Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 1(63), No. 1 - 2021, 191-208
https://doi.org/10.31926/but.mif.2021.1.63.1.13

SYMMETRIES OF FOUR-DIMENSIONAL LORENTZIAN DAMEK-RICCI SPACES

Assia MOSTEFAOUI[1] **and Noura SIDHOUMI**[*,2]

Abstract

We consider the four-dimensional Damek-Ricci spaces, equipped with the left-invariant Lorentzian metric. We obtain a full classification of Killing and affine vector fields as well as Ricci, curvature and matter collineations. In particular, we prove the non-existence of proper (that is non Killing) affine vector field.

2000 *Mathematics Subject Classification:* 53C50, 53B30.
Key words: Damek-Ricci spaces, Killing vector fields, affine vector fields, Lorentzian metrics, left-invariant metrics, curvature collineation, matter collineation.

1 Introduction

Study of symmetries, over different geometric spaces is one of the interesting topics in geometry and mathematical physics. As we know, several tensor fields live on a (pseudo-) Riemannian manifold (M, g), each of them codifying some geometric or physical quantity of the space. A symmetry of a tensor field T, is a one-parameter group of diffeomorphisms of (M, g), which leaves T invariant. By this definition, each symmetry corresponds to a vector field X which satisfies $L_X T = 0$, where L_X denotes the Lie derivative. Some famous symmetries are: symmetries of the metric tensor g which correspond to the Killing vector fields. A typical use of the Killing field is to express a symmetry in General relativity (in which the geometry of spacetime as distorted by gravitational fields is viewed as a 4-dimensional pseudo-Riemannian manifold). In a static configuration, in which nothing changes with time, the time vector will be a Killing vector, and thus the Killing field will point in the direction of forward motion in time. Furthermore, symmetries of the Levi-Civita connection ∇ correspond to the affine vector

[1]École Nationale Polytechnique d'Oran-Maurice Audin, B.P 1523 El M'naouar Oran 31000, Algeria, e-mail: assia.mostefaoui@enp-oran.dz
[2*] *Corresponding author* École Nationale Polytechnique d'Oran-Maurice Audin, B.P 1523 El M'naouar Oran 31000, Algeria, e-mail: noura.sidhoumi@enp-oran.dz

fields. Therefore, Killing vector field is also affine and it is an interesting problem whether the converse holds for a given manifold (M, g). In particular, if (M, g) is a simply connected spacetime, the existence of a proper (that is, non Killing) affine vector field implies the existence of a second-order covariantely constant symmetric tensor, nowhere vanishing, not proportional to g. As a consequence, the holonomy group of the manifold is reducible (see for example [12]).

Some other symmetries are: symmetries of the Ricci tensor ϱ which correspond to the Ricci collineations. Symmetries of the curvature tensor R which correspond to the curvature collineations. Obviously, every affine vector field is a curvature collineation. The set of all smooth curvature collineations forms a Lie algebra under the Lie bracket operation, which may be infinite-dimensional. A matter collineation is a vector field X that satisfies the condition $L_X \mathcal{T} = 0$, where $\mathcal{T}$ is the energy-momentum tensor given by $\mathcal{T} = \varrho - \frac{1}{2}\tau g$ with τ denotes the scalar curvature. The relation between geometry and physics may be highlighted here (see [3, 6]), as the vector field X is regarded as preserving certain physical quantities along the flow lines of X, this being true for any two observers. In connection with this, it may be shown that every Killing vector field is a matter collineation (by the Einstein field equations, with or without cosmological constant). Thus, a vector field that preserves the metric necessarily preserves the corresponding energy-momentum tensor. When the energy-momentum tensor represents a perfect fluid, every Killing vector field preserves the energy density, pressure and the fluid flow vector field. When the energy-momentum tensor represents an electromagnetic field, a Killing vector field does not necessarily preserve the electric and magnetic fields.

In this paper, we shall study symmetries of the four-dimensional Damek-Ricci spaces $\mathbb{S}^4_\varepsilon$, equipped with a left-invariant Lorentzian metric. We shall Characterize affine and Killing vector fields of $\mathbb{S}^4_\varepsilon$ via a system of partial differential equations. Then, we shall respectively classify Ricci, curvature and matter collineations on the four-dimensional Damek-Ricci spaces.

The geometry of Damek-Ricci spaces, has been constructed by Damek and Ricci in [8]. These spaces are semidirect products of Heisenberg groups with the real line. The study of these spaces is particularly relevant, since they are examples of harmonic manifolds that are not symmetric, proving that the conjecture posed by Lichnerowicz fails in the non-compact case. In fact, several results regarding these spaces have been investigated by many authors. In [9], Degla and Todjihounde proved the non existence of proper (nongeodesic) biharmonic curve in the four-dimensional Damek-Ricci space although such curves exist in three-dimensional Heisenberg groups. In [1], they studied the dispersive properties of the linear wave equation on Damek-Ricci spaces and their application to nonlinear Cauchy problems. In [5] many uncountable isoparametric families of hypersurfaces in Damek-Ricci spaces were constructed, by characterizing those of them that have constant principal curvatures. In [11], Koivogui and Todjihounde gave a setting for constructing Weierstrass representation formulas for simply connected minimal surfaces into four-dimensional Riemannian Damek-Ricci spaces. This was extended to the case of spacelike and timelike minimal surfaces in 4-

dimensional Damek-Ricci spaces equipped with left-invariant Lorentzian metric [7]. In [14], Tan and Deng considered the four-dimensional Lorentzian Damek-Ricci spaces $\mathbb{S}^4_\varepsilon$ and investigated other geometrical properties. In particular, they proved the non-existence of left-invariant Ricci solitons on these spaces. More recently, the second author generalized this result proving the non-existence of non invariant vector field for which the soliton equation is satisfied [13].

The paper is organized in the following way. In Section 2, we shall report some basic information about four-dimensional Damek-Ricci space and its left-invariant metrics in global coordinates, we shall describe their Levi-Civita connection, the curvature and the Ricci tensor. In Section 3 and 4, affine and Killing vector fields of four-dimensional Damek Ricci spaces $\mathbb{S}^4_\varepsilon$ are characterized via a system of partial differential equations, proving the non-exsitence of proper (that is non Killing) affine vector field. Finally, in Section 5 we give a full classification of Ricci, curvature and matter collineations on the four-dimensional Lorentzian Damek-Ricci spaces.

2 Geometry of 4-dimensional Damek-Ricci spaces

We start with a short description of four-dimensional Damek-Ricci spaces, referring to [2] and [8] for more details and further results. For this purpose, we need to recall the so-called generalized Heisenberg group, since Damek-Ricci space depends on it.

2.1 Generalized Heisenberg group

The generalized Heisenberg algebras are defined as follows. Let b and z be real vector spaces of dimension m and n, respectively, such that $\mathfrak{n}$ is the orthogonal sum $\mathfrak{n} = b \oplus z$. We define in $\mathfrak{n}$ the bracket

$$[U + X, V + Y] = \beta(U, V),$$

where $\beta : b \times b \to z$ is a skew-symmetric bilinear map. This product defines a Lie algebra structure on $\mathfrak{n}$.

We equip b with a positive inner product and z with a positive or Lorentzian inner product and let $\langle , \rangle_{\mathfrak{n}}$ denote the product metric. Define a linear map $J : Z \in z \to J_z \in \mathrm{End}(b)$ by

$$\langle J_Z U, V\rangle_{\mathfrak{n}} = \langle \beta(U, V), Z\rangle_{\mathfrak{n}} \text{ for all } U, V \in b \text{ and } Z \in z.$$

Then, $\mathfrak{n}$ is a two-step nilpotent Lie algebra with center z.

- If the inner product in z is positive and $J_Z^2 = -\langle Z, Z\rangle_{\mathfrak{n}} \mathrm{id}_b$, for all $Z \in z$, then the Lie algebra $\mathfrak{n}$ is called a generalized Riemannian Heisenberg algebra, and the associated simply connected nilpotent Lie group, endowed with the induced left-invariant Riemannian metric, is called a generalized Riemannian Heisenberg group.

- If the inner product in z is Lorentzian and

$$J_Z^2 = \begin{cases} -\langle Z, Z\rangle_{\mathfrak{n}} \,\mathrm{id}_b, & \text{when } Z \text{ is spacelike,} \\ \langle Z, Z\rangle_{\mathfrak{n}} \,\mathrm{id}_b, & \text{when } Z \text{ is timelike,} \end{cases}$$

then the Lie algebra $\mathfrak{n}$ is called a generalized Riemannian Heisenberg algebra, and the associated simply connected nilpotent Lie group, endowed with the induced left-invariant Lorentzian metric, is called a generalized Lorentzian Heisenberg group.

2.2 Damek-Ricci spaces

Now, let $\varepsilon = \pm 1$ and $\mathfrak{a}_\varepsilon$ be a one-dimensional pseudo-Riemannian real vector space, which is Riemannian when $\varepsilon = 1$ and Lorentzian when $\varepsilon = -1$, and let $\mathfrak{n}_{-\varepsilon} = b \oplus z$ be a generalized Heisenberg algebra which is Lorentzian when $\varepsilon = 1$ and Riemannian when $\varepsilon = -1$.

Consider a new vector space $\mathfrak{a}_\varepsilon \oplus \mathfrak{n}_{-\varepsilon}$ as the vector space direct sum of $\mathfrak{a}_\varepsilon$ and $\mathfrak{n}_{-\varepsilon}$. Let $s, r \in \mathbb{R}$, $U, V \in b$ and $X, Y \in z$. We define the Lorentzian product $\langle ., .\rangle$ and a Lie bracket $[., .]$ on $\mathfrak{a}_\varepsilon \oplus \mathfrak{n}_{-\varepsilon}$ by

$$\begin{aligned} \langle rA + U + X, sA + V + Y\rangle &= \langle U + X, V + Y\rangle_{\mathfrak{n}_{-\varepsilon}} + \varepsilon rs, \\ [rA + U + X, sA + V + Y] &= [U, V]_{\mathfrak{n}_{-\varepsilon}} + \frac{1}{2} rV - \frac{1}{2} sU + rY - sX, \end{aligned}$$

for a non zero vector A in $\mathfrak{a}_\varepsilon$. Therefore $\mathfrak{a}_\varepsilon \oplus \mathfrak{n}_{-\varepsilon}$ becomes a solvable Lie algebra. The corresponding simply connected Lie group, equipped with the induced left-invariant Lorentzian metric, is called a Lorentzian Damek-Ricci space and will be denoted by $\mathbb{S}_\varepsilon$.

