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GEOMETRY OF BILINEAR FORMS ON THE PLANE WITH THE OCTAGONAL NORM

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Abstract

Let $\mathbb{R}^2_{o(w)}$ be the plane with the octagonal norm with weight $0 < w, w \neq 1$

$$\|(x,y)\|_{o(w)} = \max\Big\{|x| + w|y|, |y| + w|x|\Big\}.$$

In this paper we classify all extreme, exposed and smooth points of the closed unit balls of $\mathcal{L}(^2\mathbb{R}^2_{o(w)})$ and $\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$, where $\mathcal{L}(^2\mathbb{R}^2_{o(w)})$ is the space of bilinear forms on $\mathbb{R}^2_{o(w)}$, and $\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$ is the subspace of $\mathcal{L}(^2 l^2_{\infty,\theta})$ consisting of symmetric bilinear forms.

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1 Introduction

Throughout the paper, we let $n, m \in \mathbb{N}, n, m \geq 2$. We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. An element $x \in B_E$ is called an *exposed point* of B_E if there is $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. An element $x \in B_E$ is called a *smooth point* of B_E if there is unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $\text{ext}\, B_E$, $\exp B_E$ and $\text{sm}\, B_E$ the set of extreme points, the set of exposed points and the set of smooth points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form T on the product $E \times \cdots \times E$ such that $P(x) = T(x, \cdots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by

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$\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \cdots, x_n)|$. $\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on E. Notice that $\mathcal{L}(^nE)$ is identified with the dual of n-fold projective tensor product $\hat{\bigotimes}_{\pi,n}E$. With this identification, the action of a continuous n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{\pi,n}E$ is given by

$$\Big\langle \sum_{i=1}^{k} x^{(1),i} \otimes \cdots \otimes x^{(n),i},\ T \Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i}, \cdots, x^{(n),i}\Big).$$

Notice also that $\mathcal{L}_s(^nE)$ is identified with the dual of n-fold symmetric projective tensor product $\hat{\bigotimes}_{s,\pi,n}E$. With this identification, the action of a continuous symmetric n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{s,\pi,n}E$ is given by

$$\Big\langle \sum_{i=1}^{k} \frac{1}{n!}\Big(\sum_{\sigma} x^{\sigma(1),i} \otimes \cdots \otimes x^{\sigma(n),i}\Big),\ T \Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i}, \cdots, x^{(n),i}\Big),$$

where σ goes over all permutations on $\{1, \ldots, n\}$. For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer to [8].

Let us sketch the history of classification problems of the extreme points and the exposed points of the unit ball of continuous n-homogeneous polynomials on a Banach space.

We let $l_p^n = \mathbb{R}^n$ for every $1 \le p \le \infty$ equipped with the l_p-norm. Choi et al. ([3]–[5]) initiated and classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $p = 1, 2$. Choi and Kim [7] classified $\exp B_{\mathcal{P}(^2l_p^2)}$ for $p = 1, 2, \infty$. Grecu [12] classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. Kim et al. [35] showed that if E is a separable real Hilbert space with $\dim(E) \ge 2$, then, $\operatorname{ext} B_{\mathcal{P}(^2E)} = \exp B_{\mathcal{P}(^2E)}$. Kim [16] classified $\exp B_{\mathcal{P}(^2l_p^2)}$ for $1 \le p \le \infty$. Kim ([18], [20]) characterized $\operatorname{ext} B_{\mathcal{P}(^2d_*(1,w)^2)}$, where $d_*(1,w)^2 = \mathbb{R}^2$ with an octagonal norm $\|(x,y)\|_w = \max\Big\{|x|, |y|, \frac{|x|+|y|}{1+w}\Big\}$ for $0 < w < 1$. Kim [25] classified $\exp B_{\mathcal{P}(^2d_*(1,w)^2)}$ and showed that $\exp B_{\mathcal{P}(^2d_*(1,w)^2)} \neq \operatorname{ext} B_{\mathcal{P}(^2d_*(1,w)^2)}$. Recently, Kim ([30], [33]) classified $\operatorname{ext} B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ and $\exp B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with a hexagonal norm $\|(x,y)\|_{h(\frac{1}{2})} = \max\Big\{|y|, |x| + \frac{1}{2}|y|\Big\}$.

Parallel to the classification problems of $\operatorname{ext} B_{\mathcal{P}(^nE)}$ and $\exp B_{\mathcal{P}(^nE)}$, it seems to be very natural to study the classification problems of the extreme points and the exposed points of the unit ball of continuous (symmetric) multilinear forms on a Banach space.

Kim [17] initiated and classified $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^2)}$ and $\exp B_{\mathcal{L}_s(^2l_\infty^2)}$. It was shown that $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^2)} = \exp B_{\mathcal{L}_s(^2l_\infty^2)}$.

Kim ([19], [21], [22], [24]) classified $\operatorname{ext} B_{\mathcal{L}_s(^2d_*(1,w)^2)}$, $\operatorname{ext} B_{\mathcal{L}(^2d_*(1,w)^2)}$, $\exp B_{\mathcal{L}_s(^2d_*(1,w)^2)}$, and $\exp B_{\mathcal{L}(^2d_*(1,w)^2)}$. Kim ([28], [29]) also classified $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^3)}$ and $\exp B_{\mathcal{L}_s(^3l_\infty^2)}$. It was shown that $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^3)} = \exp B_{\mathcal{L}_s(^2l_\infty^3)}$ and $\operatorname{ext} B_{\mathcal{L}_s(^3l_\infty^2)} = \exp B_{\mathcal{L}_s(^3l_\infty^2)}$. Kim [32] characterized $\operatorname{ext} B_{\mathcal{L}(^2l_\infty^n)}$ and $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)}$, and showed that

$\exp B_{\mathcal{L}(^2l_\infty^n)} = \text{ext}\, B_{\mathcal{L}(^2l_\infty^n)}$ and $\exp B_{\mathcal{L}_s(^2l_\infty^n)} = \text{ext}\, B_{\mathcal{L}_s(^2l_\infty^n)}$. Kim [34] characterized $\text{ext}\, B_{\mathcal{L}(^2l_\infty^3)}$ and $\exp B_{\mathcal{L}(^2l_\infty^3)}$. Kim [35] characterized $\text{sm}\, B_{\mathcal{L}_s(^nl_\infty^2)}$. Kim [36] studied $\text{ext}\, B_{\mathcal{L}(^2l_\infty)}$. Cavalcante et al. [2] characterized $\text{ext}\, B_{\mathcal{L}(^nl_\infty^m)}$. Recently, Kim [37] classified $\text{ext}\, B_{\mathcal{L}(^nl_\infty^2)}$ and $\text{ext}\, B_{\mathcal{L}_s(^nl_\infty^2)}$. It was shown that $|\,\text{ext}\, B_{\mathcal{L}(^nl_\infty^2)}| = 2^{(2^n)}$ and $|\,\text{ext}\, B_{\mathcal{L}_s(^nl_\infty^2)}| = 2^{n+1}$, and that $\exp B_{\mathcal{L}(^nl_\infty^2)} = \text{ext}\, B_{\mathcal{L}(^nl_\infty^2)}$ and $\exp B_{\mathcal{L}_s(^nl_\infty^2)} = \text{ext}\, B_{\mathcal{L}_s(^nl_\infty^2)}$. We refer to ([1]–[7], [9]–[52] and references therein) for some recent work about extremal properties of homogeneous polynomials and multilinear forms on Banach spaces.

