mathcal{P}$ and a pseudo-Riemannian metric $\overline{g}$ satisfying

$$\overline{g}(\mathcal{P}X, \mathcal{P}Y) = -\overline{g}(X, Y), \tag{1}$$

Clearly, signature of $\overline{g}$ is necessarily (m, m) for any vector fields X, Y tangent to $\overline{M}$. Also, Eq. (1) implies that

$$\overline{g}(\mathcal{P}X, Y) + \overline{g}(X, \mathcal{P}Y) = 0, \tag{2}$$

for any $X, Y \in \mathfrak{X}(T\overline{M})$; $\mathfrak{X}(T\overline{M})$ being Lie algebra of vector fields of $\overline{M}$. The fundamental 2-form ω of $\overline{M}$ is defined by

$$\omega(X, Y) = \overline{g}(X, \mathcal{P}Y), \quad \forall X, Y \in \mathfrak{X}(T\overline{M}). \tag{3}$$

Definition 1. *An almost para-Hermitian manifold $\overline{M}$ is called a para-Kähler manifold [10] if $\mathcal{P}$ is parallel with respect to $\overline{\nabla}$,* i.e.,

$$(\overline{\nabla}_X \mathcal{P})Y = 0, \quad \forall X, Y \in \mathfrak{X}(T\overline{M}) \tag{4}$$

where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$ with respect to $\overline{g}$.

Let M be an n-dimensional manifolds immersed in a $2m$-dimensional para-Kähler manifold $\overline{M}$. We use the notation g for the induced metric tensor on M such that $g = \overline{g}|_M$ of constant signature and rank [13]. Thus, $\forall\ p \in M$, tangent space $T_p(M)$ is a non-degenerated subspace of $T_p(\overline{M})$ such that $T_p(\overline{M}) = T_p(M) \oplus T_p(M)^{\perp}$, where $T_p(M)^{\perp}$ denotes the normal space of M. If $\mathfrak{X}(TM^{\perp})$ indicates a normal bundle to M and $\mathfrak{X}(TM)$ the tangent bundle to M, then the Gauss and Weingarten formulas are defined respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{5}$$
$$\overline{\nabla}_X \zeta = -A_\zeta X + \nabla^{\perp}_X \zeta, \tag{6}$$

for any $X, Y \in \mathfrak{X}(TM)$ and $\zeta\mathfrak{X}(TM^{\perp})$, where ∇ (resp., $\nabla^{\perp}$) is the induced tangent (resp., normal) connection on $\mathfrak{X}(TM)$ (resp., $\mathfrak{X}(TM^{\perp})$, σ is the second fundamental form, and the Weingarten map A_ζ at ζ is given in [11] by

$$g(A_\zeta X, Y) = \overline{g}(\sigma(X, Y), \zeta). \tag{7}$$

A submanifold M is called *totally geodesic* if its σ vanishes identically, *totally umbilical* if $\sigma(X, Y) = g(X, Y)\mathcal{H}$ and *minimal* if the mean curvature $\mathcal{H}$ vanishes. If we write, for all $\xi \in \mathfrak{X}(TM)$ and $\zeta \in \mathfrak{X}(TM^{\perp})$ that

$$\mathcal{P}\xi = t\xi + f\xi, \tag{8}$$
$$\mathcal{P}\zeta = t'\zeta + f'\zeta, \tag{9}$$

where $t\xi$ (resp., $f\xi$) is tangential (resp., normal) part of $\mathcal{P}\xi$ and $t'\zeta$ (resp., $f'\zeta$) is tangential (resp., normal) part of $\mathcal{P}\zeta$, then for any $X, Y \in \mathfrak{X}(TM)$ we can easily obtain from Eqs. (2) and (8) that

$$\overline{g}(X, tY) = -\overline{g}(tX, Y). \tag{10}$$

3 Pointwise slant submanifolds

The slant submanifolds of para-Hermitian manifolds have already been studied by several authors in [2, 17]. Here motivated by these we define pointwise slant submanifolds in an almost para-Hermitian manifold.

