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2D WAVEGUIDES: ASYMPTOTICS OF EIGENVALUE INDUCED BY A WINDOW IN A SEMITRANSPARENT SEPARATING WALL

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Abstract

Two-dimensional quantum waveguides with common semitransparent wall are considered. It is assumed that there is a small window in the separating wall. It leads to the appearance of an eigenvalue below the continuous spectrum. Semitransparent wall is introduced as a potential supported by a hypersurface. We use the method of asymptotic expansions of boundary problems solutions. It allows us to obtain the main term of the asymptotics explicitly.

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1 Introduction

There is a wide class of mathematical and physical problems related to systems perturbed at spatially small domains. It includes, particularly, systems of waveguides and resonators coupled through small windows. Although these system can be classical (acoustical or electro-dynamical), a special wave of interest to the problem was excited by quantum waveguides and quantum dots in nanoelectronics [19]. Physical properties of such systems are closely related with the spectral properties of the corresponding Schrödinger operator. In case of ballistic regime, the Schrödinger operator reduces to the Laplacian in some domain. There are a lot of papers devoted to the spectral problem for the Laplacian in complex domains. As for the continuous spectrum, this is related to the system behavior at infinity, e.g., periodicity. Influence of such properties on the continuous spectrum were described, e.g., in [6, 7, 25, 26, 8]. Local perturbations cannot change the continuous spectrum but can lead to the appearance of the point spectrum, i.e. bound states. Due to the importance for physical applications, this problem

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is the object of a lot of research for perturbations of different nature: waveguide curvature [13, 18], coupling of a resonator [27, 28], change of boundary condition at some places [11, 12], coupling windows between resonators or waveguides [17, 22, 29, 30, 31, 9, 10]. Eigenvalues are considered both below the continuous spectrum (or in gaps) and imbedded in the continuous spectrum [5]. Last time, a wave of interest appears to semitransparent boundaries (or potentials supported by hypersurfaces) [15, 16, 4, 24, 14, 3, 32, 35, 33, 34, 36]. In the present paper we consider two 2D waveguide separated by a semitransparent barrier. It is assumed that there is a small window in the barrier which leads to the appearance of a bound state below the threshold. We seek the asymptotics of the bound state in the window width. The technique is based on matching of asymptotic expansions of boundary value problem solutions [21].

2 Preliminaries

We consider the Laplace operator $-\Delta$ in a pair of two-dimensional quantum waveguides Ω_- and Ω_+ with widths d_- and d_+. We assume $d_- < d_+ < 2d_-$. The choice of the Cartesian coordinates is shown in Fig. 1. Semitransparent separating boundary lies on the first coordinate axis. Point $(0, 0)$ is the center of the small window of width $2a$. We assume the Dirichlet boundary at the external

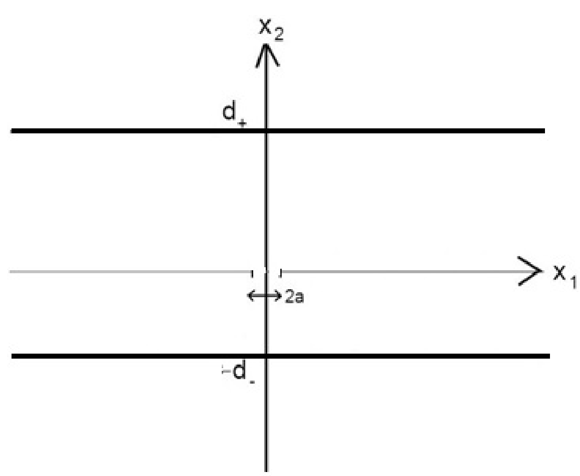

Figure 1: Geometry of the system

boundaries of the waveguides:

$$u(x_1, x_2)|_{x_2=d_\pm} = 0. \tag{1}$$

As for the common boundary, we assume that there is a delta-like potential supported by this line:

$$\begin{cases} u|_{x_2=+0} = u|_{x_2=-0}, \\ \frac{\partial u}{\partial x_2}|_{x_2=+0} - \frac{\partial u}{\partial x_2}|_{x_2=-0} = \alpha u|_{x_2=+0}. \end{cases} \tag{2}$$

The conditions of such type appear if one considers singular potential supported on hypersurface. These potentials have been intensively investigated during last two decades (see Introduction).

