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AN EVALUATION OF WEIGHTED POLYNOMIAL INTERPOLATION WITH CERTAIN CONDITIONS ON THE ROOTS OF HERMITE POLYNOMIAL

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Abstract

The purpose of this paper is to construct a polynomial R_n of degree at most $3n-2$ satisfying weighted $(0,2;0)$ interpolation under certain conditions at the zeroes of H_n and H_n', where H_n stands for Hermite polynomial. Furthermore, we prove a convergence theorem for R_n.

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1 Introduction

In 1961, J Balázs [1] initiated the study of weighted $(0,2)$ interpolation on the zeroes of n^{th} ultraspherical polynomials. L. Szili [14] extended his study by taking roots as the zeroes of n^{th} Hermite polyomials. In 1975, L.G. Pál [8] introduced a modification of Hermite-Fejér interpolation in which the function values and first derivatives were prescribed on two set of nodes $\{x_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n}$. He considered they are distributed on the real line such that

$$-\infty < x_1 < y_1 < x_2 < ..... < x_k < y_k .... < y_{n-1} < x_n < +\infty \tag{1}$$

He proved that, there exists a unique polynomial $P_n(x)$ of degree at most $2n-1$ satisfying the following condition:

$$\begin{cases} P_n(x_k) = \alpha_k & (k = 1, 2, ..., n), \\ P_n'(y_k) = \beta_k & (k = 1, 2, ..., n-1), \end{cases} \tag{2}$$

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with initial condition $P_n(x_0) = 0$, where x_0 is a given point different from the nodal points and $\{\alpha_k\}_{k=1}^{n}$, $\{\beta_k\}_{k=1}^{n-1}$ are arbitrary numbers. Later, S.A. Eneduanya [2] proved the convergence for $P_n(x)$ on the roots of $\pi_n(x)$.

In 1985, L. Szili [15] applied this interpolation process by taking the mixed zeroes of $H_n(x)$ and its derivative on infinite interval. For n even, he showed that there exists a uniquely determined polynomial $P_n^*(x)$ of degree $\leq 2n-1$, satisfying the conditions:

$$\begin{cases} P_n^*(x_k) = \alpha_k^* & (k = 1, 2, ..., n), \\ P_n^{*\prime}(y_k) = \beta_k^* & (k = 1, 2, ..., n-1), \end{cases} \tag{3}$$

$$P_n^*(0) = -2\sum_{i=0}^{n} \alpha_k^* [\frac{H_n(o)}{H_n'(x_k)}]^2 \tag{4}$$

which is given by

$$P_n^*(x) = \sum_{i=1}^{n} \alpha_k^* A_k(x) + \sum_{i=1}^{n-1} \beta_k^* B_k(x) \tag{5}$$

and uniqueness does not hold for taking n odd. Furthermore, he proved the convergence theorem for $P_n^*(x)$. In 1994, I. Joó [5] improved Szili [14] result by modifying the estimate of the fundamental polynomials.

In 1999, Z.F. Sebestyen [9] improved the result of L. Szili [14] and I. Zoo[5] by replacing the condition with an interpolatory condition $P_n^*(0) = \alpha_0$ for n even, where α_0 is an arbitrary number.

Srivastava and Mathur [12], studied mixed type weighted $(0; 0, 2)$ interpolation on the mixed zeroes of $H_n(x)$ and its derivative which means to determine a polynomial $R_n^*(x)$ of degree at most $3n-2$ satisfies the following conditions:

$$\begin{cases} P_n^{**}(x_k) = \alpha_k^{**} & (k = 1....n), \\ P_n^{**}(y_k) = \beta_k^{**} & (k = 1, .2, ..., n-1), \\ (e^{\frac{-x^2}{2}} P_n^{**})''(y_k) = \gamma_k^{**} & (k = 1, 2, ..., n-1), \end{cases} \tag{6}$$

and

$$P_n^{**}(0) = \sum_{i=0}^{n} \alpha_k^{**} \frac{H_n''(o) l_k^2(0)}{H_n'(x_k)}. \tag{7}$$

For n even, they proved that, there exists a unique polynomial of degree at most $3n-2$ satisfying (6)-(7) and for n odd, uniqueness does not exist. Furthermore they proved the convergence theorem for $R_n^*(x)$.