Definition 2. *Let ϕ be an isometric immersion $\phi : M \to \overline{M}$ into an almost para-Hermitian manifold $\overline{M}$ and D_λ be the non-degenerate distribution on M. Then*

D_λ is said to be pointwise slant distribution on M, accordingly M pointwise slant submanifold, if there exists a real valued function λ such that

$$t^2 = \lambda I, \quad g(tX, Y) = -g(X, tY)$$

for any non-null tangent vectors $X, Y \in D_\lambda$ at each given point $p \in M$. Here, λ is called slant function, independent of the choice of the non-null tangent vector fields on M.

Remark 1. *It is to note that, the manifold M is non-degenerate (that is, contains either space-like or time-like vector fields), thus our definition of pointwise slant submanifold can be thought of as a generalization of Chen-Garay definition [9] for $\lambda = cos^2\theta(X)$, where $\theta(X)$ is a slant function.*

Remark 2. *The slant function λ is sometimes chosen as $\cos^2\theta(X)$ or $\cosh^2\theta(X)$ or $-\sinh^2\theta(X)$ for vector fields tangent to M, where $\theta(X)$ is real valued function. In particular, if $\theta(X)$ is slant constant coefficient then pointwise slant is simply slant [17].*

Remark 3. *The behavior of the λ equals to $\cos^2\theta(X)$ or $-\sinh^2\theta(X)$ or $\cosh^2\theta(X)$ depending on the nature of vector fields (that is, angle between spacelike–spacelike or timelike–spacelike or timelike–timelike). A variety of different possibilities for λ as slant constant-coefficient have been addressed in [1, 2, 3], depending on the behaviour of vector fields.*

Remark 4. *Here, it is not hard to see that the holomorphic and totally real submanifolds are improper slant submanifolds with slant coefficients $\lambda = 1$ and $\lambda = 0$, respectively. Thus, a proper slant submanifold is a slant submanifold which is neither holomorphic nor totally real [8].*

Now, we derive some characterizations for pointwise slant submanifold M of $\overline{M}$.

Proposition 1. *If M be an isometric immersion of totally geodesic pointwise slant submanifold into almost para-Hermitian manifolds $\overline{M}$, then M is slant.*

Proof. We take $X \in \mathfrak{D}_\lambda$ such that $tX = \lambda\hat{X}$ where $\hat{X}$ is also a unit tangent vector field orthogonal to X. Now, for $Y \in \mathfrak{X}(TM)$ we can write by using Eqs. (8) and (6) that $\overline{\nabla}_Y(\phi X) = \overline{\nabla}_Y(tX) - A_{nX}Y + \nabla^\perp_Y(nX)$. Above equation by the use of defination of covariant differentiation becomes $\overline{\nabla}_Y(\phi X) = \lambda\overline{\nabla}_Y\hat{X} + \lambda'\hat{X} - A_{nX}Y + \nabla^\perp_Y(nX)$. Again, using the fact that structure is an almost para-Kähler manifold and Eqs. (5), (8), (9) we have that, $\overline{\nabla}_Y(\phi X) = t\nabla_Y X + n\nabla_Y X + t'\sigma(Y, X) + n'\sigma(Y, X)$. Comparing the tangential part of previous expressions we get $\lambda'\hat{X} = t\nabla_Y X + t'\sigma(Y, X) - \lambda\overline{\nabla}_Y\hat{X} + A_{nX}Y.$. Finally, taking inner product of above equation with $\hat{X}$ and the fact that M is totally geodesic, we achieve the required proposition. □

Proposition 2. *If M be a pointwise proper slant submanifold in an almost para-Kähler manifold $\overline{M}$, then ω is closed.*

Proof. The proof of this proposition is analogous to Theorem 5.2 of [9]. Hence omitted. □

4 Pointwise $\mathcal{PR}$-pseudo-slant submanifolds

In this section, similar to [19, 21], we define pointwise $\mathcal{PR}$-pseudo-slant submanifold of a para-Kähler manifold $\overline{M}$ and derive the integrability and totally geodesic foliation conditions for the distribution attached with the definition of such submanifolds.