Definition 1. *Let $-\Delta$ be the Laplace operator defined on functions from the Sobolev space $W_2^2(\Omega_+) \cup W_2^2(\Omega_-)$ satisfying conditions (1), (2).*

Lemma 1. *Operator $-\Delta$ is self-adjoint.*

It is well-known lemma (see the above mentioned papers. Wall semitransparency is described by the parameter α in (2). Generally, $\alpha \in (0; +\infty)$, where zero value means no barrier and infinity means absolutely nontransparent barrier. Zero value is not included because actually it is not our case.

Due to the window, an eigenvalue appears below the lower bound of the operator continuous spectrum. We construct its asymptotics in window size, i.e. the parameter a (half of the window size).

3 Asymptotics construction

3.1 Eigenfunctions for semitransparent wall

To satisfy the boundary (1) and matching (2) conditions, we seek the eigenfunctions in the following form:

$$\chi_n(x) = \begin{cases} A_n \sin((x_2 - d_+)\nu), & x_2 > 0, \\ B_n \sin((x_2 + d_-)\nu), & x_2 < 0. \end{cases} \tag{3}$$

The choice of the form (10) ensures condition (1). Conditions (2) lead to the following equations:

$$\begin{cases} -A_n \sin(d_+\nu) = B_n \sin(d_-\nu), \\ A_n \nu \cos(d_+\nu) - B_n \nu \cos(d_-\nu) = \alpha B_n \sin(d_-\nu). \end{cases}$$

We are interested in non-trivial solution of the system. It gives us the equation for ν

$$-\nu \cot(d_+\nu) - \nu \cot(d_-\nu) = \alpha. \tag{4}$$

The eigenfunction corresponding to a root of (4) is as follows.

$$\chi_n(x) = \begin{cases} -C_n \sin(d_-\nu) \sin((x_2 - d_+)\nu), & x_2 > 0, \\ C_n \sin(d_+\nu) \sin((x_2 + d_-)\nu), & x_2 < 0. \end{cases} \tag{5}$$

C_n is the normalizing constant.

Equation (4) is the dispersion equation. It has a countable number of roots ν. The squares of the roots are the eigenvalues of the Sturm-Liouville problem for the waveguide cross-section. Let $\lambda_n = \nu_n^2$ be numbered in increasing order. These values are the thresholds for the full operator in the waveguide. Thus, we obtain the next lemma.

Lemma 2. *The continuous spectrum of operator $-\Delta$ consists of a countable number of branches $[\lambda_n, \infty)$, $\lambda_n = \nu_n^2$, where ν_n is root of the dispersion equation (4).*

Remark 1. *One can see that if $\alpha = \infty$ (the Dirichlet separating wall) then for our choice of the system parameters, the first two thresholds are as follows: $\lambda_1 = \left(\frac{\pi}{d_+}\right)^2$, $\lambda_2 = \left(\frac{\pi}{d_-}\right)^2$. If $\alpha = 0$ (absence of separating wall) then one has $\lambda_1 = \left(\frac{\pi}{d_+ + d_-}\right)^2$, $\lambda_2 = \left(\frac{2\pi}{d_+ + d_-}\right)^2$.*

3.2 Green's function

We would like to construct the asymptotics of the eigenvalue k_a^2 close to the first threshold (and less than λ_1). We take the following form of the asymptotic expansion:

$$\sqrt{\lambda_1 - k_a^2} = \sum_{j=2}^{\infty} \sum_{i=0}^{[j/2]-1} k_{ji} a^j \ln^i \frac{a}{a_0}. \tag{6}$$