2.3 Curvature of four-dimensional Damek-Ricci spaces

Consider the four-dimensional Damek-Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$, equipped with the left-invariant Lorentzian metric g_ε. Through the paper, we will denote the coordinate basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right\}$ by $\{\partial_x, \partial_y, \partial_z, \partial_t\}$.

As it was pointed in [7], the left-invariant Lorentzian metric g_ε on the four-dimensional space $\mathbb{S}^4_\varepsilon$ is given by

$$g_\varepsilon = e^{-t}\mathrm{d}x^2 + e^{-t}\mathrm{d}y^2 + \varepsilon e^{-2t}(\mathrm{d}z + \frac{c}{2} y\mathrm{d}x - \frac{c}{2} x\mathrm{d}y)^2 - \varepsilon \mathrm{d}t^2, \tag{1}$$

where $c \in \mathbb{R}$.

Following [7], let us denote

$$e_1 = e^{\frac{t}{2}}\left(\frac{\partial}{\partial x} - \frac{cy}{2}\frac{\partial}{\partial z}\right), \; e_2 = e^{\frac{t}{2}}\left(\frac{\partial}{\partial y} + \frac{cx}{2}\frac{\partial}{\partial z}\right), \; e_3 = e^t \frac{\partial}{\partial z}, \; e_4 = \frac{\partial}{\partial t}. \tag{2}$$

Then, $\{e_1, e_2, e_3, e_4\}$ form an orthonormal basis of the Lie algebra $\mathfrak{s}^4$ of $\mathbb{S}^4_\varepsilon$ for which

$$g_\varepsilon(e_1, e_1) = g_\varepsilon(e_2, e_2) = 1, \quad g_\varepsilon(e_3, e_3) = -g_\varepsilon(e_4, e_4) = \varepsilon.$$

The bracket operation in $\mathfrak{s}^4$ is given by the formulas

$$\begin{aligned}
[e_1,e_2] &= ce_3, \quad [e_1,e_3]=0, \quad [e_1,e_4] = -\frac{1}{2}e_1, \\
[e_2,e_3] &= 0, \qquad [e_2,e_4] = -\frac{1}{2}e_2, \quad [e_3,e_4] = -e_3.
\end{aligned} \tag{3}$$

We will denote by ∇ the Levi-Civita connection of $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$, by R its curvature tensor, taken with the sign convention:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

and ϱ the Ricci tensor of $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$, which is defined by

$$\varrho\,(X,Y) = \sum_{k=1}^{5} g_\varepsilon\,(e_k,e_k)\,g_\varepsilon(R(e_k,X)Y,e_k).$$

Using the Koszul formula to calculate the components of the Levi-Civita connection, with respect to the orthonormal basis given by (2), we find

$$\begin{aligned}
\nabla_{e_1} &= \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{\varepsilon c}{2} & 0 \\ 0 & \frac{c}{2} & 0 & 0 \\ -\frac{\varepsilon}{2} & 0 & 0 & 0 \end{pmatrix}, \qquad \nabla_{e_2} = \begin{pmatrix} 0 & 0 & \frac{c\varepsilon}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ -\frac{c}{2} & 0 & 0 & 0 \\ 0 & -\frac{\varepsilon}{2} & 0 & 0 \end{pmatrix}, \\
\nabla_{e_3} &= \begin{pmatrix} 0 & \frac{\varepsilon c}{2} & 0 & 0 \\ -\frac{\varepsilon c}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \nabla_{e_4} = 0.
\end{aligned} \tag{4}$$

Denoting by R_{ij} the matrix describing $R(e_i,e_j)$ with respect to the orthonormal basis given by (2), we have

$$\begin{aligned}
R_{12} &= \begin{pmatrix} 0 & -\frac{\varepsilon}{2} & 0 & 0 \\ \frac{\varepsilon}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{c}{2} \\ 0 & 0 & \frac{c}{2} & 0 \end{pmatrix}, \qquad R_{13} = \begin{pmatrix} 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{\varepsilon c}{4} \\ -\frac{3\varepsilon}{4} & 0 & 0 & 0 \\ 0 & \frac{c}{4} & 0 & 0 \end{pmatrix} \\
R_{14} &= \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & -\frac{\varepsilon c}{4} & 0 \\ 0 & \frac{c}{4} & 0 & 0 \\ -\frac{\varepsilon}{4} & 0 & 0 & 0 \end{pmatrix}, \quad R_{23} = \begin{pmatrix} 0 & 0 & 0 & -\frac{\varepsilon c}{4} \\ 0 & 0 & \frac{3}{4} & 0 \\ 0 & -\frac{3\varepsilon}{4} & 0 & 0 \\ -\frac{c}{4} & 0 & 0 & 0 \end{pmatrix} \\
R_{24} &= \begin{pmatrix} 0 & 0 & \frac{\varepsilon c}{4} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \\ -\frac{c}{4} & 0 & 0 & 0 \\ 0 & -\frac{\varepsilon}{4} & 0 & 0 \end{pmatrix}, \quad R_{34} = \begin{pmatrix} 0 & \frac{\varepsilon c}{2} & 0 & 0 \\ -\frac{\varepsilon c}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\end{aligned} \tag{5}$$

The non-zero components $\varrho_{ij} = \varrho(e_i,e_j)$ of the Ricci tensor are:

$$\varrho_{11} = \varrho_{22} = \frac{\varepsilon}{2}, \quad \varrho_{33} = \frac{5}{2}, \quad \varrho_{44} = -\frac{3}{2}. \tag{6}$$

The scalar curvature is then given by

$$\tau = 5\varepsilon. \tag{7}$$

3 Affine and Killing vector fields of Four-dimensional Lorentzian Damek-Ricci spaces

In this section we completely classify affine and Killing vector fields of four-dimensional Lorentzian Damek-Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$. Let $X = f_1e_1 + f_2e_2 + f_3e_3 + f_4e_4$ be a vector field on $\mathbb{S}^4_\varepsilon$, where f_1, f_2, f_3, f_4 are smooth functions of the variables x, y, z, t. A vector field X tangent to $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ is said to be affine if it satisfies one of the following equivalent conditions:

(i) its local fluxes are given by affine maps,

(ii) $L_X\nabla = 0$, where ∇ is the Levi Civita connection of $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$,

(iii) for all vector fields Y, Z tangent to $\mathbb{S}^4_\varepsilon$:

$$[X, \nabla_Y Z] = \nabla_{[X,Y]}Z + \nabla_Y[X, Z].$$

We first determine the Lie derivative $L_X\nabla$ of the Levi-Civita connection (4), with respect to X. Long but direct calculation yields the following description of the components $(L_X\nabla)_{ij} = (L_X\nabla)(e_i, e_j)$, for all indices $i \leq j$, as follows

$$\begin{aligned}
&(L_X\nabla)_{11}\\
= &\left(\left(\partial_x^2 f_1 - cy\partial_x\partial_z f_1 + \frac{1}{4}y^2\partial_z^2 f_1\right)e^t + \left(\frac{c}{2}y\partial_z f_4 - \partial_x f_4\right)e^{\frac{t}{2}} + \frac{\varepsilon}{2}\partial_t f_1 + \frac{\varepsilon}{4}f_1\right)e_1\\
+ &\left(\left(\partial_x^2 f_2 - cy\partial_x\partial_z f_2 + \frac{1}{4}y^2\partial_z^2 f_2\right)e^t + \left(\frac{\varepsilon}{2}y\partial_z f_3 - c\varepsilon\partial_x f_3\right)e^{\frac{t}{2}} + \frac{\varepsilon}{2}\partial_t f_2 - \frac{3\varepsilon}{4}f_2\right)e_2\\
+ &\left(\left(\partial_x^2 f_3 - cy\partial_x\partial_z f_3 + \frac{1}{4}y^2\partial_z^2 f_3\right)e^t + \left(c\partial_x f_2 - \frac{1}{2}y\partial_z f_2\right)e^{\frac{t}{2}} + \frac{\varepsilon}{2}f_3 + \frac{\varepsilon}{2}\partial_t f_3\right)e_3\\
+ &\left(\left(\partial_x^2 f_4 - cy\partial_x\partial_z f_4 + \frac{1}{4}y^2\partial_z^2 f_4\right)e^t + \left(\frac{\varepsilon c}{2}y\partial_z f_1 - \varepsilon\partial_x f_1\right)e^{\frac{t}{2}} + \frac{\varepsilon}{2}f_4 + \frac{\varepsilon}{2}\partial_t f_4\right)e_4,
\end{aligned}$$

$$\begin{aligned}
(L_X\nabla)_{12} = &\begin{pmatrix} \left(\partial_x\partial_y f_1 + \frac{c}{2}x\partial_z\partial_x f_1 - \frac{c}{2}y\partial_z\partial_y f_1 - \frac{1}{4}xy\partial_z^2 f_1\right)e^t \\ + \left(\frac{c\varepsilon}{2}\partial_x f_3 - \frac{\varepsilon}{4}y\partial_z f_3 - \frac{1}{2}\partial_y f_4 - \frac{c}{4}x\partial_z f_4\right)e^{\frac{t}{2}} + \frac{\varepsilon}{2}f_2 \end{pmatrix} e_1\\
&+ \begin{pmatrix} \left(\partial_x\partial_y f_2 + \frac{c}{2}x\partial_x\partial_z f_2 - \frac{c}{2}y\partial_z\partial_y f_2 - \frac{1}{4}xy\partial_z^2 f_2\right)e^t \\ + \left(\frac{c}{4}y\partial_z f_4 - \frac{c\varepsilon}{2}\partial_y f_3 - \frac{\varepsilon}{4}x\partial_z f_3 - \frac{1}{2}\partial_x f_4\right)e^{\frac{t}{2}} + \frac{\varepsilon}{2}f_1 \end{pmatrix} e_2\\
&+ \begin{pmatrix} \left(\partial_x\partial_y f_3 + \frac{c}{2}x\partial_x\partial_z f_3 - \frac{c}{2}y\partial_z\partial_y f_3 - \frac{1}{4}xy\partial_z^2 f_3\right)e^t \\ + \left(\frac{c}{2}\partial_y f_2 - \frac{c}{2}\partial_x f_1 + \frac{1}{4}y\partial_z f_1 + \frac{1}{4}x\partial_z f_2\right)e^{\frac{t}{2}} \end{pmatrix} e_3\\
&+ \begin{pmatrix} \left(\partial_x\partial_y f_4 + \frac{c}{2}x\partial_x\partial_z f_4 - \frac{c}{2}y\partial_z\partial_y f_4 - \frac{1}{4}xy\partial_z^2 f_4\right)e^t \\ + \left(\frac{c\varepsilon}{4}y\partial_z f_2 - \frac{c\varepsilon}{4}x\partial_z f_1 - \frac{\varepsilon}{2}\partial_y f_1 - \frac{\varepsilon}{2}\partial_x f_2\right)e^{\frac{t}{2}} \end{pmatrix} e_4,
\end{aligned}$$