Definition 3. *Let $M \to \overline{M}$ be an isometric immersion of non-degenerate submanifold M in a para-Kähler manifold $\overline{M}$. Then we say that M is a pointwise $\mathcal{PR}$-pseudo-slant submanifold if it admits the pair of orthogonal distributions i.e., totally real $\mathfrak{D}^{\perp}$ and pointwise slant $\mathfrak{D}_{\lambda}$ with slant function λ satisfying $TM = \mathfrak{D}^{\perp} \oplus \mathfrak{D}_{\lambda}$ such that $\mathcal{P}(\mathfrak{D}^{\perp}) \subseteq \mathfrak{X}(TM)^{\perp}$.*

Let us denote by d_1 and d_2 the dimension of $\mathfrak{D}^{\perp}$ and $\mathfrak{D}_{\lambda}$, respectively then we can see that pointwise $\mathcal{PR}$-pseudo-slant submanifold M of $\overline{M}$ is

- pointwise slant (resp., pointwise pseudo-slant) submanfold, if $d_1 = 0$ and $\mathfrak{D}_{\lambda}$ (resp., the pair $\mathfrak{D}^{\perp}$, $\mathfrak{D}_{\lambda}$) indicates on M with slant function $\lambda = \cos^2(\theta)$ [21].

- $\mathcal{PR}$-pseudo slant submanifold, if $d_1.d_2 \neq 0$ and slant function λ is globally constant [1, 19], in particular if $d_1 \neq 0$ and $\lambda = 1$ then M is a $\mathcal{PR}$-submanifold [8].

Hence, we can say that pointwise $\mathcal{PR}$-pseudo-slant submanifold M of $\overline{M}$ *proper*, if $d_1.d_2 \neq 0$ and λ is non-constant function. Next, we give some numerical examples to validate proper pointwise $\mathcal{PR}$-pseudo-slant submanifolds in para-Kähler manifolds $\overline{M} = \mathbb{R}^6$. Let $\overline{M} = \mathbb{R}^6$ be a 6-dimensional manifold with the standard Cartesian coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$. Define a structure $(\mathcal{P}, \overline{g})$ on $\overline{M}$ by

$$\mathcal{P}e_1 = e_4,\ \mathcal{P}e_2 = e_5,\ \mathcal{P}e_3 = e_6,\ \mathcal{P}e_4 = e_1,\ \mathcal{P}e_5 = e_2, \mathcal{P}e_6 = e_3, \tag{11}$$

$$\overline{g} = \sum_{i=1}^{3}(dx_i)^2 - \sum_{j=4}^{6}(dx_j)^2, \tag{12}$$

where $e_1 = \dfrac{\partial}{\partial x_1}$, $e_2 = \dfrac{\partial}{\partial x_2}$, $e_3 = \dfrac{\partial}{\partial x_3}$, $e_4 = \dfrac{\partial}{\partial x_4}$, $e_5 = \dfrac{\partial}{\partial x_5}$ and $e_6 = \dfrac{\partial}{\partial x_6}$. By straightforward calculations, one verifies that the structure is an almost para-Hermitian manifold. For Levi-Civita connection $\overline{\nabla}$ with respect to $\overline{g}$, we readily conclude that the manifold $(\overline{M}, \mathcal{P}, \overline{g})$ is a para-Kähler manifold.