Also, several authors [12], [10], [6] have studied mixed type interpolation with different conditions on different nodes.

In this paper, we studying the $(0, 2; 0)$- interpolation on the zeroes of $H_n(x)$ and its derivative with Z.F Sebestyen's[9] conditions.

We have given the following problem.

Problem:

Let $x_0 = 0$ be a real number differing from the interscaled system of nodal points (1) where $\{x_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n-1}$ are the zeroes of $H_n(x)$ and $H'_n(x)$ respectively. We search for a possible minimal degree polynomial $R_n(x)$ which satisfies the following interpolation conditions:

$$\begin{cases} R_n(x_k) = g_k & (k = 0, 1....n), \\ R_n(y_k) = g_k^* & (k = 1, .2, ..., n-1), \\ (e^{\frac{-x^2}{2}} R_n)''(y_k) = g_k^{**} & (k = 1, 2, ..., n-1), \end{cases} \tag{8}$$

2 Preliminaries

In this section, we gave some well-known results, which we will use to prove Theorem 1, Lemma 1, Lemma 2, Lemma 3 and Theorem 2.

The differential equation satisfied by $H_n(x)$ is given by

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0 \tag{9}$$

$$H'_n(x) = 2nH_{n-1}(x) \tag{10}$$

The fundamental polynomials of Lagrange interpolation corresponding to the nodal point x_k and y_k are given by

$$l_k(x) = \frac{H_n(x)}{H'_n(x_k)(x - x_k)} \quad k = 1.....n \tag{11}$$

$$L_k(x) = \frac{H'_n(x)}{H''_n(y_k)(x - y_k)} \quad k = 1....n-1 \tag{12}$$

and they satisfy the conditions given below

$$l_k(x_j) = \begin{cases} 0 & for \;\; j \neq k \\ 1 & for \;\; j = k \end{cases} \quad for \;\; k = 1....n \tag{13}$$

$$L_k(y_j) = \begin{cases} 0 & for \;\; j \neq k \\ 1 & for \;\; j = k \end{cases} \quad for \;\; k = 1....n-1 \tag{14}$$

$$l'_k(x_j) = \begin{cases} \frac{H'_n(x_j)}{H_n(x_k)(x_j - x_k)} & for \;\; j \neq k \\ x_k & for \;\; j = k \end{cases} \quad for \;\; k = 1....n \tag{15}$$

$$L'_k(y_j) = \begin{cases} \frac{H''_n(y_j)}{H''_n(y_k)(x_j - x_k)} & for \;\; j \neq k \\ y_k & for \;\; j = k \end{cases} \quad for \;\; k = 1....n-1 \tag{16}$$

$$L''_k(y_j) = \begin{cases} \frac{2H''_n(y_j)}{H''_n(y_k)(x_j - x_k)}\{y_j - \frac{1}{y_j - y_k}\} & for \;\; j \neq k \\ \frac{4y_k^2 - 2(n-2)}{3} & for \;\; j = k \end{cases} \quad for \;\; k = 1....n \tag{17}$$

G. Szegö, [13] gave following results:
For the roots of $H_n(x)$, we have

$$x_k^2 \sim \frac{k^2}{n} \tag{18}$$

$$H_n(x) = O(n^{\frac{1}{4}}\sqrt{2^n n!}(1+\sqrt[3]{|x|})e^{\frac{x_k^2}{2}}) \quad x \in R \tag{19}$$

$$|l_k(x)| = O(1)\frac{2^{n+1}n!\sqrt{n}e^{\frac{\nu(x^2+x_k^2)}{2}}}{{H_n'}^2(x_k)} \quad \nu > 1 \quad and \quad k = 1...n \tag{20}$$