Here a_0 is the unit of length. As for the asymptotic expansion for the eigenfunction, it has the following form:

$$\psi_a(x) = \begin{cases} \sum_{j=-1}^{\infty} a^j \, P_{j+1}\left(D_y, \ln \frac{a}{a_0}\right) G^-(x, y, k)\Big|_{y=0}, x \in \Omega^- \setminus S_{a_0(a/a_0)^{1/2}}, \\ \sum_{j=1}^{\infty} \sum_{i=0}^{[(j-1)/2]} v_{ji}\left(\frac{x}{a}\right) a^j \ln^i \frac{a}{a_0}, x \in S_{2a_0(a/a_0)^{1/2}}, \\ -\sum_{j=-1}^{\infty} a^j \, P_{j+1}\left(D_y, \ln \frac{a}{a_0}\right) G^+(x, y, k)\Big|_{y=0}, x \in \Omega^+ \setminus S_{a_0(a/a_0)^{1/2}}, \end{cases} \tag{7}$$

where S_r is the disk of radius r with the center at the center of the window, $G^+(G^-)$ is the Green function for the unperturbed operator (i.e. without coupling window) in $\Omega^+(\Omega_-)$, P_n is a polynomial of n-th degree D_y is the differentiation with respect to y. Functions $v_{ji}\left(\frac{x}{a}\right)$ will be determined later.

Let us describe the Green function. The general expression for waveguide is well known:

$$G(x, y, k) = \sum_{n=1}^{\infty} \frac{\chi_n(x_2) \cdot \chi_n(y_2)}{2p_n} e^{-p_n(x_1 - y_1)},$$

where $p_n = \sqrt{\lambda_n - k_a^2}$, $\chi_n(x_2)$ is the $n-$th eigenfunction of the Sturm-Liouville problem for the waveguide cross-section. Keeping in mind expression (5) for χ_n, one obtains the following expression for the Green function:

$$\begin{cases} G^+(x, y, k) = \sum_{n=1}^{\infty} \frac{C_n^2 (\sin(d_- \nu_n))^2 \sin((x_2 - d_+)\nu_n) \sin((y_2 - d_+)\nu_n)}{2p_n} \cdot e^{-p_n(x_1 - y_1)}, \\ G^-(x, y, k) = \sum_{n=1}^{\infty} \frac{C_n^2 (\sin(d_+ \nu_n))^2 \sin((x_2 + d_-)\nu_n) \sin((y_2 + d_-)\nu_n)}{2p_n} \cdot e^{-p_n(x_1 - y_1)}. \end{cases}$$

Differential operator P_n from formula (7) can be described as follows:

$$P_0\left(D_y, \ln \frac{a}{a_0}\right) = a_{10}^{(0)} I, P_1\left(D_y, \ln \frac{a}{a_0}\right) = a_{10}^{(1)} D_y^1, D_y^n = \frac{\partial^n}{\partial n_y^n},$$

$$P_m\left(D_y, \ln\frac{a}{a_0}\right) = \sum_{q=1}^{m-1}\sum_{i=0}^{[(q-1)/2]} a_{qi}^{(m)}\left(\ln\frac{a}{a_0}\right)^i D_y^{m-q+1}, m \geq 2$$

We obtain the following representation for $D_y^n G$:

$$\begin{array}{l} D_y^j G^{\pm}(x,0,k) = \left.\frac{C_{1,\pm}^2(\sin(x_2\nu)\cos(d_\pm\nu)\mp\cos(x_2\nu)\sin(d_\pm\nu))D_y^j\sin((y_2\mp d_\pm)\nu)}{2p_1}\right|_{y=0} e^{-x_1p_1} \\ +\Phi_j(x,k)\ln\frac{r}{a_0} + g_j^{\pm}(x,k) + \sum_{i=0}^{[j/2]}\sum_{t=0}^{j-2i-1} b_{it}^{(j)} r^{-j+2(i+t)}\phi_{j-2i}(\theta) \end{array} \tag{8}$$

where (r,θ) are the polar coordinates with the center at $(0,0)$,

$$\phi_{j-2i}(\theta) = \cos((j-2i)\theta) + \frac{\alpha}{2(j-2i)}\sin((j-2i)\theta).$$