$$
\begin{aligned}
(L_X\nabla)_{13} \;=\;& \begin{pmatrix} \left(\partial_x\partial_z f_1-\frac{cy}{2}\partial_z^2 f_1\right)e^{\frac{3t}{2}}-\frac{1}{2}\partial_z f_4 e^t \\ +\left(\frac{\varepsilon c}{2}\partial_x f_2-\frac{\varepsilon y}{4}\partial_z f_2+\frac{c\varepsilon}{2}\partial_y f_1+\frac{\varepsilon x}{4}\partial_z f_1\right)e^{\frac{t}{2}} \end{pmatrix} e_1 \\
&+\begin{pmatrix} \left(\partial_x\partial_z f_2-\frac{cy}{2}\partial_z^2 f_2\right)e^{\frac{3t}{2}}-\frac{c\varepsilon}{2}\partial_z f_3 e^t+\frac{\varepsilon c}{2}f_4 \\ +\left(\frac{c\varepsilon}{2}\partial_y f_2-\frac{\varepsilon c}{2}\partial_x f_1+\frac{\varepsilon y}{4}\partial_z f_1+\frac{\varepsilon x}{4}\partial_z f_2\right)e^{\frac{t}{2}} \end{pmatrix} e_2 \\
&+\begin{pmatrix} \left(\partial_x\partial_z f_3-\frac{cy}{2}\partial_z^2 f_3\right)e^{\frac{3t}{2}}+\frac{c}{2}\partial_z f_2 e^t-\frac{\varepsilon}{2}f_1 \\ +\left(\frac{c\varepsilon}{2}\partial_y f_3+\frac{\varepsilon x}{4}\partial_z f_3-\partial_x f_4+\frac{cy}{2}\partial_z f_4\right)e^{\frac{t}{2}} \end{pmatrix} e_3 \\
&+\begin{pmatrix} \left(\frac{cy}{2}\partial_z f_3-\frac{cy}{2}\partial_z^2 f_4+\frac{\varepsilon c}{2}\partial_y f_4+\frac{\varepsilon x}{4}\partial_z f_4-\partial_x f_3\right)e^{\frac{t}{2}} \\ +\partial_x\partial_z f_4 e^{\frac{3t}{2}}-\frac{\varepsilon}{2}e^t\partial_z f_1-cf_2 \end{pmatrix} e_4,
\end{aligned}
$$

$$
\begin{aligned}
(L_X\nabla)_{14} =& \left(\left(\partial_x\partial_t f_1-\frac{cy}{2}\partial_z\partial_t f_1+\frac{1}{2}\partial_x f_1-\frac{cy}{4}\partial_z f_1\right)e^{\frac{t}{2}}-\frac{1}{2}\partial_t f_4\right)e_1 \\
&+\left(\left(\partial_x\partial_t f_2-\frac{cy}{2}\partial_z\partial_t f_2+\frac{1}{2}\partial_x f_2-\frac{cy}{4}\partial_z f_2\right)e^{\frac{t}{2}}-\frac{c\varepsilon}{2}\partial_t f_3-\frac{c\varepsilon}{2}f_3\right)e_2 \\
&+\left(\left(\partial_x\partial_t f_3-\frac{cy}{2}\partial_z\partial_t f_3+\frac{1}{2}\partial_x f_3-\frac{cy}{4}\partial_z f_3\right)e^{\frac{t}{2}}+\frac{c}{2}\partial_t f_2-\frac{c}{4}f_2\right)e_3 \\
&+\left(\left(\partial_x\partial_t f_4-\frac{cy}{2}\partial_z\partial_t f_4+\frac{1}{2}\partial_x f_4-\frac{cy}{4}\partial_z f_4\right)e^{\frac{t}{2}}-\frac{\varepsilon}{4}f_1-\frac{\varepsilon}{2}\partial_t f_1\right)e_4,
\end{aligned}
$$

$$
\begin{aligned}
(L_X\nabla)_{22} \;=\;& \begin{pmatrix} \left(\partial_y^2 f_1+cx\partial_y\partial_z f_1+\frac{x^2}{4}\partial_z^2 f_1\right)e^t \\ +\left(\varepsilon c\partial_y f_3+\frac{\varepsilon x}{2}\partial_z f_3\right)e^{\frac{t}{2}}-\frac{3\varepsilon}{4}f_1+\frac{\varepsilon}{2}\partial_t f_1 \end{pmatrix} e_1 \\
&+\begin{pmatrix} \left(\partial_y^2 f_2+cx\partial_y\partial_z f_2+\frac{x^2}{4}\partial_z^2 f_2\right)e^t \\ -\left(\partial_y f_4+\frac{cx}{2}\partial_z f_4\right)e^{\frac{t}{2}}+\frac{\varepsilon}{4}f_2+\frac{\varepsilon}{2}\partial_t f_2 \end{pmatrix} e_2 \\
&+\begin{pmatrix} \left(\partial_y^2 f_3+cx\partial_y\partial_z f_3+\frac{x^2}{4}\partial_z^2 f_3\right)e^t \\ -\left(c\partial_y f_1+\frac{x}{2}\partial_z f_1\right)e^{\frac{t}{2}}+\frac{\varepsilon}{2}f_3+\frac{\varepsilon}{2}\partial_t f_3 \end{pmatrix} e_3 \\
&+\begin{pmatrix} \left(\partial_y^2 f_4+cx\partial_y\partial_z f_4+\frac{x^2}{4}\partial_z^2 f_4\right)e^t \\ -\left(\varepsilon\partial_y f_2+\frac{c\varepsilon x}{2}\partial_z f_2\right)e^{\frac{t}{2}}+\frac{\varepsilon}{2}f_4+\frac{\varepsilon}{2}\partial_t f_4 \end{pmatrix} e_4,
\end{aligned}
$$

$$
\begin{aligned}
(L_X\nabla)_{23} \;=\;& \begin{pmatrix} \left(\partial_y\partial_z f_1+\frac{cx}{2}\partial_z^2 f_1\right)e^{\frac{3t}{2}}+\frac{c\varepsilon}{2}\partial_z f_3 e^t-\frac{\varepsilon c}{2}f_4 \\ +\left(\frac{c\varepsilon}{2}\partial_y f_2-\frac{\varepsilon c}{2}\partial_x f_1+\frac{\varepsilon y}{4}\partial_z f_1+\frac{\varepsilon x}{4}\partial_z f_2\right)e^{\frac{t}{2}} \end{pmatrix} e_1 \\
&+\begin{pmatrix} \left(\partial_y\partial_z f_2+\frac{cx}{2}\partial_z^2 f_2\right)^{\frac{3t}{2}} \\ +\left(\frac{\varepsilon y}{4}\partial_z f_2-\frac{\varepsilon c}{2}\partial_y f_1-\frac{x\varepsilon}{4}\partial_z f_1-\frac{\varepsilon c}{2}\partial_x f_2\right)e^{\frac{t}{2}}-\frac{1}{2}\partial_z f_4 e^t \end{pmatrix} e_2 \\
&+\begin{pmatrix} \left(\partial_y\partial_z f_3+\frac{cx}{2}\partial_z^2 f_3\right)e^{\frac{3t}{2}}-\frac{c}{2}\partial_z f_1 e^t-\frac{\varepsilon}{2}f_2 \\ +\left(\frac{\varepsilon y}{4}\partial_z f_3-\frac{\varepsilon c}{2}\partial_x f_3-\partial_y f_4-\frac{cx}{2}\partial_z f_4\right)e^{\frac{t}{2}} \end{pmatrix} e_3 \\
&+\begin{pmatrix} \left(\partial_y\partial_z f_4+\frac{cx}{2}\partial_z^2 f_4\right)e^{\frac{3t}{2}}-\frac{\varepsilon e^t}{2}\partial_z f_2+cf_1 \\ +\left(\frac{\varepsilon y}{4}\partial_z f_4-\frac{\varepsilon c}{2}\partial_x f_4-\partial_y f_3-\frac{cx}{2}\partial_z f_3\right)e^{\frac{t}{2}} \end{pmatrix} e_4,
\end{aligned}
$$

$$\begin{aligned}(L_X\nabla)_{24} \;=\;& \left(\left(\partial_y\partial_t f_1+\frac{cx}{2}\partial_z\partial_t f_1+\frac{1}{2}\partial_y f_1+\frac{cx}{4}\partial_z f_1\right)e^{\frac{t}{2}}+\frac{c\varepsilon}{2}\partial_t f_3+\frac{\varepsilon c}{2}f_3\right)e_1\\ &+\left(\left(\partial_y\partial_t f_2+\frac{cx}{2}\partial_z\partial_t f_2+\frac{1}{2}\partial_y f_2+\frac{cx}{4}\partial_z f_2\right)e^{\frac{t}{2}}-\frac{1}{2}\partial_t f_4 e^t\right)e_2\\ &+\left(\left(\partial_y\partial_t f_3+\frac{cx}{2}\partial_z\partial_t f_3+\frac{1}{2}\partial_y f_3+\frac{cx}{4}\partial_z f_3\right)e^{\frac{t}{2}}-\frac{c}{2}\partial_t f_1+\frac{c}{4}f_1\right)e_3\\ &+\left(\left(\partial_y\partial_t f_4+\frac{cx}{2}\partial_z\partial_t f_4+\frac{1}{2}\partial_y f_4+\frac{cx}{4}\partial_z f_4\right)e^{\frac{t}{2}}-\frac{\varepsilon}{2}\partial_t f_2-\frac{\varepsilon}{4}f_2\right)e_4,\end{aligned}$$