Let $\mathbb{R}^2_{o(w)}$ denote $\mathbb{R}^2$ with the octagonal norm with weight $0 < w, w \neq 1$

$$\|(x, y)\|_{o(w)} = \max\Big\{|x| + w|y|, |y| + w|x|\Big\}.$$

Let $\mathcal{F} = \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ or $\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$. First we present formulae for the norm of $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$. Using these formulae, we classify the extreme points of the unit ball of $\mathcal{F}$. We show that

$$\text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})} \neq \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)}) \text{ for } w \in [\sqrt{2} - 1, \sqrt{2} + 1]\backslash\{1\},$$
$$\text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})} = \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)}) \text{ for } w \in (0, \infty)\backslash[\sqrt{2} - 1, \sqrt{2} + 1].$$

We present formulae for the norm of $f \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})^*$. Using these formulae, we show that every extreme point is exposed in this space. We show that

$$\exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})} \neq \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)}) \text{ for } w \in [\sqrt{2} - 1, \sqrt{2} + 1]\backslash\{1\},$$
$$\exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})} = \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)}) \text{ for } w \in (0, \infty)\backslash[\sqrt{2} - 1, \sqrt{2} + 1].$$

We classify the smooth points of the unit balls of the spaces of symmetric bilinear forms and bilinear forms on $\mathbb{R}^2_{o(w)}$, respectively.

We show that $\text{sm}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \bigcap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$ is a proper subset of $\text{sm}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$.

2 Computation of the norm of bilinear forms of $\mathcal{L}(^2\mathbb{R}^2_{o(w)})$

Let $\mathbb{R}^2_{o(w)}$ denote $\mathbb{R}^2$ with the octagonal norm with weight $0 < w, w \neq 1$

$$\|(x, y)\|_{o(w)} = \max\Big\{|x| + w|y|, |y| + w|x|\Big\}.$$

Notice that

$$\|(x, y)\|_{o(w)} = \|(y, x)\|_{o(w)} = \|(x, -y)\|_{o(w)} \text{ for } (x, y) \in \mathbb{R}^2_{o(w)}.$$

Notice that if $0 < w < 1$, then

$$\text{ext}\, B_{\mathbb{R}^2_{o(w)}} = \Big\{ \pm (1, 0), \pm((1 + w)^{-1}, \pm(1 + w)^{-1}), \pm(0, 1)\Big\},$$

and that if $w > 1$, then

$$\text{ext}\, B_{\mathbb{R}^2_{o(w)}} = \Big\{ \pm (w^{-1}, 0), \pm((1+w)^{-1}, \pm(1+w)^{-1}), \pm(0, w^{-1}) \Big\}.$$

Let $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ be such that $T = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1$. For simplicity, we will denote T by (a, b, c, d).

Theorem 1. *Let* $0 < w, w \neq 1$ *and* $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$. *Then there exists (unique)* $T^{'}((x_1, y_1), (x_2, y_2)) = a^*x_1x_2 + b^*y_1y_2 + c^*x_1y_2 + d^*x_2y_1 \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ *such that* $a^*, b^*, c^*, d^* \in \{\pm a, \pm b, \pm c, \pm d\}$ *with* $a^* \geq b^* \geq 0, c^* \geq |d^*|$ *and* $\|T\| = \|T^{'}\|$ *and that* T *is extreme (exposed, respectively) if and only if* $T^{'}$ *is extreme (exposed, respectively).*

Proof. If $a < 0$, taking $-T$, we assume $a \geq 0$.

Case 1. $|b| > a$

$$\begin{aligned} \text{Let } T_1^{'}((x_1, y_1), (x_2, y_2)) &:= T((y_1, sign(b)x_1), (y_2, x_2)) \\ &= |b|x_1x_2 + |a|y_1y_2 + sign(b)dx_1y_2 + cx_2y_1. \end{aligned}$$

Then $\|T_1^{'}\| = \|T\|$ and T is extreme if and only if $T_1^{'}$ is extreme. If $sign(b)d \geq |c|$, then the bilinear form $T_1^{'}$ satisfies the condition of the theorem. Suppose that $sign(b)d < |c|$.

Subcase 1. $c \geq 0$

If $sign(b)d = |d|$ or $(sign(b)d = -|d|, |d| \leq |c|)$,

$$\begin{aligned} \text{let } T_2^{'}((x_1, y_1), (x_2, y_2)) &:= T_1^{'}((x_2, y_2), (x_1, y_1)) \\ &= |b|x_1x_2 + |a|y_1y_2 + |c|x_1y_2 + sign(b)dx_2y_1. \end{aligned}$$

Then $\|T_2^{'}\| = \|T\|$ and T is extreme (exposed, respectively) if and only if $T_2^{'}$ is extreme (exposed, respectively). Hence, the bilinear form $T_2^{'}$ satisfies the condition of the theorem. If $sign(b)d = -|d|, |d| > |c|$,

$$\begin{aligned} \text{let } T_2^{'}((x_1, y_1), (x_2, y_2)) &:= T_1^{'}((x_2, -y_2), (x_1, -y_1)) \\ &= |b|x_1x_2 + |a|y_1y_2 + |sign(b)d|x_1y_2 - |c|x_2y_1. \end{aligned}$$

Then $\|T_2^{'}\| = \|T\|$ and T is extreme (exposed, respectively) if and only if $T_2^{'}$ is extreme (exposed, respectively). Hence, the bilinear form $T_2^{'}$ satisfies the condition of the the theorem.

Subcase 2. $c < 0$

$$\begin{aligned} \text{Let } T_3^{'}((x_1, y_1), (x_2, y_2)) &:= T_1^{'}((-x_1, y_1), (-x_2, y_2)) \\ &= |b|x_1x_2 + |a|y_1y_2 - sign(b)dx_1y_2 + |c|x_2y_1. \end{aligned}$$

Applying Subcase 1 to $T_3^{'}$, we can find a bilinear form $T^{'}$ which satisfies the condition of the theorem.

Case 2. $|b| \leq a$

$$\begin{aligned}\text{Let } T_4'((x_1, y_1),\ (x_2, y_2)) &:= T((x_1, y_1), (x_2, sign(b)y_2)) \\ &= ax_1x_2 + |b|y_1y_2 + sign(b)cx_1y_2 + dx_2y_1.\end{aligned}$$

Applying Case 1 to T_4', we can find a bilinear form T' which satisfies the condition of the theorem. □

Theorem 2. *Let* $0 < w, w \neq 1$ *and* $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ *be such that* $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 = (a, b, c, d)$ *for some* $a, b, c, d \in \mathbb{R}$. *Then:*
(*a*) *If* $0 < w < 1$, *then*

$$\begin{aligned}\|T\| &= max\Big\{|a|, |b|, |c|, |d|, (1+w)^{-1}(|a|+|c|), (1+w)^{-1}(|a|+|d|), \\ &\quad (1+w)^{-1}(|b|+|c|), (1+w)^{-1}(|b|+|d|), (1+w)^{-2}(|a-b|+|c-d|), \\ &\quad (1+w)^{-2}(|a+b|+|c+d|)\Big\}\end{aligned}$$

(*b*) *If* $1 < w$, *then*

$$\begin{aligned}\|T\| &= \max\Big\{w^{-2}|a|, w^{-2}|b|, w^{-2}|c|, w^{-2}|d|, (w(1+w))^{-1}(|a|+|c|), \\ &\quad (w(1+w))^{-1}(|a|+|d|), (w(1+w))^{-1}(|b|+|c|), \\ &\quad (w(1+w))^{-1}(|b|+|d|), (1+w)^{-2}(|a-b|+|c-d|), \\ &\quad (1+w)^{-2}(|a+b|+|c+d|)\Big\}.\end{aligned}$$

Proof. (a). Let $0 < w < 1$. Notice that

$$\text{ext } B_{\mathbb{R}^2_{o(w)}} = \Big\{\pm(1,0), \pm((1+w)^{-1}, \pm(1+w)^{-1}), \pm(0,1)\Big\}.$$

By the bilinearity of T, we have

$$\begin{aligned}&\|T\| \\ =\ & \sup\Big\{|T((x_1, y_1), (x_2, y_2))| : (x_j, y_j) \in \text{ext } B_{\mathbb{R}^2_{o(w)}} \text{ for } j = 1, 2\Big\} \\ =\ & \max\Big\{|T((1,0),(1,0))|, |T((0,1),(0,1))|, |T((1,0),(0,1))|, |T((0,1),(1,0))|, \\ & |T((1,0), \pm((1+w)^{-1}, \pm(1+w)^{-1}))|, |T(\pm((1+w)^{-1}, \pm(1+w)^{-1}), (1,0))|, \\ & |T((0,1), \pm((1+w)^{-1}, \pm(1+w)^{-1}))|, |T(\pm((1+w)^{-1}, \pm(1+w)^{-1}), (0,1))|, \\ & \quad |T(\pm((1+w)^{-1}, \pm(1+w)^{-1}), ((1+w)^{-1}, -(1+w)^{-1})|, \\ & |T(((1+w)^{-1}, -(1+w)^{-1}), ((1+w)^{-1}, (1+w)^{-1}))|, \\ & |T(((1+w)^{-1}, (1+w)^{-1}), ((1+w)^{-1}, (1+w)^{-1}))|, \\ & |T(((1+w)^{-1}, -(1+w)^{-1}), ((1+w)^{-1}, -(1+w)^{-1}))|\Big\} \\ =\ & \max\Big\{|a|,|b|,|c|,|d|, (1+w)^{-1}(|a|+|c|), (1+w)^{-1}(|a|+|d|), (1+w)^{-1}(|b|+|c|), \\ & (1+w)^{-1}(|b|+|d|), (1+w)^{-2}(|a-b|+|c-d|), (1+w)^{-2}(|a+b|+|c+d|)\Big\}.\end{aligned}$$

(b). Let $w > 1$.