The choice of function ϕ is related to the boundary condition (2). Terms $b_{it}^{(j)}$, $\Phi_j(x,k), g_j^{\pm}(x,k)$ are analytic with respect to k in some neighborhood of the point λ_1,

$$b_{00}^j = (-1)^{[(j+1)/2]}(j-1)!/\pi, b_{10}^3 = \frac{\lambda_2}{2\pi}, \Phi_{1n}(0,k) = -\frac{\lambda_2}{2\pi}.$$

3.3 The main term of the asymptotics

3.3.1 Procedure of constructing

Boundary problems for $v_{ji}\left(\frac{x}{a}\right)$ from (7) can be obtained in the following way. We substitute the series (7) and (6) into the Helmholtz equation (for $k = k_a$) and then change variables $\xi = \frac{x}{a}$. The coefficients in the terms with the same powers of a and $\ln\frac{a}{a_0}$ should be equal. Hence, we obtain the following problems:

$$\begin{array}{l} \Delta_\xi v_{ji} = -\sum_{p=0}^{j-3}\sum_{q=0}^{[p/2]}\Lambda_{pq}v_{j-p-2,i-q}, \quad \xi\in R^2\backslash\gamma, \\ v_{ji} = 0, \quad \xi\in\gamma \end{array} \tag{9}$$

where $\gamma = \{\xi|\xi_2 = 0, \xi_1 \in (-\infty;-1]\cup[1;+\infty)\}$ and Λ_{pq} are the coefficients of the series:

$$k_a^2 = \sum_p\sum_q \Lambda_{pq}a^p\ln^q\frac{a}{a_0}.$$

One can notice that (9) gives us the homogeneous Laplace equation for v_{10} and v_{20} but for v_{30} one gets the inhomogeneous Laplace equation (the Poisson equation) and it looks like $\Delta_\xi v_{30} = -k_0^2 v_{10}$ because of the value of Λ_{00}.

As the next step one needs to introduce operator $M_{pq}(U)$. The operator changes variables in expressions U ($\xi = \frac{x}{a}$, $\rho = \frac{r}{a}$, $\ln r = \ln\rho + \ln a$) and filters summand with $a^p\ln^q\frac{a}{a_0}\varphi(\xi)$. Also $M_p = \sum_q M_{pq}$. It is used to get all summands with a^p.

3.3.2 Calculating of k_{20}

We will find the main term of the asymptotics. As for the next terms, they are determined sequently in accordance with the procedure described above. Let's find summands of order a^1 from (7) using operator M_1.

$$a^{-1}M_1\left(-\sqrt{\lambda_1-k_a^2}\cdot P_1G^+(x,0,k_a)\right)=\tfrac{1}{\pi}k_{20}a_{10}^{(1)}\rho^{-1}\phi_1(\theta)+ \\ \rho\phi_1(\theta)\cdot\tfrac{a_{10}^{(1)}(\sin(d_-\nu_1))^2\nu_1^2\cos^2(d_+\nu_1)}{2}C_1^2, \tag{10}$$

$$a^{-1}M_1\left(\sqrt{\lambda_1-k_a^2}\cdot P_1G^-(x,0,k_a)\right)=-\tfrac{1}{\pi}k_{20}a_{10}^{(1)}\rho^{-1}\phi_1(\theta)+ \\ \rho\phi_1(\theta)\cdot\tfrac{a_{10}^{(1)}(\sin(d_+\nu_1))^2\nu_1^2\cos^2(d_-\nu_1)}{2}C_1^2. \tag{11}$$

To find v_{10}, we use the following lemma [20, 23].