R. Srivastava and K.K Mathur [11], proved that

$$|L_k(x)| = O(\frac{2^n n! e^{\frac{\nu(x^2+y_k^2)}{2}}}{\sqrt{n}H_n^2(y_k)}) \quad \nu > 1 \quad and \quad k = 1...n-1 \tag{21}$$

L. Szili [15] gave following results

$$|H_n'(x_k)| \geq c_1 n^{\frac{1}{4}}\sqrt{2^{n+1}n!}e^{\frac{\delta x_k^2}{2}} \quad (i = 1....n) \tag{22}$$

$$|H_n(y_k)| \geq c_2 n^{\frac{-1}{4}}\sqrt{2^{n+1}n!}e^{\frac{\delta y_k^2}{2}} \quad (i = 1....n-1), \tag{23}$$

where c_1, c_2 are constants which are independent of n and $0 < \delta < 1$ is an arbitrarily given real number. He also proved that

$$\sum_{i=0}^{n} e^{-\epsilon x_k^2} = O(\sqrt{n}) \tag{24}$$

$$\sum_{i=0}^{n} \frac{e^{\delta x_k^2}}{{H_n'}^2(x_k)} = O(2^{n+1}n!)^{-1} \tag{25}$$

Definition: $\omega(f,\delta)$ denotes the special form of modulus of continuity introduced by G.Freud [4], given by

$$\omega(f,\delta) = \sup_{0\leq t\leq\delta}\{\| W(x+t)f(x+t) - W(x)f(x) \| + \| \tau(\delta x)W(x)f(x) \|\}, \tag{26}$$

where

$$\tau(x) = \begin{cases} |x| & for \ |x| \leq 1 \\ 1 & for \ |x| > 1 \end{cases}$$

and $\| \cdot \|$ denotes the sup-norm in $C(\mathbb{R})$, if $f \in C(\mathbb{R})$ and $\lim_{|x|\to\infty} W(x)f(x) = 0$ then $\lim_{\delta\to 0}\omega(f,\delta) = 0$.

G.Freud[3](Theorem 4) and Theorem 1[1] gave the following results:

Let $f:\mathbb{R}\to\mathbb{R}$ be continuously differentiable. Further, let

$$\begin{cases} \lim\limits_{|x|\to+\infty} x^{2k}f(x)e^{\frac{-x^2}{2}}=0 \quad (k=0,1,....) \ \ x\in\mathbb{R} \\ and \lim\limits_{|x|\to+\infty} f'(x)e^{\frac{-x^2}{2}}=0 \ \ x\in\mathbb{R} \end{cases} \tag{27}$$

then there exists a polynomial $Q_n(x)$ of degree $\leq n$ such that

$$e^{\frac{-x^2}{2}}|f(x)-Q_n(x)|=O(\frac{1}{\sqrt{n}})\omega(f';\frac{1}{\sqrt{n}}) \tag{28}$$

$$e^{\frac{-x^2}{2}}|f'(x)-Q_n'(x)|=O(1)\omega(f';\frac{1}{\sqrt{n}}) \tag{29}$$

where ω stands for modulus of continuity defined by (26).
Szili[14](Lemma 4, Theorem 4) established the following. For $x\in R$

$$e^{\frac{-x^2}{2}}|Q_n(x)|=O(1) \tag{30}$$

$$e^{\frac{-x^2}{2}}|Q_n'(x)|=O(1) \tag{31}$$

and

$$e^{\frac{-x^2}{2}}|Q_n''(x)|=O(1)\sqrt{n}\omega(f';\frac{1}{\sqrt{n}}),\ \ for\ |x|<\sqrt{2n+1} \tag{32}$$

3 Explicit representation of interpolatory polynomial

In this section, we have proved explicit representation of fundamental polynomials.