Claim. $\|T\|_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} = \left\|w^{-2}T\right\|_{\mathcal{L}(^2\mathbb{R}^2_{o(1/w)})}$.

Notice that

$$\|(w^{-1}x, w^{-1}y)\|_{o(w)} = \|(x,y)\|_{o(1/w)}$$

for $(x,y) \in \mathbb{R}^2$. It follows that

$$\begin{aligned}
& \left\|w^{-2}T\right\|_{\mathcal{L}(^2\mathbb{R}^2_{o(1/w)})} \\
= & \sup_{\|(x_j,y_j)\|_{o(1/w)}=1,\ j=1,2} \left|w^{-2}ax_1x_2 + w^{-2}by_1y_2 + w^{-2}cx_1y_2 + w^{-2}dx_2y_1\right| \\
= & \sup_{\|(w^{-1}x_j,w^{-1}y_j)\|_{o(w)}=1,\ j=1,2} \Big|a(w^{-1}x_1)(w^{-1}x_2) + w^{-2}b(w^{-1}y_1)(w^{-1}y_2) \\
+ & w^{-2}c(w^{-1}x_1)(w^{-1}y_2) + w^{-2}d(w^{-1}x_2)(w^{-1}y_1)\Big| \\
= & \|T\|_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}.
\end{aligned}$$

By (a), we have

$$\begin{aligned}
& \|T\|_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} = \left\|(w^{-2}a, w^{-2}b, w^{-2}c, w^{-2}d)\right\|_{\mathcal{L}(^2\mathbb{R}^2_{o(1/w)})} \\
= & \max\Big\{w^{-2}|a|, w^{-2}|b|, w^{-2}|c|, w^{-2}|d|, (1+w^{-1})^{-1}(w^{-2}|a| + w^{-2}|c|), \\
& (1+w^{-1})^{-1}(w^{-2}|a| + w^{-2}|d|), (1+w^{-1})^{-1}(w^{-2}|b| + w^{-2}|c|), \\
& (1+w^{-1})^{-1}(w^{-2}|b| + w^{-2}|d|), (1+w^{-1})^{-2}(w^{-2}|a-b| + w^{-2}|c-d|), \\
& (1+w^{-1})^{-2}(w^{-2}|a+b| + w^{-2}|c+d|)\Big\} \\
= & \Big\{w^{-2}|a|, w^{-2}|b|, w^{-2}|c|, w^{-2}|d|, (w(1+w))^{-1}(|a|+|c|), \\
& (w(1+w))^{-1}(|a|+|d|), (w(1+w))^{-1}(|b|+|c|), \\
& (w(1+w))^{-1}(|b|+|d|), (1+w)^{-2}(|a-b|+|c-d|), \\
& (1+w)^{-2}(|a+b|+|c+d|)\Big\}.
\end{aligned}$$

□

3 The extreme points of the unit ball of $\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$

Let $0 < w, w \neq 1$ and $T \in \mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$ be such that $T = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$. For simplicity, we will denote T by (a,b,c).

Theorem 3. *(a) If* $0 < w \leq \sqrt{2} - 1$*, then*

$$\begin{aligned} \mathrm{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})} &= \Big\{ \pm(1, w^2, \pm w), \pm(w^2, 1, \pm w), \pm(1, -(w^2+2w), \pm w), \\ &\quad \pm(-(w^2+2w), 1, \pm w), \pm\Big(\frac{(1+w)^2}{2}, -\frac{(1+w)^2}{2}, \pm\frac{1-w^2}{2}\Big) \\ &\quad \pm\Big(\frac{1-w^2}{2}, -\frac{(1-w^2)}{2}, \pm\frac{(1+w)^2}{2}\Big)\Big\}. \end{aligned}$$

(b) If $\sqrt{2} - 1 < w < 1$*, then,*

$$\begin{aligned} \mathrm{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})} &= \Big\{ \pm(1, w^2, \pm w), \pm(w^2, 1, \pm w), \pm(w, w^2+w-1, \pm 1), \\ &\quad \pm(w^2+w-1, w, \pm 1), \pm\Big(1, 1, \frac{\pm(w^2+2w-1)}{2}\Big), \\ &\quad \pm\Big(\frac{w^2+2w-1}{2}, \frac{w^2+2w-1}{2}, \pm 1\Big), \pm(1, -1, \pm w), \\ &\quad \pm(w, -w, \pm 1)\Big\}. \end{aligned}$$

(c) If $1 < w < \sqrt{2} + 1$*, then,*

$$\begin{aligned} \mathrm{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})} &= \Big\{ \pm(w^2, 1, \pm w), \pm(1, w^2, \pm w), \pm(w, -w^2+w+1, \pm w^2), \\ &\quad \pm(-w^2+w+1, w, \pm w^2), \pm\Big(w^2, w^2, \frac{\pm(-w^2+2w+1)}{2}\Big), \\ &\quad \pm\Big(\frac{-(w^2+2w-1)}{2}, \frac{-(w^2+2w-1)}{2}, \pm w^2\Big), \\ &\quad \pm(w^2, -w^2, \pm w), \pm(w, -w, \pm w^2)\Big\}. \end{aligned}$$

(d) If $\sqrt{2} + 1 < w$*, then,*

$$\begin{aligned} \mathrm{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})} &= \Big\{ \pm(w^2, 1, \pm w), \pm(1, w^2, \pm w), \pm(w^2, -(1+2w), \pm w), \\ &\quad \pm(-(1+2w), w^2, \pm w), \pm\Big(\frac{(1+w)^2}{2}, -\frac{(1+w)^2}{2}, \pm\frac{w^2-1}{2}\Big) \\ &\quad \pm\Big(\frac{w^2-1}{2}, -\frac{(w^2-1)}{2}, \pm\frac{(1+w)^2}{2}\Big)\Big\}. \end{aligned}$$

Proof. Let $T \in \mathrm{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$ be such that $T = (a, b, c)$. By Theorem 1, we may assume that $a \geq |b|$ and $c \geq 0$. Suppose that $0 < w < 1$.

Case 1. $b \geq 0$

Subcase 1. $b = a$

Suppose that $a = b = 1$. By Theorem 2(a), $c \leq w$. If $c = w$, then $T = (1, 1, w)$, which is a contradiction because $\|T\| = 1$. Hence, $c < w$. Since $T \in \mathrm{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$, we have $\frac{1}{(1+w)^2}(a+b+2c)=1$, which shows that $T=\Big(1,1,\frac{w^2+2w-1}{2}\Big)$ for $\sqrt{2}-1\le w<1$.

Claim 1. $T=\Big(1,1,\frac{w^2+2w-1}{2}\Big)\in \text{ext}\,B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1\le w<1$.

Let

$$T^{\pm}=\Big(1,1,\frac{w^2+2w-1}{2}\pm\gamma\Big)$$

be such that $1=\|T^{\pm}\|$ for some $\gamma\in\mathbb{R}$. By Theorem 2(a), we have

$$\frac{(1+w)^2\pm 2\gamma}{(1+w)^2}\le 1,$$

hence, $\gamma=0$.