$$\begin{aligned}(L_X\nabla)_{33} \;=\;& \left(\partial_z^2 f_1 e^{2t}+\varepsilon c\partial_z f_2 e^t+\frac{1}{2}f_1+\partial_t f_1\right)e_1\\ &+\left(\partial_z^2 f_2 e^{2t}-\varepsilon c\partial_z f_1 e^t+\frac{1}{2}f_2+\partial_t f_2\right)e_2\\ &+\left(\partial_z^2 f_3 e^{2t}-2\partial_z f_4 e^t+f_3+\partial_t f_3\right)e_3\\ &+\left(\partial_z^2 f_4 e^{2t}-2\partial_z f_3 e^t+2f_4+\partial_t f_4\right)e_4,\end{aligned}$$

$$\begin{aligned}(L_X\nabla)_{34} \;=\;& \left((\partial_z\partial_t f_1+\partial_z f_1)e^t+\tfrac{\varepsilon c}{2}\partial_t f_2+\tfrac{\varepsilon c}{4}f_2\right)e_1\\ &+\left((\partial_z\partial_t f_2+\partial_z f_2)e^t-\tfrac{\varepsilon c}{2}\partial_t f_1-\tfrac{\varepsilon c}{4}f_1\right)e_2\\ &+\left((\partial_z\partial_t f_3+\partial_z f_3)e^t-\partial_t f_4\right)e_3\\ &+\left((\partial_z\partial_t f_4+\partial_z f_4)e^t-\partial_t f_3-f_3\right)e_4,\end{aligned}$$

$$(L_X\nabla)_{44}=\left(\partial_t^2 f_1-\tfrac{1}{4}f_1\right)e_1+\left(\partial_t^2 f_2-\tfrac{1}{4}f_2\right)e_2+\left(\partial_t^2 f_3-\tfrac{1}{4}f_3\right)e_3+\left(\partial_t^2 f_4\right)e_4.$$

In order to determine affine vector fields, we will completely solve the system of 40 PDEs obtained by requiring that all coefficients in the above Lie derivative components are equal to zero.

From $(L_X\nabla)_{44}=0$, we prove that

$$\left\{\begin{array}{l} f_1=H(x,y,z)e^{\frac{t}{2}}+\overline{H}(x,y,z)e^{-\frac{t}{2}}\\ f_2=G(x,y,z)e^{\frac{t}{2}}+\overline{G}(x,y,z)e^{-\frac{t}{2}}\\ f_3=K(x,y,z)e^{t}+\overline{K}(x,y,z)e^{-t}\\ f_4=S(x,y,z)t+\overline{S}(x,y,z)\end{array}\right. \tag{8}$$

where $H,\overline{H},G,\overline{G},J,\overline{K},S$ and $\overline{S}$ are real-valued smooth functions depending on x,y and z.

We then replace f_1 and f_2 in the equations obtained from $\mathrm{d}x\,[(L_X\nabla)_{34}]=0$ and $\mathrm{d}y\,[(L_X\nabla)_{34}]=0$, we deduce

$$\left\{\begin{array}{l} \partial_z G=\partial_z H=0\\ \partial_z\overline{G}-\varepsilon cH=0\\ \partial_z\overline{H}+\varepsilon cG=0\end{array}\right.$$

which easily yields

$$\begin{cases} \overline{H} = -\varepsilon c G(x,y) z + \widetilde{G}(x,y) \\ \overline{G} = \varepsilon c H(x,y) z + \widetilde{H}(x,y) \end{cases}$$

where $\widetilde{G} = \widetilde{G}(x,y)$ and $\widetilde{H} = \widetilde{H}(x,y)$ are smooth functions on $\mathbb{S}^4_\varepsilon$.

Next, substituting $\overline{H}$ and $\overline{G}$ into $f_1$and f_2 and using the equations given by $\mathrm{d}x\,[(L_X\nabla)_{33}] = 0$ and $\mathrm{d}y\,[(L_X\nabla)_{33}] = 0$, we prove that

$$G = H = 0.$$

Then, from equations given by $\mathrm{d}x\,[(L_X\nabla)_{24}] = 0$, $\mathrm{d}z\,[(L_X\nabla)_{24}] = 0$ and $\mathrm{d}x\,[(L_X\nabla)_{14}] = 0$, we obtain

$$\begin{cases} K = S = 0 \\ \frac{cx}{2}\partial_z \overline{K}(x,y,z) - c\widetilde{G}(x,y) + \partial_y \overline{K}(x,y,z) = 0 \end{cases} \tag{9}$$

Hence, (8) reduces to

$$\begin{cases} f_1 = \widetilde{G}(x,y) e^{-\frac{t}{2}} \\ f_2 = \widetilde{H}(x,y) e^{-\frac{t}{2}} \\ f_3 = \overline{K}(x,y,z) e^{-t} \\ f_4 = \overline{S}(x,y,z) \end{cases}$$

Now, we replace f_3 and f_4 into the equations obtained from $\mathrm{d}z\,[(L_X\nabla)_{33}] = 0$ and $\mathrm{d}t\,[(L_X\nabla)_{33}] = 0$. We get

$$\begin{cases} \partial_z^2 \overline{K} - 2\partial_z \overline{S} = 0 \\ \partial_z^2 \overline{S} = 0 \\ \overline{S} - \partial_z \overline{K} = 0 \end{cases}$$

As a consequence, $\overline{S}$ depends only on x and y, thus

$$\overline{K} = \overline{S}(x,y) z + \widetilde{S}(x,y),$$

for some smooth function $\widetilde{S} = \widetilde{S}(x,y)$, and so the second equation of (9) together with $\mathrm{d}t\,[(L_X\nabla)_{44}] = 0$ and $\mathrm{d}x\,[(L_X\nabla)_{34}] = 0$ give

$$\begin{cases} \overline{S}(x) = k_1, \\ \frac{ck_1}{2} x - c\widetilde{G}(x,y) + \partial_y \widetilde{S}(x,y) = 0, \\ \frac{ck_1}{2} y - c\widetilde{H}(x,y) - \partial_x \widetilde{S}(x,y) = 0, \end{cases} \tag{10}$$

where k_1 is a real constant.

On the other hand, from equations $\mathrm{d}t\,[(L_X\nabla)_{11}] = 0$ and $\mathrm{d}t\,[(L_X\nabla)_{12}] = 0$, we get

$$\begin{cases} \widetilde{G}(x,y) = \frac{k_1}{2} x + \widehat{G}(y) \\ \widetilde{H}(x,y) = -\widehat{G}'(y) x + \widehat{H}(y), \end{cases}$$

for some smooth functions $\widehat{G}$ and $\widehat{H}$ of the variable y. Thus, the last two equations in (10) become

$$\begin{cases} \partial_y \widetilde{S} = c\widehat{G}(y) \\ \partial_x \widetilde{S} = \frac{ck_1}{2} y + c\widehat{G}'(y) x - c\widehat{H}(y). \end{cases} \tag{11}$$

Integrating the second equation in (11) with respect to x we deduce that

$$\frac{c}{2}\widehat{G}''(y)x^2+\left(\frac{ck_1}{2}-c\widehat{H}'(y)\right)x+\widehat{S}'(y)-c\widehat{G}(y)=0$$

for some smooth function $\widehat{S}=\widehat{S}(y)$.

The above equation must be satisfied for any value of x. Therefore, it yields

$$\begin{cases}\widehat{G}(y)=a_1y+a_2,\\ \widehat{H}(y)=\frac{k}{2}y+a_3,\\ \widehat{S}(y)=\frac{ca_1}{2}y^2+ca_2y+a_4,\end{cases}$$

for some real constants $a_1,..,a_4$.

Therefore, we prove that $(L_X\nabla)_{ij}=0$, for all indices i,j if and only if

$$\begin{cases}f_1=\left(\frac{k_1}{2}x+a_1y+a_2\right)e^{-\frac{t}{2}}\\ f_2=\left(-a_1x+\frac{k_1}{2}y+a_3\right)e^{-\frac{t}{2}}\\ f_3=\left(k_1z+\frac{ca_1}{2}(x^2+y^2)+ca_2y-ca_3x+a_4\right)e^{-t}\\ f_4=k_1.\end{cases}$$

We now determine $(L_Xg_\varepsilon)_{ij}=(L_Xg_\varepsilon)(e_i,e_j), i\leq j$, the components of the Lie derivative of the metric (1) with respect to X. We have the following.

$$\begin{aligned}
(L_Xg_\varepsilon)_{11}&=(2\partial_xf_1-cy\partial_zf_1)e^{\frac{t}{2}}-f_4,\\
(L_Xg_\varepsilon)_{12}&=\left(\partial_xf_2-\frac{cy}{2}\partial_zf_2+\partial_yf_1+\frac{cx}{2}\partial_zf_1\right)e^{\frac{t}{2}},\\
(L_Xg_\varepsilon)_{13}&=\left(\varepsilon\partial_xf_3-\frac{cy\varepsilon}{2}\partial_zf_3\right)e^{\frac{t}{2}}+\partial_zf_1e^t+\varepsilon cf_2,\\
(L_Xg_\varepsilon)_{14}&=\left(\frac{c\varepsilon y}{2}\partial_zf_4-\varepsilon\partial_xf_4\right)e^{\frac{t}{2}}+\frac{1}{2}f_1+\partial_tf_1,\\
(L_Xg_\varepsilon)_{22}&=(2\partial_yf_2+cx\partial_zf_2)e^{\frac{t}{2}}-f_4,\\
(L_Xg_\varepsilon)_{23}&=\left(\varepsilon\partial_yf_3+\frac{\varepsilon cx}{2}\partial_zf_3\right)e^{\frac{t}{2}}+\partial_zf_2e^t-\varepsilon cf_1,\\
(L_Xg_\varepsilon)_{24}&=\frac{1}{2}f_2+\partial_tf_2-\left(\varepsilon\partial_yf_4+\frac{c\varepsilon x}{2}\partial_zf_4\right)e^{\frac{t}{2}},\\
(L_Xg_\varepsilon)_{33}&=2\varepsilon e^t\partial_zf_3-2\varepsilon f_4,\\
(L_Xg_\varepsilon)_{34}&=\varepsilon\left(f_3+\partial_tf_3-e^t\partial_zf_4\right),\\
(L_Xg_\varepsilon)_{44}&=-2\varepsilon\partial_tf_4,
\end{aligned}$$