Theorem 1. *There exists a polynomial*

$$R_n(x)=\sum_{k=0}^{n}g_kA_k(x)+\sum_{k=1}^{n-1}g_k^{*}B_k(x)+\sum_{k=1}^{n-1}g_k^{**}C_k(x) \tag{33}$$

of degree 3n-2 satisfying condition (8), where $A_k(x)$ $(k=0,1,2....,n)$ *and* $B_k(x)$ $(k=1,2.....,n-1)$ *are the fundamental polynomial of first kind and* $C_k(x)$ $(k=1,2.....,n-1)$ *are fundamental polynomials of second kind of weighted (0,2;0) interpolation. Each such fundamental polynomials of degree at most* $3n-2$ *is given by*

$$A_0(x)=\frac{{H_n}'(x)H_n(x)}{{H_n}'(0)H_n(0)} \tag{34}$$

$$A_k(x)=\frac{x^n{H_n}'(x)l_k(x)}{x_k^n{H_n}'(x_k)}+\frac{2H_n(x)H_n'(x)}{x_k^n(H_n'(x_k))^2}\int_0^x\frac{(n+1)t^n-nt^{n-1}x_k}{(t-y_k)^2}dt \tag{35}$$

$$\begin{aligned} B_k(x) =& \frac{x^n H_n(x) L_k(x)}{y_k^n H_n(y_k)} + \frac{H_n(x) H_n'(x)}{n y_k^n H_n^2(y_k)} \int_0^x \frac{(n-1)t^n - y_k t^{n-1}}{(t-y_k)^2} dt \\ & - e^{\frac{-y_k^2}{2}} (y_k^2 + (n-1) y_k^{-2} - (3n+2)) C_k(x) \end{aligned} \tag{36}$$

$$C_k(x) = -\frac{H_n(x) {H_n}'(x) e^{\frac{y_k^2}{2}}}{4n H_n^2(y_k)} \int_0^x L_k(t) dt \tag{37}$$

Proof. It is enough to show that the polynomials $A_k(x)$ $(k = 0, 1, 2...., n)$, $B_k(x)$ $(k = 1, 2...., n-1)$, and $C_k(x)$ $(k = 1, 2...., n-1)$ have the following properties:

$$A_k(x_j) = \begin{cases} 0 & for \;\; j \neq k \\ 1 & for \;\; j = k \end{cases} \quad for \;\; (j, k = 0, ..., n), \tag{38}$$

$$\begin{gathered} A_k(y_j) = 0 \;\; (j = 1...n-1, k = 0, ..., n), \\ (e^{\frac{-x^2}{2}} A_k)''(y_j) = 0 \quad (j = 1, ...n-1, k = 0, ..., n) \end{gathered}$$

$$B_k(y_j) = \begin{cases} 0 & for \;\; j \neq k \\ 1 & for \;\; j = k \end{cases} \quad for \;\; (j, k = 1, ..., n-1), \tag{39}$$

$$\begin{gathered} B_k(x_j) = 0 \;\; (j = 0, ..n, k = 1, .., n-1), \\ (e^{\frac{-x^2}{2}} B_k)''(y_j) = 0 \quad (j, k = 1, ..., n-1) \end{gathered}$$

and

$$(e^{\frac{-x^2}{2}} C_k)''(y_j) = \begin{cases} 0 & for \;\; j \neq k \\ 1 & for \;\; j = k \end{cases} \quad for \;\; (j, k = 1, ..., n-1), \tag{40}$$

$$\begin{gathered} C_k(y_j) = 0, \quad (j, k = 0, ..., n-1), \\ C_k(x_j) = 0 \;\; (j = 0, ...n, k = 1, .., n-1). \end{gathered}$$