Suppose that $a=b<1$. If $c<1$, since $T\in \text{ext}\,B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$, we have $\frac{1}{1+w}(a+c)=\frac{1}{(1+w)^2}(a+b+2c)=1$, which shows that $w^2=1$, which is a contradiction. Hence, $c=1$. Since $T\in \text{ext}\,B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$, we have $\frac{1}{1+w}(a+c)=1$ or $\frac{1}{1+w}(a+c)=\frac{1}{(1+w)^2}(a+b+2c)=1$. If $\frac{1}{1+w}(a+c)=1$, then $T=(w,w,1)$, which is a contradiction because $\|T\|=1$. If $\frac{1}{1+w}(a+c)=\frac{1}{(1+w)^2}(a+b+2c)=1$, then $T=(w,w^2+w-1,1)$, which is impossible because $a=b$.

Subcase 2: $b<a$

Suppose that $a=1$. By Theorem 2(a), $c\le w$. If $c=w$, then $\frac{1}{(1+w)^2}(a+b+2c)=1$, hence, $T=(1,w^2,w)$ for $0<w<1$.

Claim 2. $T=(1,w^2,w)\in \text{ext}\,B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ for $0<w<1$.

Let

$$T^{\pm}=(1,w^2\pm\delta,w\pm\gamma)$$

be such that $1=\|T^{\pm}\|$ for some $\delta,\gamma\in\mathbb{R}$. By Theorem 2(a), we have

$$\frac{1}{1+w}(1+w\pm\delta)\le 1,\ \frac{1}{(1+w)^2}((1+w)^2\pm(\delta+2\gamma))\le 1,$$

hence, $\delta=\gamma=0$. If $c<w$, then $\frac{1}{(1+w)^2}(a+b+2c)=1$. Let

$$T^{\pm}=(a,b\pm\frac{2}{n},c\mp\frac{1}{n})$$

so that $1=\|T^{\pm}\|$ for some big $n\in\mathbb{N}$, which shows that T is not extreme. It is a contradiction.

Suppose that $a<1$. If $c<1$, then $1=\frac{1}{1+w}(a+c)$ or $1=\frac{1}{(1+w)^2}(a+b+2c)$, which is a contradiction because $T\in \text{ext}\,B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$. Hence, $c=1$. Since $T\in \text{ext}\,B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$, we have $\frac{1}{1+w}(a+c)=\frac{1}{(1+w)^2}(a+b+2c)=1$, then $T=(w,w^2+w-1,1)$ for $\frac{\sqrt{5}-1}{2}\le w<1$.

Claim 3. $T=(w,w^2+w-1,1)\in \operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$ for $\frac{\sqrt{5}-1}{2}\le w<1$.

Let

$$T^{\pm}=(w\pm\epsilon, w^2+w-1\pm\delta,1)$$

be such that $1=\|T^{\pm}\|$ for some $\epsilon,\delta\in\mathbb{R}$. By Theorem 2(a), we have

$$1=\frac{1}{1+w}(1+w\pm\epsilon)\le 1, \frac{1}{(1+w)^2}((1+w)^2\pm(\epsilon+\delta))\le 1,$$

hence, $\epsilon=\delta=0$.

Case 2: $b<0$

Subcase 1: $|b|=a$

Suppose that $a=|b|=1$. By Theorem 2(a), $c\le w$. If $c=w$, then $T=(1,-1,w)$ for $\sqrt{2}-1\le w<1$.

Claim 4. $T=(1,-1,w)\in \operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1\le w<1$.

Let

$$T^{\pm}=(1,-1,w\pm\gamma)$$

be such that $1=\|T^{\pm}\|$ for some $\gamma\in\mathbb{R}$. By Theorem 2(a), we have

$$\frac{1}{1+w}(1+w\pm\gamma)\le 1,$$

hence, $\gamma=0$.

If $c<w$, then $\frac{1}{(1+w)^2}(a-b)=1$, hence, $T=(1,-1,c)$ for $0\le c<w=\sqrt{2}-1$, which is a contradiction because $T\in \operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(\sqrt{2}-1)})}$.

Suppose that $a=|b|<1$. Suppose that $c<1$. Note that if $\frac{1}{1+w}(a+c)<1$, then $\frac{1}{(1+w)^2}(a-b)=1$ or $\frac{1}{(1+w)^2}(a+b+2c)=1$, which is a contradiction because $T\in \operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$. Hence, $\frac{1}{1+w}(a+c)=1$.

If $\frac{1}{(1+w)^2}(a-b)=1$, then $T=\Big(\frac{(1+w)^2}{2},-\frac{(1+w)^2}{2},\frac{1-w^2}{2}\Big)$ for $0<w\le\sqrt{2}-1$.

Claim 5. $T=\Big(\frac{(1+w)^2}{2},-\frac{(1+w)^2}{2},\frac{1-w^2}{2}\Big)\in \operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$ for $0<w\le\sqrt{2}-1$.

Let

$$T^{\pm}=\Big(\frac{(1+w)^2}{2}\pm\epsilon,-\frac{(1+w)^2}{2}\pm\delta,\frac{1-w^2}{2}\pm\gamma\Big)$$

be such that $1=\|T^{\pm}\|$ for some $\epsilon,\delta,\gamma\in\mathbb{R}$. Since

$$\Big|T^{\pm}\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\le 1\ (j=1,2),$$

we have $\epsilon+\gamma=0$. Since

$$\Big|T^{\pm}\Big((0,1),\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big)\Big)\Big|\le 1\ (j=1,2),$$

we have $-\delta+\gamma=0$. Since

$$\Big|T^{\pm}\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big)\Big)\Big|\leq 1\ (j=1,2),$$

we have $\epsilon-\delta=0$. Hence, $\epsilon=\delta=\gamma=0$.

If $\frac{1}{(1+w)^2}(a+b+2c)=1$, then $T=\Big(\frac{1-w^2}{2},-\frac{(1-w^2)}{2},\frac{(1+w)^2}{2}\Big)$ for $0<w\leq\sqrt{2}-1$.

Claim 6. $T=\Big(\frac{1-w^2}{2},-\frac{(1-w^2)}{2},\frac{(1+w)^2}{2}\Big)\in \operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$ for $0<w\leq\sqrt{2}-1$.

Let

$$T^{\pm}=\Big(\frac{1-w^2}{2}\pm\epsilon,-\frac{(1-w^2)}{2}\pm\delta,\frac{(1+w)^2}{2}\pm\gamma\Big)$$

be such that $1=\|T^{\pm}\|$ for some $\epsilon,\delta,\gamma\in\mathbb{R}$. Since

$$\Big|T^{\pm}\Big((1,0),\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big)\Big)\Big|\leq 1\ (j=1,2),$$

we have $\epsilon-\gamma=0$. Since

$$\Big|T^{\pm}\Big((0,1),\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big)\Big)\Big|\leq 1\ (j=1,2),$$

we have $-\delta+\gamma=0$. Since

$$\Big|T^{\pm}\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\leq 1\ (j=1,2),$$

we have $\epsilon+\delta+2\gamma=0$. Hence, $\epsilon=\delta=\gamma=0$.

Suppose that $c=1$. By Theorem 2(a), $a\leq w$. If $a=w$, then $T=(w,-w,1)$ $\sqrt{2}-1\leq w<1$.

Claim 7. $T=(w,-w,1)\in \operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1\leq w<1$

Let

$$T^{\pm}=(w\pm\epsilon,-w\pm\delta,1)$$

be such that $1=\|T^{\pm}\|$ for some $\epsilon,\delta\in\mathbb{R}$. By Theorem 2(a), we have

$$\frac{1}{1+w}(1+w\pm\epsilon)\leq 1,\frac{1}{1+w}(1+|-w\pm\delta|)\leq 1,$$

hence, $\epsilon=\delta=0$.