In order to determine the Killing vector fields, we must solve the system of PDEs obtained by requiring that all coefficients in the above Lie derivative are equal to zero. Routine but long computations lead proving the following:

Theorem 1. *Let X be an arbitrary vector field on the four-dimensional Lorentzian Damek-Ricci spaces $(\mathbb{S}^4_\varepsilon,g_\varepsilon)$. Then, the following are equivalent*

- *X is an affine vector field*
- *X is a Killing vector*
- *X is given by*

$$X = e^{-\frac{t}{2}}\left(\frac{k_1}{2}x - k_2 y + k_4\right)e_1 + e^{-\frac{t}{2}}\left(k_2 x + \frac{k_1}{2}y + k_3\right)e_2$$
$$+ e^{-t}\left(k_1 z - \frac{ck_2}{2}(x^2+y^2) + ck_4 y - ck_3 x + k_5\right)e_3 + k_1 e_4,$$

for some real constants $k_1, .., k_5$.

4 Ricci, curvature and matter collineations

In this section, we shall investigate other symmetries of the four-dimensional Lorentzian Damek-Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ where g_ε is given by (1).

Let $X = f_1e_1 + f_2e_2 + f_3e_3 + f_4e_4$ denote an arbitrary vector field on $\mathbb{S}^4_\varepsilon$ where f_1, f_2, f_3, f_4 are smooth functions of the variables x, y, z, t. We first determine the components $(L_X\varrho)_{ij} = (L_X\varrho)(e_i, e_j)$, $i \leq j$, of the Lie derivative of the Ricci tensor ϱ. We have the following

$$\begin{aligned}
(L_X\varrho)_{11} &= e^{\frac{t}{2}}(\varepsilon\partial_x f_1 - \tfrac{c\varepsilon y}{2}\partial_z f_1) - \frac{\varepsilon}{2}f_4,\\
(L_X\varrho)_{12} &= \tfrac{\varepsilon}{2}\left(\partial_y f_1 + \tfrac{cx}{2}\partial_z f_1 + \partial_x f_2 - \tfrac{cy}{2}\partial_z f_2\right)e^{\frac{t}{2}},\\
(L_X\varrho)_{13} &= \tfrac{5}{2}\left(\left(\partial_x f_3 - \tfrac{cy}{2}\partial_z f_3\right)e^{\frac{t}{2}} + cf_2\right) + \tfrac{\varepsilon}{2}\partial_z f_1 e^t,\\
(L_X\varrho)_{14} &= \tfrac{3}{2}\left(\tfrac{cy}{2}\partial_z f_4 - \partial_x f_4\right)e^{\frac{t}{2}} + \tfrac{\varepsilon}{4}f_1 + \tfrac{\varepsilon}{2}\partial_t f_1,\\
(L_X\varrho)_{22} &= \varepsilon(\partial_y f_2 + \tfrac{cx}{2}\partial_z f_2)e^{\frac{t}{2}} - \tfrac{\varepsilon}{2}f_4,\\
(L_X\varrho)_{23} &= -\tfrac{5}{2}\left(cf_1 - \left(\partial_y f_3 + \tfrac{cx}{2}\partial_z f_3\right)e^{\frac{t}{2}}\right) + \tfrac{\varepsilon}{2}\partial_z f_2 e^t,\\
(L_X\varrho)_{24} &= -\tfrac{3}{2}\left(\partial_y f_4 + \tfrac{cx}{2}\partial_z f_4\right)e^{\frac{t}{2}} + \tfrac{\varepsilon}{4}f_2 + \tfrac{\varepsilon}{2}\partial_t f_2,\\
(L_X\varrho)_{33} &= 5\left(\partial_z f_3 e^t - f_4\right),\\
(L_X\varrho)_{34} &= \tfrac{5}{2}(f_3 + \partial_t f_3) - \tfrac{3}{2}\partial_z f_4 e^t,\\
(L_X\varrho)_{44} &= \ -3\partial_t f_4.
\end{aligned}$$

Next, using (5), we calculate the Lie derivative of the curvature tensor R, with respect to the orthonormal basis (2). Setting $(L_XR)_{ijk} = (L_XR)(e_i, e_j, e_k)$ for all indices $i < j, k$, we have

$$\begin{aligned}
(L_XR)_{121} \;=\; & e^{\frac{t}{2}}\left(\tfrac{\varepsilon cy}{4}\partial_z f_2 - \tfrac{\varepsilon}{2}\partial_y f_1 - \tfrac{c\varepsilon x}{4}\partial_z f_1 - \tfrac{\varepsilon}{2}\partial_x f_2\right)e_1\\
& + \left(e^{\frac{t}{2}}(\varepsilon\partial_x f_1 - \tfrac{\varepsilon cy}{2}\partial_z f_1) - \tfrac{\varepsilon}{2}f_4\right)e_2\\
& + \left(e^{\frac{t}{2}}\left(\tfrac{3c}{4}\partial_x f_4 - \tfrac{3y}{8}\partial_z f_4 - \tfrac{5\varepsilon}{4}\partial_y f_3 - \tfrac{5\varepsilon cx}{8}\partial_z f_3\right) + \tfrac{5c\varepsilon}{4}f_1\right)e_3\\
& + \left(e^{\frac{t}{2}}\left(\tfrac{3c}{4}\partial_x f_3 - \tfrac{3\varepsilon}{4}\partial_y f_4 - \tfrac{3\varepsilon cx}{8}\partial_z f_4 - \tfrac{3y}{8}\partial_z f_3\right) + \tfrac{3}{4}f_2\right)e_4,
\end{aligned}$$

$$\begin{aligned}(L_XR)_{122} &= \left(e^{\frac{t}{2}}\left(-\varepsilon\partial_y f_2 - \tfrac{\varepsilon cx}{2}\partial_z f_2\right) + \tfrac{\varepsilon}{2} f_4\right) e_1 \\ &\quad + e^{\frac{t}{2}}\left(\tfrac{\varepsilon}{2}\partial_y f_1 + \tfrac{\varepsilon cx}{4}\partial_z f_1 + \tfrac{\varepsilon}{2}\partial_x f_2 - \tfrac{\varepsilon cy}{4}\partial_z f_2\right) e_2 \\ &\quad + \left(e^{\frac{t}{2}}\left(\tfrac{5\varepsilon}{4}\partial_x f_3 - \tfrac{5\varepsilon cy}{8}\partial_z f_3 + \tfrac{3c}{4}\partial_y f_4 + \tfrac{3x}{8}\partial_z f_4\right) + \tfrac{5\varepsilon c}{4} f_2\right) e_3 \\ &\quad + \left(e^{\frac{t}{2}}\left(\tfrac{3\varepsilon}{4}\partial_x f_4 - \tfrac{3\varepsilon cy}{8}\partial_z f_4 + \tfrac{3c}{4}\partial_y f_3 + \tfrac{3x}{8}\partial_z f_3\right) - \tfrac{3}{4} f_1\right) e_4,\end{aligned}$$

$$\begin{aligned}&(L_XR)_{123} \\ &= \left(e^{\frac{t}{2}}\left(\tfrac{y\varepsilon}{8}\partial_z f_4 - \tfrac{\varepsilon c}{4}\partial_x f_4 + \tfrac{3}{4}\partial_y f_3 + \tfrac{3cx}{8}\partial_z f_3\right) - \tfrac{\varepsilon}{2} e^t\partial_z f_2 - \tfrac{c}{2}\partial_t f_1 - c f_1\right) e_1 \\ &+ \left(e^{\frac{t}{2}}\left(-\tfrac{3}{4}\partial_x f_3 + \tfrac{3cy}{8}\partial_z f_3 - \tfrac{\varepsilon c}{4}\partial_y f_4 - \tfrac{\varepsilon x}{8}\partial_z f_4\right) + \tfrac{\varepsilon}{2} e^t\partial_z f_1 - \tfrac{c}{2}\partial_t f_2 - c f_2\right) e_2 \\ &+ \left(\tfrac{c}{2} e^t\partial_z f_4 - \tfrac{c}{2}\partial_t f_3 - \tfrac{c}{2} f_3\right) e_3 \\ &+ \left(e^{\frac{t}{2}}\left(\tfrac{c}{2}\partial_x f_1 - \tfrac{y}{4}\partial_z f_1 + \tfrac{c}{2}\partial_y f_2 + \tfrac{x}{4}\partial_z f_2\right) + \tfrac{c}{2} e^t\partial_z f_3 - \tfrac{c}{2}\partial_t f_4 - c f_4\right) e_4,\end{aligned}$$

$$\begin{aligned}(L_XR)_{124} &= \left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_x f_3 - \tfrac{\varepsilon y}{8}\partial_z f_3 - \tfrac{1}{4}\partial_y f_4 - \tfrac{cx}{8}\partial_z f_4\right) - \tfrac{c}{2} e^t\partial_z f_1 - \tfrac{\varepsilon}{2}\partial_t f_2\right) e_1 \\ &\quad + \left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_y f_3 + \tfrac{\varepsilon x}{8}\partial_z f_3 + \tfrac{1}{4}\partial_x f_4 - \tfrac{cy}{8}\partial_z f_4\right) - \tfrac{c}{2} e^t\partial_z f_2 + \tfrac{\varepsilon}{2}\partial_t f_1\right) e_2 \\ &\quad + \left(e^{\frac{t}{2}}\left(\tfrac{c}{2}\partial_x f_1 - \tfrac{y}{4}\partial_z f_1 + \tfrac{c}{2}\partial_y f_2 + \tfrac{x}{4}\partial_z f_2\right) - \tfrac{c}{2} e^t\partial_z f_3 + \tfrac{c}{2}\partial_t f_4\right) e_3 \\ &\quad + \left(\tfrac{c}{2}\partial_t f_3 + \tfrac{c}{2} f_3 - \tfrac{c}{2} e^t\partial_z f_4\right) e_4,\end{aligned}$$