First, we construct the polynomials $C_k(x)$. Let k be fixed (k∈{1,..., n-1}), from (40) it follows that

$$C_k(x) = H_n(x) {H_n}'(x) q_k(x), \tag{41}$$

where $p_k(x)$ is the polynomial such that,

$$p_k(0) = 0. \tag{42}$$

By (41), we get

$$(e^{\frac{-x^2}{2}} C_k)''(y_j) = 4n e^{\frac{-y_k^2}{2}} H_n^2(y_j) q_k'(y_j), \tag{43}$$

(43) satisfies (40), only if,

$$q_k(x) = \frac{1}{4n e^{\frac{-y_k^2}{2}} H_n^2(y_j) q_k'(y_j)} \int_0^x L_k(t) dt. \tag{44}$$

Combining (44), (41), we obtain (37). Obviously, $C_k(x)$ is a polynomial of degree $3n-2$, which satisfies (40). Second, we construct $B_k(x)$, k be fixed (k∈{1,...n-1}). We look for $B_k(x)$ in the following form

$$B_k(x) = c_1 x^n H_n L_k(x) + H_n(x) H_n'(x) w_k(x) + c_2 . C_k(x) \tag{45}$$

where $w_k(x)$ is the suitable polynomial for which

$$w_k(0) = 0 \tag{46}$$

and c_1, c_2 are arbitrary constants. According to (39) $q_k(x)$, for j≠k

$$B_k(y_j) = 0 \tag{47}$$

and for $j = k$

$$B_k(y_k) = 1 \implies c_1 = \frac{1}{y_k^n H_n(y_k)} \tag{48}$$

from (45) and (39) we get for $j \neq k$

$$\begin{aligned} &(e^{\frac{-x^2}{2}} B_k)''(y_j) \\ = \quad & \frac{2e^{\frac{-y_j^2}{2}} H_n^2(y_j)}{H_n(y_k)} [\frac{(n-1)y_j^n - y_k y_j^{n-1}}{(y_j - y_k)^2}] - 2ne^{\frac{-y_j^2}{2}} H_n^2(y_j) w_k'(x) = 0 \end{aligned} \tag{49}$$

for $j = k$

$$(e^{\frac{-x^2}{2}} B_k)''(y_k) = e^{\frac{-y_k^2}{2}} (y_k^2 + (n-1) - (3n-2)) + c_2 = 0 \tag{50}$$

From (49) and (50), we conclude that

$$w_k(x) = \frac{1}{n} \int_0^x \frac{(n-1)t^n - y_k t^{n-1}}{(t - y_k)^2} dt \tag{51}$$

and

$$c_2 = -e^{\frac{-y_k^2}{2}} (y_k^2 + (n-1) y_k^{-2} - (3n-2)) \tag{52}$$

.
Combining (48), (51) and (52), we get (36). It is easy to see that, $B_k(x)$ is a polynomial of degree $3n - 2$, which satisfies (39).
Proof of $A_k(x)$ is like proof of $B_k(x)$. □

4 Order of convergence of fundamental polynomials

In this Section, we the compute order of convergence of fundamental polynomials, which is required to prove theorem 2

Lemma 1. *For $k = 0, 1....n$ and $x \in (-\infty, +\infty)$*

$$\sum_{i=0}^{n} e^{\beta x_k^2} |A_k(x)| = O(\sqrt{n} \log n) e^{\nu x^2} \quad for \quad \nu > \frac{3}{2}, \ 0 < \beta < 1 \tag{53}$$

where $A_k(x)$is given by (35)