Subcase 2. $|b|<a$

Suppose that $a=1$. By Theorem 2(a), $c\leq w$. If $c=w$, then $\frac{1}{(1+w)^2}(a-b)=1$, hence, $T=(1,-(2w+w^2),w)$ for $0<w\leq\sqrt{2}-1$.

Claim 8. $T=(1,-(w^2+2w),w)\in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$ for $0<w\leq\sqrt{2}-1$.

Let

$$T^{\pm}=(1,-(w^2+2w)\pm\delta,w\pm\gamma)$$

be such that $1 = \|T^{\pm}\|$ for some $\delta, \gamma \in \mathbb{R}$. By Theorem 2(a), we have

$$\frac{1}{1+w}(1+w\pm\gamma) \leq 1, \frac{1}{(1+w)^2}((1+w)^2 \pm \delta) \leq 1,$$

hence, $\delta = \gamma = 0$.

If $c < w$, then $\frac{1}{(1+w)^2}(a-b) \leq 1$ and $\frac{1}{(1+w)^2}(a+b+2c) < 1$, which is a contradiction because $T \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$. Suppose that $a < 1$. Suppose that $c = 1$. By Theorem 2(a), $a \leq w$. If $a < w$, then $\frac{1}{(1+w)^2}(a-b) < 1$ and $\frac{1}{(1+w)^2}(a+b+2c) = 1$, which is a contradiction because $T \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$. Hence, $a = w$ and $\frac{1}{(1+w)^2}(a+b+2c) = 1$, for $\sqrt{2}-1 < w < \frac{\sqrt{5}-1}{2}$.

Claim 9. $T = (w, w^2+w-1, 1) \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1 < w < \frac{\sqrt{5}-1}{2}$.

Let

$$T^{\pm} = (w \pm \epsilon, w^2+w-1\pm\delta, 1)$$

be such that $1 = \|T^{\pm}\|$ for some $\epsilon, \delta \in \mathbb{R}$. By Theorem 2(a), we have

$$1 = \frac{1}{1+w}(1+w\pm\epsilon) \leq 1, \frac{1}{(1+w)^2}((1+w)^2 \pm (\epsilon+\delta)) \leq 1,$$

hence, $\epsilon = \delta = 0$. If $c < 1$, then $\frac{1}{1+w}(a+c) = \frac{1}{(1+w)^2}(a-b) = \frac{1}{(1+w)^2}(a+b+2c) = 1$, which is a contradiction.

Suppose that $1 < w$. By the claim in the proof (b) of Theorem 2,

$$\text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})} = \Big\{w^2 T : T \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(1/w)})}\Big\}.$$

By (a) and (b) in the case of $0 < w < 1$, (c) and (d) follow. Therefore, we complete the proof. □

4 The extreme points of the unit ball of $\mathcal{L}({}^2\mathbb{R}^2_{o(w)})$

Theorem 4. *Let* $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 = (a, b, c, d) \in \mathcal{L}({}^2\mathbb{R}^2_{o(w)})$. *Then the following are equivalent:*

(*a*) $T \in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})}$;

(*b*) $(-a, -b, -c, -d) \in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})}$;

(*c*) $(a, b, -c, -d) \in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})}$;

(*d*) $(a, -b, c, -d) \in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})}$;

(*e*) $(a, -b, -c, d) \in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})}$;

(*f*) $(b, a, c, d) \in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})}$;

(*g*) $(d, c, a, b) \in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})}$.

Proof. Notice that

$$\begin{aligned}(-a,-b,-c,-d) &= T((x_1,y_1),\ (-x_2,-y_2)),\\ (a,b,-c,-d) &= T((x_1,-y_1),\ (x_2,-y_2)),\\ (a,-b,c,-d) &= T((x_1,-y_1),\ (x_2,y_2)),\\ (a,-b,-c,d) &= T((x_1,y_1),\ (x_2,-y_2)),\\ (b,a,c,d) &= T((y_2,x_2),\ (y_1,x_1)),\\ (d,c,a,b) &= T((y_2,x_2),\ (x_1,y_1)),\end{aligned}$$

and that

$$\|(x_j,y_j)\|_{o(w)} = \|(y_j,x_j)\|_{o(w)} = \|(x_j,-y_j)\|_{o(w)}$$

for $(x_j,y_j)\in\mathbb{R}^2$ and $j=1,2$. We complete the proof. □

For $T\in\mathcal{L}(^2\mathbb{R}^2_{o(w)})$, we let

$$\begin{aligned}&\mathrm{Norm}(T)\\ =\ &\Big\{((x_1,y_1),(x_2,y_2))\in \mathrm{ext}\,B_{\mathbb{R}^2_{o(w)}}\times \mathrm{ext}\,B_{\mathbb{R}^2_{o(w)}} : |T((x_1,y_1),(x_2,y_2))| = \|T\|\Big\}.\end{aligned}$$

We call $\mathrm{Norm}(T)$ *the norming set* of T. By Theorems 2 and 4, it suffices to consider only $T=(a,b,c,d)\in\mathcal{L}(^2\mathbb{R}^2_{o(w)})$ with $a\geq b\geq 0$ and $c\geq |d|$ in order to classify the extreme points of $B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$.

Theorem 5. *Let $0<w, w\neq 1$ and $T\in\mathcal{L}(^2\mathbb{R}^2_{o(w)})$ be such that $T=ax_1x_2+by_1y_2+cx_1y_2+dx_2y_1$ with $a\geq b\geq 0$ and $c\geq |d|$. Then:*

(*a*) *Let $0<w\leq\sqrt{2}-1$. Then, $T\in \mathrm{ext}\,B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ if and only if*

$$\begin{aligned}T\ \in\ &\Big\{(1,w^2,w,w),(w,w,1,w^2),(1,w^2+2w,w,-w),\\ &\Big(w,w,1,-(w^2+2w)\Big),\Big(\frac{(1+w)^2}{2},\frac{(1+w)^2}{2},\frac{1-w^2}{2},-(\frac{1-w^2}{2})\Big),\\ &\Big(\frac{1-w^2}{2},\frac{1-w^2}{2},\frac{(1+w)^2}{2},-\frac{(1+w)^2}{2}\Big)\Big\}.\end{aligned}$$

(*b*) *Let $\sqrt{2}-1<w\leq\frac{\sqrt{5}-1}{2}$. Then, $T\in \mathrm{ext}\,B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ if and only if*

$$\begin{aligned}T\ \in\ &\Big\{(1,w^2,w,w),(w,w,1,w^2),(1,1,w,w^2+w-1),\\ &(w,-(w^2+w-1),1,-1),(1,1,w,-w),(w,w,1,-1)\Big\}.\end{aligned}$$

(*c*) *Let $\frac{\sqrt{5}-1}{2}<w<1$. Then, $T\in \mathrm{ext}\,B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ if and only if*

$$\begin{aligned}T\ \in\ &\Big\{(1,w^2,w,w),(w,w,1,w^2),(1,1,w,w^2+w-1),\\ &(w,w^2+w-1,1,1),(1,1,w,-w),(w,w,1,-1)\Big\}.\end{aligned}$$

(*d*) *Let* $1 < w \leq \frac{\sqrt{5}+1}{2}$. *Then,* $T \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ *if and only if*

$$\begin{aligned} T \in & \Big\{(w^2, 1, w, w), (w, w, w^2, 1), (w, -w^2 + w + 1, w^2, w^2), \\ & (w^2, w^2, w, -w^2 + w + 1), (w^2, w^2, w, -w), (w, w, w^2, -w^2)\Big\}. \end{aligned}$$

(*e*) *Let* $\frac{\sqrt{5}+1}{2} < w \leq \sqrt{2} + 1$. *Then,* $T \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ *if and only if*

$$\begin{aligned} T \in & \Big\{(w^2, 1, w, w), (w, w, w^2, 1), (w, -(-w^2 + w + 1), w^2, -w^2), \\ & (w^2, w^2, w, -w^2 + w + 1), (w^2, w^2, w, -w), (w, w, w^2, -w^2)\Big\}. \end{aligned}$$

(*f*) *Let* $\sqrt{2} + 1 < w$ *Then,* $T \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ *if and only if*

$$\begin{aligned} T \in & \Big\{(w^2, 1, w, w), (w, w, w^2, 1), (w^2, 1 + 2w, w, -w), \\ & \Big(w, w, w^2, -(1 + 2w)\Big), \Big(\frac{(1+w)^2}{2}, \frac{(1+w)^2}{2}, \frac{w^2-1}{2}, -(\frac{w^2-1}{2})\Big), \\ & \Big(\frac{w^2-1}{2}, \frac{w^2-1}{2}, \frac{(1+w)^2}{2}, -\frac{(1+w)^2}{2}\Big)\Big\}. \end{aligned}$$

Proof. Suppose that $0 < w < 1$.