Lemma 3. *There exist harmonic functions $Y_{q1}(\xi),Y_{q2}(\xi)$ in $\mathbb{R}^2\backslash(\mathbb{R}\backslash(-1;1))$, $Y_{qs}|_{\mathbb{R}\backslash(-1;1)}=0$, $Y_{qs}\in W^1_{2,loc}(\mathbb{R}^2)$, which have the following differentiable asymptotics by $\rho,\rho\to\infty$:*

$$X_q=\begin{cases}-\sum_{j=1}^{\infty}\rho^{-j}a_{qj}^+\cos j\theta, & \xi_2>0\\ \rho^q a_q^0\cos q\theta+\sum_{j=1}^{\infty}\rho^{-j}a_{qj}^-\cos j\theta, & \xi_2<0\end{cases}$$
$$Y_q=\begin{cases}-\sum_{j=1}^{\infty}\rho^{-j}b_{qj}^+\sin j\theta, & \xi_2>0\\ \rho^q b_q^0\sin q\theta+\sum_{j=1}^{\infty}\rho^{-j}b_{qj}^-\sin j\theta,\xi_2<0\end{cases}$$

Each harmonic in $\mathbb{R}^2\setminus(\mathbb{R}\setminus(-1;1))$ function V that is 0 on $\mathbb{R}\setminus(-1;1)$ and has the order $O(\rho^q)$ is a linear combination of $X_j(\xi),Y_j(\xi),X_j(\xi^*),Y_j(\xi^*)$ for $j\leq q$, where $\xi^*=(\xi_1,-\xi_2)$.

To find the main term of the asymptotics, we need only two functions which can be obtained by using the complex variable $\zeta=\xi_1+i\xi_2$:

$$X=\Re\ln(\zeta+\sqrt{\zeta^2-1},\quad Y=\Im(\zeta+\sqrt{\zeta^2-1},$$

and, correspondingly, $\tilde{X}(\zeta)=X(\zeta^*),\tilde{Y}(\zeta)=Y(\zeta^*)$. To construct the term ϕ_j satisfying the boundary condition at the semitransparent boundary ($\phi_j=\cos(j\theta)+\frac{\alpha}{2j}\sin(j\theta)$), we use a combination $\phi=X+\frac{\alpha}{2}Y$ and $\tilde{\phi}=\tilde{X}+\frac{\alpha}{2}\tilde{Y}$.

To match terms increasing for $\rho\to\infty$ in accordance with (10), (11), we choose $v_{10}(\xi)$ in such a way:

$$v_{10}(\xi)=\frac{a_{10}^{(1)}(\sin(d_+\nu_1))^2\nu_1^2\cos^2(d_-\nu_1)}{2}C_1^2\phi(\xi)+\frac{a_{10}^{(1)}(\sin(d_-\nu_1))^2\nu_1^2\cos^2(d_+\nu_1)}{2}C_1^2\tilde{\phi}(\xi).$$

Hence, matching terms of order $\rho^{-1}\phi_1(\theta)$ in (10), (11) with v_{10} leads to the following relation:

$$\frac{a_{10}^{(1)}(\sin(d_+\nu_1))^2\nu_1^2\cos^2(d_-\nu_1)}{2}C_1^2+\frac{a_{10}^{(1)}(\sin(d_-\nu_1))^2\nu_1^2\cos^2(d_+\nu_1)}{2}C_1^2=\frac{1}{\pi}k_{20}a_{10}^{(1)}.$$

Finally:

$$k_{20}=\frac{(\sin(d_+\nu_1))^2\nu_1^2\cos^2(d_-\nu_1)}{2}C_1^2+\frac{(\sin(d_-\nu_1))^2\nu_1^2\cos^2(d_+\nu_1)}{2}C_1^2. \tag{12}$$

Thus, we come to the main theorem of this paper.

Theorem 1. *The asymptotics of the eigenvalue k_a^2 of the operator $-\Delta$ induced by a small coupling window of width $2a$ in semitransparent barrier is as follows:*

$$k_a^2 \approx \lambda_1 - k_{20}^2 a^4,$$

where k_{20} is given by formula (12).

3.3.3 Discrepancy estimation

One can obtain the discrepancy estimation using the following lemma from [20].