Proof. From (35) we have

$$\begin{aligned}\sum_{k=0}^{n} e^{\beta x_k^2}|A_k(x)| \leq & \sum_{k=0}^{n} e^{\beta x_k^2}\frac{|x^n||H_n{}'(x)||l_k(x)|}{|x_k^n||H_n{}'(x_k)|} \\ & + \sum_{k=0}^{n} e^{\beta x_k^2}\frac{2|H_n(x)||H_n'(x)|}{|x_k^n||(H_n'(x_k))^2|}\left|\int_0^x \frac{(n+1)t^n - nt^{n-1}x_k}{(t-y_k)^2}dt\right| \\ \leq & \zeta_1 + \zeta_2 \end{aligned} \tag{54}$$

using (10), (18), (19), (20), (22) and (24), we have

$$\zeta_1 = \sum_{k=0}^{n} e^{\beta x_k^2}\frac{2n|x^n||H_{n-1}(x)||l_k(x)|}{|x_k^n||H_n{}'(x_k)|} = O(\sqrt{n})e^{\nu x^2} \ \mathit{for}\ \nu > \frac{3}{2} \tag{55}$$

Using (10), (18) (19), and (25), we have

$$\zeta_2 = O(\log n)\sum_{k=0}^{n} e^{\beta x_k^2}\frac{4n|H_n(x)||H_{n-1}(x)|}{|x_k^n||(H_n'(x_k))^2|} = O(\sqrt{n}\log n)e^{\nu x^2} \ \mathit{for}\ \nu > \frac{3}{2} \tag{56}$$

Thus, by using (55) and (56) in (54), we get the required lemma. □

Lemma 2. *For $k = 1....n-1$ and $x \in (-\infty, +\infty)$*

$$\sum_{k=1}^{n-1} e^{\beta y_k^2}|B_k(x)| = O(\sqrt{n}\log n)e^{\nu x^2} \quad \mathit{where} \quad \nu > \frac{3}{2},\ 0 < \beta < 1 \tag{57}$$

where $B_k(x)$ is given by (36)

Proof. From (36), we have

$$\begin{aligned}\sum_{k=1}^{n-1} e^{\beta y_k^2}|B_k(x)| \leq & \sum_{k=1}^{n-1} e^{\beta y_k^2}\frac{|x^n||H_n(x)||L_k(x)|}{|y_k^n||H_n(y_k)|} \\ & - \sum_{k=1}^{n-1} e^{\beta y_k^2}\frac{|H_n(x)||H_n'(x)|}{n|y_k^n||H_n^2(y_k)|}\left|\int_0^x \frac{(n-1)t^n - y_k t^{n-1}}{(t-y_k)^2}dt\right| \\ & + \sum_{k=1}^{n-1} e^{\beta y_k^2}|(y_k^2 + (n-1)y_k^{-2} - (3n+2))||C_k(x)| \\ \leq & \zeta_1 + \zeta_2 + \zeta_3 \end{aligned} \tag{58}$$

By using (10), (19), (21), (23) and (24), we have

$$\zeta_1 = O(\sqrt{n})e^{\nu x^2} \ \mathit{for}\ \nu > \frac{3}{2} \tag{59}$$

By using (10), (18) (19), (24), we have

$$\zeta_2 = O(\log n)\sum_{k=0}^{n} e^{\beta x_k^2}\frac{2|H_n(x)||H_{n-1}(x)|}{|y_k^n||H_n^2(y_k)|} = O(\sqrt{n}\log n)e^{\nu x^2} \ \mathit{for}\ \nu > \frac{3}{2} \tag{60}$$

By using (18), (26) and [11](lemma 5.3), we have

$$\zeta_3 = O(\sqrt{n}\log n)e^{\nu x^2} \ for\ \nu > \frac{3}{2} \tag{61}$$

Thus, by using (59)-(61) in (58), lemma follows. □

Lemma 3. *For $k = 1, 2, ..., n-1$ and $x \in (-\infty, +\infty)$*

$$\sum_{k=1}^{n} e^{\beta y_k^2}|C_k(x)| = O(\frac{\log n}{\sqrt{n}})e^{\nu x^2} \quad where \quad \nu > \frac{3}{2},\ 0 < \beta < 1 \tag{62}$$

where $C_k(x)$ is given by (37)

Proof. Lemma follows from[11](lemma 5.3) □

5 Main result: Convergence theorem of interpolatory polynomial

In this section, we have proved convergence theorem for interpolatory polynomial $R_n(x)$.