Case 1. $c = |d|$.

First, suppose that $c = d$.
Since $T \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$, we have $T \in \text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$. By Theorem 3, we have

$$\begin{aligned} T = & (1, w^2, w, w) \ (0 < w < 1), \\ & \Big(1, 1, \frac{w^2 + 2w - 1}{2}, \frac{w^2 + 2w - 1}{2}\Big) \ (\sqrt{2} - 1 \leq w < 1), \\ & (w, -(w^2 + w - 1), 1, -1) \ (\sqrt{2} - 1 < w \leq \frac{\sqrt{5} - 1}{2}) \text{ or} \\ & (w, w^2 + w - 1, 1, 1) \ (\frac{\sqrt{5} - 1}{2} < w < 1). \end{aligned}$$

Claim 1. $T = (1, w^2, w, w) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w < 1$.

Note that

$$\begin{aligned} \text{Norm}(T) = & \Big\{\Big((1,0), (1,0)\Big), \Big((1,0), \Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big), \Big(\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), (1,0)\Big), \\ & \Big(\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), \Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}. \end{aligned}$$

Let

$$T^{\pm} = (1 \pm \epsilon, w^2 \pm \delta, w \pm \gamma, w \pm \rho)$$

be such that $\|T^{\pm}\| = 1$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since

$$|T^{\pm}((1,0),(1,0))| \leq 1, \ \Big|T^{\pm}\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big| \leq 1,$$
$$\Big|T^{\pm}\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),(1,0)\Big)\Big| \leq 1, \ \Big|T^{\pm}\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big| \leq 1,$$

we have $0 = \epsilon = \delta = \gamma = \rho$. By Theorem 4, $(w, w, 1, w^2) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w < 1$.

Claim 2. $T = \Big(1, 1, \frac{w^2+2w-1}{2}, \frac{w^2+2w-1}{2}\Big) \notin \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1 \leq w < 1$.

Let $n \in \mathbb{N}$ be such that

$$\frac{w^2+2w-1}{2} + \frac{1}{n} < w, \frac{2}{n(1+w)^2} < 1.$$

Let

$$T^{\pm} = \Big(1, 1, \frac{w^2+2w-1}{2} \pm \frac{1}{n}, \frac{w^2+2w-1}{2} \mp \frac{1}{n}\Big).$$

By Theorem 2(a), $\|T^{\pm}\| = 1, \ T = \frac{1}{2}(T^+ + T^-)$. Since $T \neq T^{\pm}$, $T \notin \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$.

Claim 3. $T = (w, -(w^2+w-1), 1, -1) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1 < w \leq \frac{\sqrt{5}+1}{2}$.

Note that

$$\begin{aligned}\text{Norm}(T) &= \Big\{((1,0),(0,1)),((0,1),(1,0)),\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big),\\ &\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),(1,0)\Big),\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big\}.\end{aligned}$$

Let

$$T^{\pm} = (w \pm \epsilon, -(w^2+w-1) \pm \delta, 1 \pm \gamma, -1 \pm \rho)$$

be such that $\|T^{\pm}\| = 1$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since

$$|T^{\pm}((1,0),(0,1))| \leq 1, \ |T^{\pm}((0,1),(1,0))| \leq 1,$$
$$\Big|T^{\pm}\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big| \leq 1, \ \Big|T^{\pm}\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big| \leq 1,$$

we have $0 = \epsilon = \delta = \gamma = \rho$.

Claim 4. $T = (w, w^2+w-1, 1, 1) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\frac{\sqrt{5}+1}{2} < w < 1$.

Note that

$$\begin{aligned}\text{Norm}(T) &= \Big\{((1,0),(0,1)),((0,1),(1,0)),((1,0),\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big),\\ &\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),(1,0)\Big),\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big\}.\end{aligned}$$

Let

$$T^{\pm} = (w \pm \epsilon, -(w^2+w-1) \pm \delta, 1 \pm \gamma, -1 \pm \rho)$$

be such that $\|T^{\pm}\| = 1$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since

$$|T^{\pm}((1,0),(0,1))| \leq 1,\ |T^{\pm}((0,1),(1,0))| \leq 1,$$
$$\Big|T^{\pm}\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big| \leq 1,\ \Big|T^{\pm}\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big| \leq 1,$$

we have $0 = \epsilon = \delta = \gamma = \rho$.

By Theorem 4, $(1, 1, w, w^2 + w - 1) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2} - 1 < w < 1$.

Suppose that $c = -d$.

By Theorem 4, $S = (a, -b, c, c) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$. Hence, $S \in \text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$. By Theorem 3, we have

$$\begin{aligned} S &= \Big(\frac{(1+w)^2}{2}, -\frac{(1+w)^2}{2}, \frac{1-w^2}{2}, \frac{1-w^2}{2}\Big)\ (0 < w \leq \sqrt{2}-1), \\ &\quad (1, -(w^2+2w), w, w)\ \big(0 < w \leq \sqrt{2}-1\big), (1, -1, w, w)\ (\sqrt{2}-1 \leq w < 1). \end{aligned}$$

Claim 5. $S = \Big(\frac{(1+w)^2}{2}, -\frac{(1+w)^2}{2}, \frac{1-w^2}{2}, \frac{1-w^2}{2}\Big) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w \leq \sqrt{2} - 1$.

Notice that

$$\begin{aligned} \text{Norm}(S) &= \Big\{\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big), \Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),(1,0)\Big), \\ &\quad \Big((0,1),\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big)\Big), \Big(\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big),(0,1)\Big), \\ &\quad \Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big)\Big), \\ &\quad \Big(\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big\}. \end{aligned}$$

Let

$$S^{\pm} = \Big(\frac{(1+w)^2}{2} \pm \epsilon, -\frac{(1+w)^2}{2} \pm \delta, \frac{1-w^2}{2} \pm \gamma, \frac{1-w^2}{2} \pm \rho\Big)$$

be such that $\|S^{\pm}\| = 1$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since

$$\Big|S^{\pm}\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big| \leq 1,\ \Big|S^{\pm}\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),(1,0)\Big)\Big| \leq 1,$$
$$\Big|S^{\pm}\Big((0,1),\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big)\Big)\Big| \leq 1,\ \Big|S^{\pm}\Big(\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big),(0,1)\Big)\Big| \leq 1,$$

we have $0 = \epsilon = \delta = \gamma = \rho$.

Claim 6. $S = (1, -(w^2 + 2w), w, w) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w \leq \sqrt{2} - 1$.

Note that

$$\begin{aligned} \text{Norm}(S) &= \Big\{((1,0),(1,0)), \Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big), \\ &\quad \Big(\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big),(1,0)\Big), \Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big\}. \end{aligned}$$

Let

$$S^{\pm}=(1\pm\epsilon, w^2+2w\pm\delta, w\pm\gamma, -w\pm\rho)$$

be such that $\|S^{\pm}\|=1$ for some $\epsilon,\delta,\gamma,\rho\in\mathbb{R}$. Since

$$|S^{\pm}((1,0),(1,0))|\le 1,\ \Big|S^{\pm}\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\le 1,$$
$$\Big|S^{\pm}\Big(\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big),(1,0)\Big)\Big|\le 1,\ \Big|S^{\pm}\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\le 1,$$

we have $0=\epsilon=\delta=\gamma=\rho$.

By Theorem 4, $(w,w,1,-(w^2+2w))\in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0<w\le\sqrt{2}-1$.