Lemma 4. *Let $\hat{\psi}_{a,N}, \hat{k}_{a,N}$ be the parts of the corresponding asymptotic series up to the order N. Then, the following correlations take place:*

$$\hat{\psi}_{a,N}(x, \hat{k}_{a,N}) - M_N(\hat{\psi}_{a,N}(x, \hat{k}_{a,N})) = O(r^{N+1} + a^{N+1}(\ln a)^N),$$

$$(\hat{\psi}_{a,N}(x, \hat{k}_{a,N}) - M_N(\hat{\psi}_{a,N}(x, \hat{k}_{a,N})))'_{x_i} = O(r^N + a^{N+1}(\ln a/r)^N).$$

Let us present $\hat{\psi}_{a,1}$ in the form

$$\hat{\psi}_{a,1}(x, \hat{k}_{a,1}) = \kappa(ra^{-1/2})(\hat{\psi}^+_{a,1}(x, \hat{k}_{a,1}) + \hat{\psi}^-_{a,1}(x, \hat{k}_{a,1})) + (1 - \kappa(ra^{-1/2}))\hat{v}_{10},$$

where κ is a cutting function: $\kappa \in C^\infty$,

$$\kappa(t) = \begin{cases} 0, & t \leq 1, \\ 1, & .t \geq 2. \end{cases}$$

Then,

$$\begin{gathered}(\Delta + \hat{k}^2_{a,1})\hat{\psi}_{a,1}(x, \hat{k}_{a,1}) = (1 - \kappa(ra^{-1/2}))(\Delta + \hat{k}^2_{a,1})\hat{v}_{10} + \\ \kappa(ra^{-1/2})(\Delta + \hat{k}^2_{a,1})(\hat{\psi}^+_{a,1}(x, \hat{k}_{a,1}) + \hat{\psi}^-_{a,1}(x, \hat{k}_{a,1})) - \\ \sum_{i=1}^{2} \kappa_{x_i x_i}(ra^{-1/2})(\hat{v}_{10} - \hat{\psi}^+_{a,1}(x, \hat{k}_{a,1}) - \hat{\psi}^-_{a,1}(x, \hat{k}_{a,1})) - \\ \sum_{i=1}^{2} \kappa_{x_i}(ra^{-1/2})(\hat{v}_{10} - \hat{\psi}^+_{a,1}(x, \hat{k}_{a,1}) - \hat{\psi}^-_{a,1}(x, \hat{k}_{a,1}))_{x_i}.\end{gathered}$$

The first term is of order $O(a^{1/2})$, the second gives one zero. The last two terms can be transformed to the following form allowing one to use lemma 4:

$$\begin{gathered}\hat{v}_{10} - \hat{\psi}^+_{a,1}(x, \hat{k}_{a,1}) - \hat{\psi}^-_{a,1}(x, \hat{k}_{a,1}) = \hat{v}_{10} - M_1(\hat{\psi}^+_{a,1}(x, \hat{k}_{a,1}) + \hat{\psi}^-_{a,1}(x, \hat{k}_{a,1})) + \\ M_1(\hat{\psi}^+_{a,1}(x, \hat{k}_{a,1})) - \hat{\psi}^+_{a,1}(x, \hat{k}_{a,1}) + M_1(\hat{\psi}^-_{a,1}(x, \hat{k}_{a,1})) - \hat{\psi}^-_{a,1}(x, \hat{k}_{a,1}).\end{gathered}$$

Using Lemma 4 one obtains that the last two terms has the order $O(a^{1/2})$ in $L_2(\Omega^- \cup \Omega^+)$ and, finally, one comes to the discrepancy estimation:

$$\|(\Delta + \hat{k}^2_{a,1})\hat{\psi}_{a,1}(x, \hat{k}_{a,1})\|_{L_2(\Omega^- \Cup \Omega^+)} \leq C a^{1/2}.$$

4 Conclusion

The suggested procedure can be continued to obtain terms of the asymptotic expansion of any order. The obtained results give one an estimation of the shift of the eigenvalue with respect to the threshold. These results can be useful for the description of "quantum waveguide – quantum dot – quantum waveguide" systems. One can find such systems in different nanotechnological applications.

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