$$\begin{aligned}(L_XR)_{131} &= \left(e^{\frac{t}{2}}\left(\tfrac{3}{4}\partial_x f_3 - \tfrac{3cy}{8}\partial_z f_3\right) + \tfrac{3\varepsilon}{4} e^t\partial_z f_1 + \tfrac{3c}{4} f_2\right) e_1 \\ &\quad + \left(e^{\frac{t}{2}}\left(\tfrac{3\varepsilon c}{4}\partial_x f_4 - \tfrac{3\varepsilon y}{8}\partial_z f_4\right) + \tfrac{5\varepsilon}{4} e^t\partial_z f_2\right) e_2 \\ &\quad + \left(e^{\frac{t}{2}}\left(-\tfrac{3\varepsilon}{2}\partial_x f_1 + \tfrac{3\varepsilon cy}{4}\partial_z f_1\right) + \tfrac{3\varepsilon}{4} f_4\right) e_3 \\ &\quad + \left(\tfrac{\varepsilon}{2} e^t\partial_z f_4\right) e_4,\end{aligned}$$

$$\begin{aligned}&(L_XR)_{132} \\ =\; &\left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon y}{4}\partial_z f_4 - \tfrac{\varepsilon c}{2}\partial_x f_4 + \tfrac{3}{4}\partial_y f_3 + \tfrac{3cx}{8}\partial_z f_3\right) - \tfrac{\varepsilon}{2} e^t\partial_z f_2 - \tfrac{c}{4}\partial_t f_1 - \tfrac{7c}{8} f_1\right) e_1 \\ &+ \left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_y f_4 + \tfrac{\varepsilon x}{8}\partial_z f_4\right) - \tfrac{c}{4}\partial_t f_2 - \tfrac{c}{8} f_2\right) e_2 \\ &+ \left(e^{\frac{t}{2}}\left(\tfrac{3\varepsilon cy}{8}\partial_z f_2 - \tfrac{3\varepsilon}{4}\partial_x f_2 - \tfrac{3\varepsilon}{4}\partial_y f_1 - \tfrac{3\varepsilon cx}{8}\partial_z f_1\right) + \tfrac{c}{4} e^t\partial_z f_4 - \tfrac{c}{4}\partial_t f_3 - \tfrac{c}{4} f_3\right) e_3 \\ &+ \left(e^{\frac{t}{2}}\left(\tfrac{c}{4}\partial_x f_1 - \tfrac{y}{8}\partial_z f_1 + \tfrac{c}{4}\partial_y f_2 + \tfrac{x}{8}\partial_z f_2\right) + \tfrac{c}{4} e^t\partial_z f_3 - \tfrac{c}{4}\partial_t f_4 - \tfrac{c}{2} f_4\right) e_4,\end{aligned}$$

$$\begin{aligned}(L_XR)_{133} = &\left(\tfrac{3}{2} e^t\partial_z f_3 - \tfrac{3}{2} f_4\right) e_1 \\ &+ \left(e^{\frac{t}{2}}\left(\tfrac{3cy}{8}\partial_z f_3 - \tfrac{3}{4}\partial_x f_3\right) - \tfrac{3\varepsilon}{4} e^t\partial_z f_1 - \tfrac{3c}{4} f_2\right) e_3 \\ &+ \left(e^{\frac{t}{2}}\left(\tfrac{1}{4}\partial_x f_4 - \tfrac{cy}{8}\partial_z f_4\right) + \tfrac{3c}{4} e^t\partial_z f_2\right) e_4,\end{aligned}$$

$$
\begin{aligned}
&(L_X R)_{134}\\
= \; & \Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon y}{8}\partial_z f_2 - \tfrac{\varepsilon c}{4}\partial_x f_2 - \tfrac{\varepsilon c}{4}\partial_y f_1 - \tfrac{\varepsilon x}{8}\partial_z f_1\right) - \tfrac{1}{4}e^t\partial_z f_4 + \tfrac{3}{4}\partial_t f_3 + \tfrac{3}{4}f_3\Big)\, e_1\\
&+ \Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_x f_1 - \tfrac{\varepsilon y}{8}\partial_z f_1 - \tfrac{\varepsilon c}{4}\partial_y f_2 - \tfrac{\varepsilon x}{8}\partial_z f_2\right) + \tfrac{\varepsilon c}{4}e^t\partial_z f_3 + \tfrac{\varepsilon c}{4}\partial_t f_4 - \tfrac{\varepsilon c}{4}f_4\Big)\, e_2\\
&+ \Big(e^{\frac{t}{2}}\left(\partial_x f_4 - \tfrac{cy}{2}\partial_z f_4 - \tfrac{\varepsilon c}{4}\partial_y f_3 - \tfrac{\varepsilon x}{8}\partial_z f_3\right) + \tfrac{c}{2}e^t\partial_z f_2 - \tfrac{3\varepsilon}{4}\partial_t f_1 - \tfrac{\varepsilon}{8}f_1\Big)\, e_3\\
&+ \Big(e^{\frac{t}{2}}\left(-\tfrac{\varepsilon c}{4}\partial_y f_4 - \tfrac{\varepsilon x}{8}\partial_z f_4\right) + \tfrac{c}{8}f_2 + \tfrac{c}{4}\partial_t f_2\Big)\, e_4,
\end{aligned}
$$

$$
\begin{aligned}
(L_X R)_{141} \;=\; & \Big(e^{\frac{t}{2}}\left(\tfrac{cy}{8}\partial_z f_4 - \tfrac{1}{4}\partial_x f_4\right) + \tfrac{\varepsilon}{4}\partial_t f_1 + \tfrac{\varepsilon}{8}f_1\Big)\, e_1\\
&+ \Big(e^{\frac{t}{2}}\Big(\tfrac{3\varepsilon y}{8}\partial_z f_3 - \tfrac{3\varepsilon c}{4}\partial_x f_3\Big) + \tfrac{3\varepsilon}{4}\partial_t f_2 - \tfrac{3\varepsilon}{8}f_2\Big)\, e_2\\
&+ \tfrac{\varepsilon}{2}\left(-f_3 - \partial_t f_3\right) e_3\\
&+ \Big(e^{\frac{t}{2}}\left(\tfrac{cy\varepsilon}{4}\partial_z f_1 - \tfrac{\varepsilon}{2}\partial_x f_1\right) + \tfrac{\varepsilon}{4}f_4\Big)\, e_4,
\end{aligned}
$$

$$
\begin{aligned}
&(L_X R)_{142}\\
= \; & \Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{2}\partial_x f_3 - \tfrac{\varepsilon y}{4}\partial_z f_3 - \tfrac{1}{4}\partial_y f_4 - \tfrac{cx}{8}\partial_z f_4\right) - \tfrac{c}{4}e^t\partial_z f_1 - \tfrac{\varepsilon}{2}\partial_t f_2 + \tfrac{\varepsilon}{4}f_2\Big)\, e_1\\
&+ \Big(e^{\frac{t}{2}}\left(-\tfrac{\varepsilon c}{4}\partial_y f_3 - \tfrac{\varepsilon x}{8}\partial_z f_3\right) - \tfrac{c}{4}e^t\partial_z f_2 + \tfrac{\varepsilon}{4}f_1\Big)\, e_2\\
&+ \Big(e^{\frac{t}{2}}\left(\tfrac{c}{4}\partial_x f_1 - \tfrac{y}{8}\partial_z f_1 + \tfrac{c}{4}\partial_y f_2 + \tfrac{x}{8}\partial_z f_2\right) - \tfrac{c}{4}e^t\partial_z f_3 + \tfrac{c}{4}\partial_t f_4\Big)\, e_3\\
&+ \Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon cy}{8}\partial_z f_2 - \tfrac{\varepsilon}{4}\partial_x f_2 - \tfrac{\varepsilon}{4}\partial_y f_1 - \tfrac{c\varepsilon x}{8}\partial_z f_1\right) - \tfrac{c}{4}e^t\partial_z f_4 + \tfrac{c}{4}\partial_t f_3 + \tfrac{c}{4}f_3\Big)\, e_4,
\end{aligned}
$$

$$
\begin{aligned}
&(L_X R)_{143}\\
= \; & \Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_y f_1 + \tfrac{\varepsilon x}{8}\partial_z f_1 + \tfrac{\varepsilon c}{4}\partial_x f_2 - \tfrac{\varepsilon y}{8}\partial_z f_2\right) - \tfrac{1}{4}e^t\partial_z f_4 + \tfrac{3}{4}\partial_t f_3 + \tfrac{3}{4}f_3\Big)\, e_1\\
&+ \Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_y f_2 + \tfrac{\varepsilon x}{8}\partial_z f_2 - \tfrac{\varepsilon c}{4}\partial_x f_1 + \tfrac{\varepsilon y}{8}\partial_z f_1\right) - \tfrac{\varepsilon c}{4}e^t\partial_z f_3 - \tfrac{\varepsilon c}{4}\partial_t f_4 + \tfrac{\varepsilon c}{4}f_4\Big)\, e_2\\
&+ \Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_y f_3 + \tfrac{\varepsilon x}{8}\partial_z f_3\right) + \tfrac{c}{4}e^t\partial_z f_2 - \tfrac{\varepsilon}{4}f_1\Big)\, e_3\\
&+ \Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_y f_4 + \tfrac{\varepsilon x}{8}\partial_z f_4 - \partial_x f_3 + \tfrac{cy}{2}\partial_z f_3\right) - \tfrac{\varepsilon}{4}e^t\partial_z f_1 + \tfrac{c}{2}\partial_t f_2 - \tfrac{3c}{4}f_2\Big)\, e_4,
\end{aligned}
$$