Example 1. *Let $M \to \overline{M} = \mathbb{R}^6$ is an immersion satisfying Eqs.* (11) *and* (12) *given by*

$$\phi_2(v, \alpha, \beta, t) = (v\cosh(\alpha), v\beta, t, v\sinh(\alpha), k_1, k_2) \tag{13}$$

where k_1, k_2 are constants and $v, \beta \in \mathbb{R} - \{0\}$. Then the TM spanned by the vectors

$$Z_1 = \cosh(\alpha)e_1 + \beta e_2 + \sinh(\alpha)e_4,\ Z_2 = v\sinh(\alpha)e_1 + v\cosh(\alpha)e_4, \quad (14)$$
$$Z_3 = ve_2,\ Z_4 = e_3$$

where $Z_1, Z_2, Z_3, Z_4 \in \mathfrak{X}(TM)$. Therefore from Eqs. (11)*, we obtain that*

$$\mathcal{P}(Z_1) = \sinh(\alpha)e_1 + cosh(\alpha)e_4 + \beta e_5, \mathcal{P}(Z_2) = v\cosh(\alpha)e_1 + v\sinh(\alpha)e_4, \quad (15)$$
$$\mathcal{P}(Z_3) = ve_5, \mathcal{P}(Z_4) = e_6.$$

From Eqs. (12)*,* (14) *and* (15)*, we obtain that $\mathfrak{D}^\perp$ and $\mathfrak{D}_\lambda$ are the subspaces spanned by span$\{Z_3, Z_4\}$ and span$\{Z_1, Z_2\}$ respectively, where $\mathfrak{D}^\perp$ is a totally real distribution and $\mathfrak{D}_\lambda$ is a pointwise slant distribution with slant function $\lambda = \frac{1}{\sqrt{1+\beta^2}}$. Thus, M becomes a proper pointwise $\mathcal{PR}$-pseudo-slant submanifold of $\overline{M}$.*

Example 2. *Let $M \to \overline{M} = \mathbb{R}^6$ is an immersion satisfying Eqs.* (11) *and* (12) *given by*

$$\phi_2(t, s, v, u) = (t + \cos(v), s, \sin(v), k_1, k_2, u^2) \quad (16)$$

where k_1, k_2 are constants and $u, v \in \mathbb{R} - \{0, 1\}$. Then the TM spanned by the vectors

$$Z_1 = e_1,\ Z_2 = e_2,\ Z_3 = -\sin(v)e_1 + \cos(v)e_3,\ Z_4 = 2ue_6, \quad (17)$$

where $Z_1, Z_2, Z_3, Z_4 \in \mathfrak{X}(TM)$. Therefore from Eqs. (11) *and* (12)*, we obtain that*

$$\mathcal{P}(Z_1) = e_4, \mathcal{P}(Z_2) = e_5, \mathcal{P}(Z_3) = -\sin(v)e_4 + \cos(v)e_6, \mathcal{P}(Z_4) = 2ue_3. \quad (18)$$

From Eqs. (17) *and* (18)*, we obtain that $\mathfrak{D}^\perp$ and $\mathfrak{D}_\lambda$ are the subspaces spanned by span$\{Z_1, Z_2\}$ and span$\{Z_3, Z_4\}$ respectively, where $\mathfrak{D}^\perp$ is a totally real distribution and $\mathfrak{D}_\lambda$ is a pointwise slant distribution with slant function $\lambda = \cos(v)$. Thus, M becomes a proper pointwise $\mathcal{PR}$-pseudo-slant submanifold of $\overline{M}$.*

Furthermore, if the distributions $\mathfrak{D}^\perp$ and $\mathfrak{D}_\lambda$ are represented by the projections $P^\perp$ and P_λ, respectively. Then $X = P^\perp X + P_\lambda X$ by definition of M for any $X \in \mathfrak{X}(TM)$. Applying $\mathcal{P}$ and using Eq. (8), we have $\mathcal{P}X = fP^\perp X + tP_\lambda X + fP_\lambda X$. From previous expression we obtain that

$$fP^\perp X \in \mathfrak{X}(\mathfrak{D}^\perp), \quad tP^\perp X = 0, \quad (19)$$
$$tP_\lambda X \in \mathfrak{X}(\mathfrak{D}_\lambda), \quad fP_\lambda X \in \mathfrak{X}(TM^\perp). \quad (20)$$