Claim 7. $S=(1,-1,w,w)\in extB_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1\le w<1$.

Note that

$$\begin{aligned}\text{Norm}(S) &= \Big\{((1,0),(1,0)),((0,1),(0,1)),\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big),\\ &\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),(1,0)\Big),\Big((0,1),\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big)\Big)\\ &\Big(\Big(\frac{1}{1+w},-\frac{1}{1+w}\Big),(0,1)\Big)\Big\}.\end{aligned}$$

Let

$$S^{\pm}=(1\pm\epsilon,-1\pm\delta, w\pm\gamma, w\pm\rho)$$

be such that $\|S^{\pm}\|=1$ for some $\epsilon,\delta,\gamma,\rho\in\mathbb{R}$. Since

$$|S^{\pm}((1,0),(1,0))|\le 1, |S^{\pm}((0,1),(0,1))|\le 1,$$
$$\Big|S^{\pm}\Big((1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\le 1,\Big|S^{\pm}\Big(\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),(1,0)\Big)\Big|\le 1,$$

we have $0=\epsilon=\delta=\gamma=\rho$.

By Theorem 4, $(1,1,w,-w)$, $(w,w,1,-1)\in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1\le w<1$.

Case 2. $c>|d|$.

Suppose that $a=1$.
Note that

$$\begin{aligned}\|T\| &= 1=\max\Big\{a,b,c,\frac{1}{(1+w)}(a+c),\frac{1}{(1+w)}(b+c),\\ &\frac{1}{(1+w)^2}(a-b+c-d),\frac{1}{(1+w)^2}(a+b+c+d)\Big\}.\end{aligned}$$

Hence, $c\le w$. We claim that if $a=1, c<w$, then T is not extreme. Without a loss of generality we may assume that $b=1$. Then, $\frac{1}{(1+w)^2}(a-b+c-d)<1$. Hence,

$$\|T\|=1=\max\Big\{a,b,\frac{1}{(1+w)^2}(a+b+c+d)\Big\}.$$

Note that Norm(T) has at most 3 elements. Hence, T is not extreme, which is a contradiction. Hence, $a = b = 1, c = w, 1 = \frac{1}{(1+w)^2}(a+b+c+d)$. Therefore, $T = (1, 1, w, w^2+w-1)$ for $\sqrt{2}-1 < w < 1$. Since $(w, w^2+w-1, 1, 1) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1 < w < 1$, by Theorem 4, $T \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2}-1 < w < 1$.

Suppose that $a < 1$.

Note that if $c = 1$, then $a = w$. Indeed, if $a < w$, then

$$\|T\| = 1 = \max\Big\{c, \frac{1}{(1+w)^2}(a-b+c-d), \frac{1}{(1+w)^2}(a+b+c+d)\Big\},$$

which shows that T is not extreme because Norm(T) has at most 3 elements. Hence, $a = w, c = 1$. If $0 \le b < w$, then $\frac{1}{(1+w)^2}(a-b+c-d) < 1$. Hence,

$$\|T\| = 1 = \max\Big\{c, \frac{1}{(1+w)}(a+c), \frac{1}{(1+w)^2}(a+b+c+d)\Big\},$$

which shows that T is not extreme because Norm(T) has at most 3 elements. Therefore, $a = b = w, c = 1$ and $T = (w, w, 1, w^2)$ for $0 < w < 1$. Since $(1, w^2, w, w) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w < 1$, by Theorem 4, $T \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w < 1$.

Suppose that $1 < w$.

By the claim in the proof (b) of Theorem 2,

$$\operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} = \Big\{w^2 T : T \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(1/w)})}\Big\}.$$

By (a), (b) and (c) in the case of $0 < w < 1$, (d), (e) and (f) follow. Therefore, we complete the proof. □

Notice that $(\operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)})) \subseteq \operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ for all $0 < w, w \neq 1$.

Theorem 6. (*a*) *If* $w \in [\sqrt{2}-1, \sqrt{2}+1]\backslash\{1\}$, *then*

$$\begin{aligned} &\operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}\backslash(\operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)})) \\ = \ &\Big\{\pm\Big(1, 1, \pm\frac{w^2+2w-1}{2}, \pm\frac{w^2+2w-1}{2}\Big), \\ &\qquad \pm\Big(\frac{w^2+2w-1}{2}, \frac{w^2+2w-1}{2}, \pm 1, \pm 1\Big)\Big\}. \end{aligned}$$

(*b*) *If* $w \in (0, \infty)\backslash[\sqrt{2}-1, \sqrt{2}+1]$, *then*

$$\operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}\backslash(\operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)})) = \emptyset.$$

Proof. It follows from Claim 2 in the proof of Theorem 5. □

Kim [38] showed that for $n, m \ge 2$, $\operatorname{ext} B_{\mathcal{L}_s(^n l^2_\infty)} = \operatorname{ext} B_{\mathcal{L}(^n l^2_\infty)} \cap \mathcal{L}_s(^n l^2_\infty)$ and $\operatorname{ext} B_{\mathcal{L}_s(^2 l^{m+1}_\infty)} \neq \operatorname{ext} B_{\mathcal{L}(^2 l^{m+1}_\infty)} \cap \mathcal{L}_s(^2 l^{m+1}_\infty)$.

Corollary 1. (*a*) *If* $w \in [\sqrt{2}-1, \sqrt{2}+1]\backslash\{1\}$, *then*

$$\text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})} \neq \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)});$$

(*b*) *If* $w \in (0,\infty)\backslash[\sqrt{2}-1, \sqrt{2}+1]$, *then*

$$\text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})} = \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)}).$$

5 The exposed points of the unit balls of $\mathcal{L}({}^2\mathbb{R}^2_{o(w)})$ and $\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$

Lemma 1. *Let* $f \in \mathcal{L}({}^2\mathbb{R}^2_{o(w)})^*$ *for some* $w > 1$. *Then,* $\|f\| = w^2\|f\|_{\mathcal{L}({}^2\mathbb{R}^2_{o(\frac{1}{w})})}$.

Proof. It follows that

$$\begin{aligned}\|f\| &= \sup_{T\in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})}} |f(T)| = \sup_{R\in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(1/w)})}} |f(w^2R)| \\ &= w^2 \sup_{R\in \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{o(1/w)})}} |f(R)| = w^2\|f\|_{\mathcal{L}({}^2\mathbb{R}^2_{o(1/w)})}.\end{aligned}$$

□

Theorem 7. *Let* $f \in \mathcal{L}({}^2\mathbb{R}^2_{o(w)})^*$ *be such that* $\alpha = f(x_1x_2), \beta = f(y_1y_2), \gamma = f(x_1y_2), \delta = f(x_2y_1)$.

(*a*) *Let* $w \leq \sqrt{2}-1$. *Then,*

$$\begin{aligned}\|f\| = \max\Big\{&|\alpha \pm w^2\beta| + w|\gamma \pm \delta|, |w^2\alpha \pm \beta| + w|\gamma \pm \delta|, \\ &w|\alpha \pm \beta| + |\gamma \pm w^2\delta|, w|\alpha \pm \beta| + |w^2\gamma \pm \delta|, \\ &|\alpha \pm (w^2+2w)\beta| + w|\gamma \mp \delta|, |\beta \pm (w^2+2w)\alpha| + w|\gamma \mp \delta|, \\ &w|\alpha \pm \beta| + |\gamma \mp (w^2+2w)\delta|, w|\alpha \pm \beta| + |\delta \mp (w^2+2w)\gamma|, \\ &\frac{(1+w)^2}{2}|\alpha \pm \beta| + \frac{1-w^2}{2}|\gamma \mp \delta|, \frac{1-w^2}{2}|\alpha \pm \beta| + \frac{(1+w)^2}{2}|\gamma \mp \delta|\Big\}.\end{aligned}$$