$$
\begin{aligned}
(L_X R)_{144} \;=\; & \left(-\tfrac{1}{2}\partial_t f_4\right) e_1\\
&+ \Big(e^{\frac{t}{2}}\Big(\tfrac{3cy}{8}\partial_z f_3 - \tfrac{3}{8}\partial_x f_3\Big) + \tfrac{3c}{4}\partial_t f_2 - \tfrac{3c}{8}f_2\Big)\, e_3\\
&+ \Big(e^{\frac{t}{2}}\left(\tfrac{1}{4}\partial_x f_4 - \tfrac{cy}{8}\partial_z f_4\right) - \tfrac{\varepsilon}{4}\partial_t f_1 - \tfrac{\varepsilon}{8}f_1\Big)\, e_4,
\end{aligned}
$$

$$
\begin{aligned}
&(L_XR)_{231}\\
=\ &\left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon y}{8}\partial_z f_4-\tfrac{\varepsilon c}{4}\partial_x f_4\right)+\tfrac{c}{4}\partial_t f_1+\tfrac{c}{8}f_1\right)e_1\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{2}\partial_y f_4+\tfrac{\varepsilon x}{4}\partial_z f_4+\tfrac{3}{4}\partial_x f_3-\tfrac{3cy}{8}\partial_z f_3\right)-\tfrac{\varepsilon}{2}e^t\partial_z f_1+\tfrac{c}{4}\partial_t f_2+\tfrac{7c}{8}f_2\right)e_2\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{3\varepsilon cy}{8}\partial_z f_2-\tfrac{3\varepsilon}{4}\partial_x f_2-\tfrac{3\varepsilon}{4}\partial_y f_1-\tfrac{3\varepsilon cx}{8}\partial_z f_1\right)-\tfrac{c}{4}e^t\partial_z f_4+\tfrac{c}{4}\partial_t f_3+\tfrac{c}{4}f_3\right)e_3\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{y}{8}\partial_z f_1-\tfrac{c}{4}\partial_x f_1-\tfrac{c}{4}\partial_y f_2-\tfrac{x}{8}\partial_z f_2\right)-\tfrac{c}{4}e^t\partial_z f_3+\tfrac{c}{4}\partial_t f_4+\tfrac{c}{2}f_4\right)e_4,
\end{aligned}
$$

$$
\begin{aligned}
(L_XR)_{232}=\ &\left(\tfrac{5\varepsilon}{4}e^t\partial_z f_1-e^{\frac{t}{2}}\left(\tfrac{3\varepsilon c}{4}\partial_y f_4+\tfrac{3\varepsilon x}{8}\partial_z f_4\right)\right)e_1\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{3}{4}\partial_y f_3+\tfrac{3cx}{8}\partial_z f_3\right)+\tfrac{3\varepsilon}{4}e^t\partial_z f_2-\tfrac{3c}{4}f_1\right)e_2\\
&+\left(\tfrac{3\varepsilon}{4}f_4-e^{\frac{t}{2}}\left(\tfrac{3\varepsilon}{2}\partial_y f_2+\tfrac{3\varepsilon cx}{4}\partial_z f_2\right)\right)e_3\\
&+\left(\tfrac{\varepsilon}{2}e^t\partial_z f_4\right)e_4,
\end{aligned}
$$

$$
\begin{aligned}
(L_XR)_{233}\ =\ &\left(\tfrac{3}{2}e^t\partial_z f_3-\tfrac{3}{2}f_4\right)e_2\\
&+\left(\tfrac{3c}{4}f_1-e^{\frac{t}{2}}\left(\tfrac{3}{4}\partial_y f_3+\tfrac{3cx}{8}\partial_z f_3\right)-\tfrac{3\varepsilon}{4}e^t\partial_z f_2\right)e_3\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{1}{4}\partial_y f_4+\tfrac{cx}{8}\partial_z f_4\right)-\tfrac{3c}{4}e^t\partial_z f_1\right)e_4,
\end{aligned}
$$

$$
\begin{aligned}
&(L_XR)_{234}\\
=\ &\left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_x f_1-\tfrac{\varepsilon y}{8}\partial_z f_1-\tfrac{\varepsilon c}{4}\partial_y f_2-\tfrac{\varepsilon x}{8}\partial_z f_2\right)-\tfrac{\varepsilon c}{4}e^t\partial_z f_3-\tfrac{\varepsilon c}{4}\partial_t f_4+\tfrac{\varepsilon c}{4}f_4\right)e_1\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_x f_2-\tfrac{\varepsilon y}{8}\partial_z f_2+\tfrac{\varepsilon c}{4}\partial_y f_1+\tfrac{\varepsilon x}{8}\partial_z f_1\right)-\tfrac{1}{4}e^t\partial_z f_4+\tfrac{3}{4}\partial_t f_3+\tfrac{3}{4}f_3\right)e_2\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_x f_3-\tfrac{\varepsilon y}{8}\partial_z f_3+\partial_y f_4+\tfrac{cx}{2}\partial_z f_4\right)-\tfrac{c}{2}e^t\partial_z f_1-\tfrac{3\varepsilon}{4}\partial_t f_2-\tfrac{\varepsilon}{8}f_2\right)e_3\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_x f_4-\tfrac{\varepsilon y}{8}\partial_z f_4\right)-\tfrac{c}{4}\partial_t f_1-\tfrac{c}{8}f_1\right)e_4,
\end{aligned}
$$

$$
\begin{aligned}
&(L_XR)_{241}\\
=\ &\left(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_x f_3-\tfrac{\varepsilon y}{8}\partial_z f_3\right)+\tfrac{c}{4}e^t\partial_z f_1+\tfrac{\varepsilon}{4}f_2\right)e_1\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{cy}{8}\partial_z f_4-\tfrac{1}{4}\partial_x f_4-\tfrac{\varepsilon c}{2}\partial_y f_3-\tfrac{\varepsilon x}{4}\partial_z f_3\right)+\tfrac{c}{4}e^t\partial_z f_2-\tfrac{\varepsilon}{2}\partial_t f_1+\tfrac{\varepsilon}{4}f_1\right)e_2\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{y}{8}\partial_z f_1-\tfrac{c}{4}\partial_x f_1-\tfrac{c}{4}\partial_y f_2-\tfrac{x}{8}\partial_z f_2\right)+\tfrac{c}{4}e^t\partial_z f_3-\tfrac{c}{4}\partial_t f_4\right)e_3\\
&+\left(e^{\frac{t}{2}}\left(\tfrac{c\varepsilon y}{8}\partial_z f_2-\tfrac{\varepsilon}{4}\partial_x f_2-\tfrac{\varepsilon}{4}\partial_y f_1-\tfrac{c\varepsilon x}{8}\partial_z f_1\right)+\tfrac{c}{4}e^t\partial_z f_4-\tfrac{c}{4}\partial_t f_3-\tfrac{c}{4}f_3\right)e_4,
\end{aligned}
$$

$$
\begin{aligned}
(L_XR)_{242}\ =\ &\left(e^{\frac{t}{2}}\left(\tfrac{3\varepsilon c}{4}\partial_y f_3+\tfrac{3\varepsilon x}{8}\partial_z f_3\right)+\tfrac{3\varepsilon}{4}\partial_t f_1-\tfrac{3\varepsilon}{8}f_1\right)e_1\\
&+\left(e^{\frac{t}{2}}\left(-\tfrac{1}{4}\partial_y f_4-\tfrac{cx}{8}\partial_z f_4\right)+\tfrac{\varepsilon}{8}f_2+\tfrac{\varepsilon}{4}\partial_t f_2\right)e_2\\
&+\tfrac{\varepsilon}{2}\left(-\partial_t f_3-f_3\right)e_3\\
&+\left(\tfrac{\varepsilon}{4}f_4-e^{\frac{t}{2}}\left(\tfrac{\varepsilon}{2}\partial_y f_2+\tfrac{\varepsilon cx}{4}\partial_z f_2\right)\right)e_4,
\end{aligned}
$$

$$\begin{aligned}
&(L_XR)_{243}\\
=\;& \Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{4}\partial_y f_2+\tfrac{\varepsilon x}{8}\partial_z f_2-\tfrac{\varepsilon c}{4}\partial_x f_1+\tfrac{\varepsilon y}{8}\partial_z f_1\right)+\tfrac{\varepsilon c}{4}e^t\partial_z f_3+\tfrac{\varepsilon c}{4}\partial_t f_4-\tfrac{\varepsilon c}{4}f_4\Big)\,e_1\\
&+\Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon y}{8}\partial_z f_2-\tfrac{\varepsilon c}{4}\partial_x f_2-\tfrac{\varepsilon c}{4}\partial_y f_1-\tfrac{\varepsilon x}{8}\partial_z f_1\right)-\tfrac{1}{4}e^t\partial_z f_4+\tfrac{3}{4}\partial_t f_3+\tfrac{3}{4}f_3\Big)\,e_2\\
&+\Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon y}{8}\partial_z f_3-\tfrac{\varepsilon c}{4}\partial_x f_3\right)-\tfrac{c}{4}e^t\partial_z f_1-\tfrac{\varepsilon}{4}f_2\Big)\,e_3\\
&+\Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon y}{8}\partial_z f_4-\tfrac{\varepsilon c}{4}\partial_x f_4-\partial_y f_3-\tfrac{cx}{2}\partial_z f_3\right)-\tfrac{\varepsilon}{4}e^t\partial_z f_2-\tfrac{c}{2}\partial_t f_1+\tfrac{3c}{4}f_1\Big)\,e_4,
\end{aligned}$$

$$\begin{aligned}
(L_XR)_{244}\;=\;&\left(-\tfrac{1}{2}\partial_t f_4\right)e_2\\
&+\Big(\tfrac{3c}{8}f_1-\tfrac{3c}{4}\partial_t f_1-e^{\frac{t}{2}}\left(\tfrac{3}{4}\partial_y f_3+\tfrac{3cx}{8}\partial_z f_3\right)\Big)\,e_3\\
&+\Big(e^{\frac{t}{2}}\left(\tfrac{1}{4}\partial_y f_4+\tfrac{cx}{8}\partial_z f_4\right)-\tfrac{\varepsilon}{4}\partial_t f_2-\tfrac{\varepsilon}{8}f_2\Big)\,e_4,
\end{aligned}$$