Again, we have $tX = tP_\lambda X, \quad fX = fP^\perp X + fP_\lambda X$ for $X \in \mathfrak{X}(TM)$ by the use of Eq. (8). Since, $\mathfrak{D}_\lambda$ is pointwise slant distribution, hence we conclude that $\forall$ $X \in \mathfrak{D}_\lambda$ and real-valued function λ defined on M that

$$t^2 X = \lambda X. \quad (21)$$

Now, we derive the characterizations for pointwise $\mathcal{PR}$-pseudo-slant submanifold M of $\overline{M}$:

Theorem 1. *In order for a submanifold of a $\overline{M}$ to be a pointwise $\mathcal{PR}$-pseudo-slant M, it is necessary and sufficient that there exists a λ and a distribution D on M such that*

(i) $D = \{X \in \mathfrak{X}(TM) \mid (t_D)^2 X = \lambda X\}$,

(ii) $tX = 0$, *if* $X \perp D$.

where λ denotes the slant function of M.

Proof. Let M be a pointwise $\mathcal{PR}$-pseudo-slant submanifold of $\overline{M}$. Using Eqs. (19), (20) and (21) we have that $D = \mathfrak{D}_\lambda$, which follows (i) and (ii). Conversely (i) and (ii) implies that $TM = \mathfrak{D}_\lambda \oplus \mathfrak{D}^\perp$. From (ii) , we received that $\mathcal{P}(\mathfrak{D}^\perp) = \mathfrak{D}^\perp$. This completes the proof. □

From theorem 1 we have the following corollary :

Corollary 1. *If M be a pointwise $\mathcal{PR}$-pseudo-slant submanifold in $\overline{M}$. Then for all $X, Y \in \mathfrak{X}(\mathfrak{D}_\lambda)$, we have*

$$g(tX, tY) = \lambda\, g(X, Y) \tag{22}$$

$$g(fX, fY) = g(X, Y) - \lambda\, g(X, Y). \tag{23}$$

Further, we prove an important lemma for later use:

Lemma 1. *Let M be a pointwise $\mathcal{PR}$-pseudo-slant submanifold of a $\overline{M}$. Then canonical structures* (i) $t'f = I - \lambda$ *and* (ii) $f'f = -ft$.

Proof. Formula-(i) can be obtained easily from Eqs.(9), (2) and (23). For formula-(ii), replacing X by tX and employing Eq. (21) in Eq. (8) we obtain that $\mathcal{P}tX = t^2X + ftX = \lambda X + ftX$. On the other hand, using Eq. (9) and formula-(i) we achieve that $\mathcal{P}fX = t'fX + f'fX = (1 - \lambda)X + f'fX$. From previous two expressions, we deduce $\mathcal{P}tX + \mathcal{P}fX = X + ftX + f'fX$. Employing Eqs. (2) and (8) in previous expression we get formula-(ii). This completes the proof. □

Here, we investigate the conditions for distributions linked with the definition of pointwise $\mathcal{PR}$-pseudo-slant submanifold of a para-Kähler manifold being integrable and totally geodesic foliation.

Theorem 2. *Let $M \to \overline{M}$ be a proper pointwise $\mathcal{PR}$-pseudo-slant immersion in a para-Kähler manifold. In order for a totally real distribution $\mathfrak{D}^\perp$ on M to be integrable, it is necessary and sufficient that $g(\sigma(X, tZ), \mathcal{P}Y) = g(\sigma(Y, tZ), \mathcal{P}X)$, $\forall\ X, Y \in \mathfrak{D}^\perp$ and $Z, W \in \mathfrak{D}_\lambda$.*