Theorem 2. *Let the interpolated function $f : R \longrightarrow R$ be continuously differentiable such that*

$$\begin{cases} \lim\limits_{|x|\to+\infty} x^{2k} f(x)\rho(x) = 0 & (k = 0, 1, ...) \\ \lim\limits_{|x|\to+\infty} f'(x)\rho(x) = 0 & ,where \ \ \rho(x) = e^{\frac{-x^2}{2}} \end{cases} \tag{63}$$

further taking the number δ_k such that

$$\delta_k = O(\sqrt{n}e^{\delta y_k^2}\omega(f'; \frac{1}{\sqrt{n}})), k = 1, ..., n-1 \tag{64}$$

where ω is modulus of continuity of f'.Then

$$R_n(f, x) = \sum_{k=0}^{n} f(x_k)A_k(x) + \sum_{k=1}^{n-1} f(y_k)B_k(x) + \sum_{k=1}^{n-1} \delta_k C_k(x) \tag{65}$$

satisfies the relation

$$e^{-\nu x^2}|f(x) - R_n(x)| = O(\log n)\omega(f; \frac{1}{\sqrt{n}}), \nu > \frac{3}{2} \tag{66}$$

Proof. Since $R_n(x)$ given by (33) is exact for all polynomials $Q_n(x)$ of degree $\leq$ 3n-2, we have

$$Q_n(x) = \sum_{k=0}^{n} Q_n(x_k)A_k(x) + \sum_{k=1}^{n-1} Q_n(y_k)B_k(x) + \sum_{k=1}^{n-1}(e^{\frac{-x^2}{2}}Q_n)''(y_k)C_k(x) \tag{67}$$

Using lemma (1)-(2), (65), (67) (26) and (28)-(31), it can be easily seen that,

$$\begin{aligned} e^{-\nu x^2}|R_n(x)-f(x)| \leq & e^{-\nu x^2}|R_n(x)-f(x)| + e^{-\nu x^2}\sum_{k=0}^{n}|f(x_k)-Q_n(x_k)||A_k(x)| \\ & + e^{-\nu x^2}\sum_{k=1}^{n-1}|f(y_k)-Q_n(y_k)||B_k(x)| \\ & + e^{-\nu x^2}\sum_{k=1}^{n-1}|(e^{\frac{-x^2}{2}}Q_n)''(y_k)-\delta_k||C_k(x)| \\ \leq & O(1)\omega(f';\frac{1}{\sqrt{n}}) + O(\log n)\omega(f';\frac{1}{\sqrt{n}}) \\ & + e^{-\nu x^2}\sum_{k=1}^{n-1}|e^{\frac{-y_k^2}{2}}Q_n''(y_k)||C_k(x)| \\ & + e^{-\nu x^2}\sum_{k=1}^{n-1}|(e^{\frac{-y_k^2}{2}})'Q_n'(y_k)||C_k(x)| \\ & + e^{-\nu x^2}\sum_{k=1}^{n-1}|(e^{\frac{-y_k^2}{2}})''Q_n(y_k)||C_k(x)| \\ & + e^{-\nu x^2}\sum_{k=1}^{n-1}|\delta_k C_k(x)| \end{aligned} \tag{68}$$

Thus by using lemma (3), (30)-(32) and (64) in (68) , we get the proof of the required theorem. □

Conclusion:

Let $\{x_k\}_{k=1}^{n}$ and $\{y_k\}_{k=1}^{n-1}$ be the roots of Hermite polynomial $H_n(x)$ and its derivative ${H_n}'(x)$ respectively. If f(x) is a continuously differentiable function on $(-\infty,+\infty)$ satisfying (63), then their exist a polynomial $R_n(x)$(33) satisfying condition (8), which uniformly converges to f(x) on $(-\infty,+\infty)$ as $n\to\infty$.

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