$$\begin{aligned}
&(L_XR)_{341}\\
=\;& e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{2}\partial_y f_1+\tfrac{\varepsilon x}{4}\partial_z f_1+\tfrac{\varepsilon c}{2}\partial_x f_2-\tfrac{\varepsilon y}{4}\partial_z f_2\right)e_1\\
&+\Big(e^{\frac{t}{2}}\left(-\tfrac{\varepsilon c}{2}\partial_x f_1+\tfrac{\varepsilon y}{4}\partial_z f_1+\tfrac{\varepsilon c}{2}\partial_y f_2+\tfrac{\varepsilon x}{4}\partial_z f_2\right)-\tfrac{\varepsilon c}{2}e^t\partial_z f_3-\tfrac{\varepsilon c}{2}\partial_t f_4+\tfrac{\varepsilon c}{2}f_4\Big)\,e_2\\
&+\Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{2}\partial_y f_3+\tfrac{\varepsilon x}{4}\partial_z f_3-\partial_x f_4+\tfrac{cy}{2}\partial_z f_4\right)-\tfrac{c}{4}e^t\partial_z f_2+\tfrac{3\varepsilon}{4}\partial_t f_1-\tfrac{\varepsilon}{8}f_1\Big)\,e_3\\
&+\Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{2}\partial_y f_4+\tfrac{\varepsilon x}{4}\partial_z f_4-\partial_x f_3+\tfrac{cy}{2}\partial_z f_3\right)-\tfrac{\varepsilon}{4}e^t\partial_z f_1+\tfrac{c}{4}\partial_t f_2-\tfrac{7c}{8}f_2\Big)\,e_4,
\end{aligned}$$

$$\begin{aligned}
(L_XR)_{342}=\;&\Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon c}{2}\partial_y f_2+\tfrac{\varepsilon x}{4}\partial_z f_2-\tfrac{\varepsilon c}{2}\partial_x f_1+\tfrac{\varepsilon y}{4}\partial_z f_1\right)+\tfrac{\varepsilon c}{2}e^t\partial_z f_3+\tfrac{\varepsilon c}{2}\partial_t f_4-\tfrac{\varepsilon c}{2}f_4\Big)\,e_1\\
&+e^{\frac{t}{2}}\left(\tfrac{\varepsilon y}{4}\partial_z f_2-\tfrac{\varepsilon c}{2}\partial_x f_2-\tfrac{\varepsilon c}{2}\partial_y f_1-\tfrac{\varepsilon x}{4}\partial_z f_1\right)e_2\\
&+\Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon y}{4}\partial_z f_3-\tfrac{\varepsilon c}{2}\partial_x f_3-\partial_y f_4-\tfrac{cx}{2}\partial_z f_4\right)+\tfrac{c}{4}e^t\partial_z f_1+\tfrac{3\varepsilon}{4}\partial_t f_2-\tfrac{\varepsilon}{8}f_2\Big)\,e_3\\
&+\Big(e^{\frac{t}{2}}\left(\tfrac{\varepsilon y}{4}\partial_z f_4-\tfrac{\varepsilon c}{2}\partial_x f_4-\partial_y f_3-\tfrac{cx}{2}\partial_z f_3\right)-\tfrac{\varepsilon}{4}e^t\partial_z f_2-\tfrac{c}{4}\partial_t f_1+\tfrac{7c}{8}f_1\Big)\,e_4,
\end{aligned}$$

$$\begin{aligned}
(L_XR)_{343}\;=\;&\left(\tfrac{3\varepsilon c}{4}e^t\partial_z f_2+\tfrac{1}{4}\partial_t f_1+\tfrac{1}{8}f_1\right)e_1\\
&+\left(\tfrac{1}{8}f_2+\tfrac{1}{4}\partial_t f_2-\tfrac{3\varepsilon c}{4}e^t\partial_z f_1\right)e_2\\
&+\left(f_3+\partial_t f_3-e^t\partial_z f_4\right)e_3\\
&+\left(2f_4-2e^t\partial_z f_3\right)e_4,
\end{aligned}$$

$$\begin{aligned}
(L_XR)_{344}\;=\;&\left(\tfrac{3}{4}e^t\partial_z f_1+\tfrac{3\varepsilon c}{4}\partial_t f_2+\tfrac{3\varepsilon c}{8}f_2\right)e_1\\
&+\left(\tfrac{3}{4}e^t\partial_z f_2-\tfrac{3\varepsilon c}{8}f_1-\tfrac{3\varepsilon c}{4}\partial_t f_1\right)e_2\\
&+\left(-2\partial_t f_4\right)e_3\\
&+\left(e^t\partial_z f_4-f_3-\partial_t f_3\right)e_4.
\end{aligned}$$

Finally, we study matter collineations, using equations (1), (6) and (7), we obtain the tensor field $\mathcal{T} = \varrho - \frac{\tau}{2} g_\varepsilon$ as follows

$$\mathcal{T} = \begin{pmatrix} -2\varepsilon & 0 & 0 & 0 \\ 0 & -2\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Lie derivative components $(L_X \mathcal{T})_{ij} = (L_X \mathcal{T})(e_i, e_j)$ of $\mathcal{T}$ are then given by

$$\begin{aligned}
(L_X \mathcal{T})_{11} &= 2\varepsilon f_4 - e^{\frac{t}{2}} \left(4\varepsilon \partial_x f_1 - 2\varepsilon c y \partial_z f_1\right), \\
(L_X \mathcal{T})_{12} &= e^{\frac{t}{2}} \left(\varepsilon c y \partial_z f_2 - 2\varepsilon \partial_x f_2 - 2\varepsilon \partial_y f_1 - \varepsilon c x \partial_z f_1\right), \\
(L_X \mathcal{T})_{13} &= -2\varepsilon e^{t} \partial_z f_1, \\
(L_X \mathcal{T})_{14} &= e^{\frac{t}{2}} \left(\partial_x f_4 - \frac{cy}{2} \partial_z f_4\right) - 2\varepsilon \left(\frac{1}{2} f_1 + \partial_t f_1\right),
\end{aligned}$$

$$\begin{aligned}
&(L_X \mathcal{T})_{22} = \left(2\varepsilon f_4 - e^{\frac{t}{2}} \left(4\varepsilon \partial_y f_2 + 2\varepsilon c x \partial_z f_2\right)\right), \\
&(L_X \mathcal{T})_{23} = -2\varepsilon e^{t} \partial_z f_2, \\
&(L_X \mathcal{T})_{24} = e^{\frac{t}{2}} \left(\partial_y f_4 + \frac{cx}{2} \partial_z f_4\right) - \left(\varepsilon f_2 + 2\varepsilon \partial_t f_2\right), \\
&(L_X \mathcal{T})_{33} = 0, \\
&(L_X \mathcal{T})_{34} = e^{t} \partial_z f_4, \\
&(L_X \mathcal{T})_{44} = 2\partial_t f_4.
\end{aligned}$$

Ricci, curvature and Matter collineations are then calculated by solving the system of PDE obtained by requiring that all the above components of $L_X \varrho$, $L_X R$ and $L_X \mathcal{T}$ vanish, respectively. A very long computation leads to proving the following:

Theorem 2. *Let X be an arbitrary vector field on the four-dimensional Lorentzian Damek-Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ where g_ε is given by (1). Then, the following are equivalent*

- *X is a Ricci collineation,*
- *X is a curvature collineation,*
- *X is a Killing vector field,*
- *X is an affine vector field.*

Furthermore, X is a matter collineation if and only if X is given by

$$\begin{aligned}
X = {} & e^{-\frac{t}{2}} \left(\frac{k_1}{2} x - k_2 y + k_4\right) e_1 + e^{-\frac{t}{2}} \left(k_2 x + \frac{k_1}{2} y + k_3\right) e_2 \\
& + f(x, y, z, t) e_3 + k_1 e_4,
\end{aligned}$$

for some smooth real valued function f and real constants $k_1, .., k_4$.

References

[1] Anker, J-P., Pierfelice, V. and Vallarino, M., *The wave equation on Damek-Ricci spaces*, Ann. di Mat. Pura ed Appl., Springer Verlag, **194** (2015), no. 3, 731–758.

[2] Berndt, J., Tricerri, F. and Vanhecke, L., *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture Notes in Mathematics, 1598, Springer, Heidelberg, 1995.

[3] Camci, U. and Sharif, M., *Matter collineations of spacetime homogeneous Godel-type metrics*, Class. Quantum Gravity **20** (2003), 2169-2179.

[4] Cao, H. D., *Recent progress on Ricci solitons*, Recent Advances in Geometric Analysis, **11** (2009), 1–38.

[5] Carlos Díaz-Ramos, J. and Domínguez-Vazquez, M., *Isoparametric hypersurfaces in Damek-Ricci spaces*, Adv. Math. (NY), **239** (2013), 1–17.

[6] Carot, J., da Costa, J. and Vaz, E.G.L.R., *Matter collineations: the inverse "symmetry inheritance"problem*, J. Math. Phys. **35** (1994), 4832-4838.

[7] Cintra, A., Mercuri, F. and Onnis, I., *Minimal surfaces in 4-dimensional Lorentzian Damek-Ricci spaces*, 2015, arXiv:1501.03427.

[8] Damek, E. and Ricci, F., *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. **27** (1992), no. 1, 139–142.

[9] Degla, S., and Todjihounde, L., *Biharmonic curve in four-dimensional Damek-Ricci spaces*, J. Math. Scien.: Advances and Applications, **5** (2010), no. 1, 19-27.

[10] Hamilton, R.S., *Three manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255–306.

[11] Koivogui, M. and Todjihounde, L., *Weierstrass representation for minimal immersions into Damek-Ricci spaces*, Int. Electron. J. Geom., **6** (2013), 1–7.

[12] Shabbir, G., *Proper affine vector fields in spherically symmetic static spacetimes*, Differ. Geom. Dyn. Syst. **8** (2006), 244-252.

[13] Sidhoumi, N., *Ricci solitons of four-dimensional Lorentzian Damek–Ricci spaces*, J. Math. Phys. Anal. Geom., **16**, (2020), no. 2, 1-10.

[14] Tan, J. and Deng, S., *Some geometrical properties of four dimensional Lorentzian Damek-Ricci spaces*, Publ. Math. Debrecen, **89** (2016), no. 1-2, 105–124.