Proof. We have from Gauss and Weingarten formulas and the fact that structure is para-Hermition, that $g([X, Y], Z) = -\overline{g}(\mathcal{P}\overline{\nabla}_X Y, \mathcal{P}Z) + \overline{g}(\mathcal{P}\overline{\nabla}_Y X, \mathcal{P}Z)$. Next, applying Eqs. (4), (8) and (9) we achieve $-\overline{g}(\mathcal{P}\overline{\nabla}_X Y, \mathcal{P}Z) = -\overline{g}(\overline{\nabla}_X \mathcal{P}Y, tZ) + \overline{g}(\overline{\nabla}_X Y, t'fZ) + \overline{g}(\overline{\nabla}_X Y, f'fZ)$. Using lemma 1, Eqs. (6), (21) and the fact that σ is symmetric in previous expression, we obtain that $-\overline{g}(\mathcal{P}\overline{\nabla}_X Y, \mathcal{P}Z) =$

$g(A_{\mathcal{P}Y}X, tZ)+(1-\lambda)\overline{g}(\overline{\nabla}_X Y, Z)-\overline{g}(\overline{\nabla}_X Y, ftZ)$. Now again from Eqs. (4), (8), (9) and lemma 1, we arrive at $\overline{g}(\mathcal{P}\overline{\nabla}_Y X, \mathcal{P}Z) = -g(A_{\mathcal{P}X}Y, tZ) - (1-\lambda)\,\overline{g}(\overline{\nabla}_Y X, Z) + \overline{g}(\overline{\nabla}_Y X, ftZ)$. Hence from above expressions and Eq. (7) we derive that $\lambda g([X,Y], Z) = g(\sigma(X, tZ), \mathcal{P}Y) - g(\sigma(Y, tZ), \mathcal{P}X)$ This completes the proof. □

Theorem 3. *Let $M \to \overline{M}$ be a proper pointwise $\mathcal{PR}$-pseudo-slant immersion in a para-Kähler manifold. In order for a pointwise slant distribution $\mathfrak{D}_\lambda$ of a M to be integrable, it is necessary and sufficient that $g(\sigma(X, W), ftZ) = g(\sigma(X, Z), ftW)$, $\forall\ X \in \mathfrak{D}^\perp$ and $Z, W \in \mathfrak{D}_\lambda$.*

Proof. The proof is identical to that of Theorem 2 and hence omitted. □

Theorem 4. *Let $M \to \overline{M}$ be a proper pointwise $\mathcal{PR}$-pseudo-slant immersion in a para-Kähler manifold. In order for a totally real distribution $\mathfrak{D}^\perp$ of a M defines a totally geodesic foliation, it is necessary and sufficient that $g(\sigma(X, Z), \mathcal{P}Y) = g(\sigma(X, Y), fZ)$, $\forall\ X, Y \in \mathfrak{D}^\perp$ and $Z \in \mathfrak{D}_\lambda$.*

Proof. We can write $g(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \mathcal{P}Y, tZ) + \overline{g}(\overline{\nabla}_X Y, t'fZ + f'fZ)$, by the virtue of Eqs. (2)-(9). Now previous expression by the use of Eq. (6) and lemma 1 can be represented as $g(\nabla_X Y, Z) = g(A_{\mathcal{P}Y}X, tZ) + \overline{g}(\overline{\nabla}_X Y, Z) - \lambda\overline{g}(\overline{\nabla}_X Y, Z) - g(A_{ftZ}X, Y)$. Finally, using Eqs. (5) and (7), in previous expression we completes the proof. □

Theorem 5. *Let $M \to \overline{M}$ be a proper pointwise $\mathcal{PR}$-pseudo-slant immersion in a para-Kähler manifold. In order for a pointwise slant distribution $\mathfrak{D}_\lambda$ of a M defines a totally geodesic foliation, it is necessary and sufficient that $g(\sigma(tW, Z), \mathcal{P}X) = g(\sigma(X, Z), ftW)$, $\forall\ X \in \mathfrak{D}^\perp$ and $Z, W \in \mathfrak{D}_\lambda$.*

Proof. The proof is identical to that of Theorem 4 and hence omitted. □

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