(*b*) *Let* $\sqrt{2}-1 < w < 1$. *Then,*

$$\begin{aligned}\|f\| = \max\Big\{&|\alpha \pm w^2\beta| + w|\gamma \pm \delta|, |w^2\alpha \pm \beta| + w|\gamma \pm \delta|, \\ &w|\alpha \pm \beta| + |\gamma \pm w^2\delta|, w|\alpha \pm \beta| + |w^2\gamma \pm \delta|, \\ &|\alpha \pm \beta| + |w\gamma \pm (w^2+w-1)\delta|, |\beta \pm \alpha| + |w\delta \pm (w^2+w-1)\gamma|, \\ &|w\alpha \pm (w^2+w-1)\beta| + |\gamma \pm \delta|, |w\beta \pm (w^2+w-1)\alpha| + |\gamma \pm \delta|, \\ &|\alpha \pm \beta| + w|\gamma \mp \delta|, w|\alpha \pm \beta| + |\gamma \mp \delta|\Big\}.\end{aligned}$$

(*c*) *Let* $1 < w < \sqrt{2}+1$. *Then,*

$$\begin{aligned}\|f\| &= \max\Big\{|\alpha \pm w^2\beta| + w|\gamma \pm \delta|, |w^2\alpha \pm \beta| + w|\gamma \pm \delta|,\\ &\quad w|\alpha \pm \beta| + |\gamma \pm w^2\delta|, w|\alpha \pm \beta| + |w^2\gamma \pm \delta|,\\ &\quad |w\alpha \pm (-w^2+w+1)\beta| + w^2|\gamma \pm \delta|, |w\beta \pm (-w^2+w+1)\alpha| + w^2|\gamma \pm \delta|,\\ &\quad w^2|\alpha \pm \beta| + |w\gamma \pm (-w^2+w+1)\delta|, w^2|\alpha \pm \beta| + |w\delta \pm (-w^2+w+1)\gamma|,\\ &\quad w^2|\alpha \pm \beta| + w|\gamma \mp \delta|, w|\alpha \pm \beta| + w^2|\gamma \mp \delta|\Big\}.\end{aligned}$$

(*d*) *Let* $\sqrt{2}+1 \le w$. *Then,*

$$\begin{aligned}\|f\| &= \max\Big\{|\alpha \pm w^2\beta| + w|\gamma \pm \delta|, |w^2\alpha \pm \beta| + w|\gamma \pm \delta|,\\ &\quad w|\alpha \pm \beta| + |\gamma \pm w^2\delta|, w|\alpha \pm \beta| + |w^2\gamma \pm \delta|,\\ &\quad |w^2\alpha \pm (1+2w)\beta| + w|\gamma \mp \delta|, |w^2\beta \pm (1+2w)\alpha| + w|\gamma \mp \delta|,\\ &\quad w|\alpha \pm \beta| + |w^2\gamma \mp (1+2w)\delta|, w|\alpha \pm \beta| + |w^2\delta \mp (1+2w)\gamma|,\\ &\quad \frac{(1+w)^2}{2}|\alpha \pm \beta| + \frac{w^2-1}{2}|\gamma \mp \delta|, \frac{w^2-1}{2}|\alpha \pm \beta| + \frac{(1+w)^2}{2}|\gamma \mp \delta|\Big\}.\end{aligned}$$

Proof. (a) and (b). It follows from Theorems 4, 5 and the fact that

$$\|f\| = \sup_{T \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}} |f(T)|.$$

(c) and (d). It follows from Lemma 1, (a) and (b). □

Theorem 8. *Let* $T = (a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$. *Then the following are equivalent:*

(*a*) $T \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$;
(*b*) $(-a, -b, -c, -d) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$;
(*c*) $(a, b, -c, -d) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$;
(*d*) $(a, -b, c, -d) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$;
(*e*) $(a, -b, -c, d) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$;
(*f*) $(b, a, c, d) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$;
(*g*) $(d, c, a, b) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$.

Proof. It follows from the arguments in the proof of Theorem 4. □

Theorem 9. (*[22]*) *Let* E *be a real Banach space such that* $extB_E$ *is finite. Suppose that* $x \in \operatorname{ext} B_E$ *satisfies that there exists an* $f \in E^*$ *with* $f(x) = 1 = \|f\|$ *and* $|f(y)| < 1$ *for every* $y \in \operatorname{ext} B_E \setminus \{\pm x\}$. *Then* $x \in \exp B_E$.

Theorem 10. *The equality* $\exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} = \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ *holds.*

Proof. First, suppose that $0 < w < 1$.

Claim 1. $T = (1, w^2, w, w) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w < 1$.

Let $f \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})^*$ be such that $\alpha = 1 - \frac{w^2+4w}{3n}, \beta = \frac{1}{3n}, \gamma = \delta = \frac{2}{3n}$, where $n \in \mathbb{N}$ is big such that $\|f\| = 1$. By Theorem 7, $1 = \|f\| = f(T)$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \backslash \{\pm T\}$. By Theorem 9, $T \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$. Hence, by Theorem 8, $(w, w, 1, w^2) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w < 1$.

Claim 2. $T = (1, w^2 + 2w, w, -w) \in \exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ for $w \leq \sqrt{2} - 1$.

Let $f \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})^*$ be such that $\alpha = 1 - \frac{w^2+4w}{2n}, \beta = \frac{1}{2n}, \gamma = -\delta = \frac{1}{n}$, where $n \in \mathbb{N}$ is big such that $\|f\| = 1$. By Theorem 7, $1 = \|f\| = f(T)$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \backslash \{\pm T\}$. By Theorem 9, $T \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$. Hence, by Theorem 8, $(w, -w, 1, w^2 + 2w) \in \exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ for $w \leq \sqrt{2} - 1$

Claim 3. $T = (1, 1, w, w^2 + w - 1) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2} - 1 < w < 1$.

Let $f \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})^*$ be such that $\alpha = \beta = \frac{1}{2}(1 - \frac{1}{n}), \gamma = \frac{2}{n(w^2+3w-1)}, \delta = \frac{1}{n(w^2+3w-1)}$, where $n \in \mathbb{N}$ is big such that $\|f\| = 1$. By Theorem 7, $1 = \|f\| = f(T)$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \backslash \{\pm T\}$. By Theorem 9, $T \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$. Hence, by Theorem 8, $T = (w, w^2 + w - 1, 1, 1) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2} - 1 < w < 1$.

Claim 4. $T = \left(\frac{(1+w)^2}{2}, \frac{(1+w)^2}{2}, \frac{1-w^2}{2}, -\frac{(1-w^2)}{2}\right) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w \leq \sqrt{2} - 1$.

Let $f \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})^*$ be such that $\alpha = \beta = \frac{1}{(1+w)^2}(1 - \frac{1}{n}), \gamma = \frac{1}{n(1-w^2)} = -\delta$, where $n \in \mathbb{N}$ is big such that $\|f\| = 1$. By Theorem 7, $1 = \|f\| = f(T)$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \backslash \{\pm T\}$. By Theorem 9, $T \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$. Hence, by Theorem 8, $(\frac{1-w^2}{2}, \frac{1-w^2}{2}, \frac{(1+w)^2}{2}, -\frac{(1+w)^2}{2}) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $0 < w \leq \sqrt{2} - 1$.

Claim 5. $T = (1, 1, w, w) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2} - 1 \leq w < 1$.

Let $f \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})^*$ be such that $\alpha = 1 - \frac{2}{n}, \beta = \frac{1}{n}, \gamma = \frac{1}{2nw} = -\delta$, where $n \in \mathbb{N}$ is big such that $\|f\| = 1$. By Theorem 7, $1 = \|f\| = f(T)$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \backslash \{\pm T\}$. By Theorem 9, $T \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$. Hence, by Theorem 8, $T = (w, w, 1, -1) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ for $\sqrt{2} - 1 \leq w < 1$. We have shown that if $0 < w < 1$, then $\exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} = \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$.

Suppose that $1 < w$.

It follows that

$$\begin{aligned}\exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} &= \left\{w^2 T : T \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(1/w)})}\right\} = \left\{w^2 T : T \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(1/w)})}\right\} \\ &= \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}.\end{aligned}$$

Therefore, we complete the proof. □

Theorem 11. *The following equalities hold:*
(*a*) $\exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})} = \mathrm{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$;
(*b*) $\exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})} \backslash (\exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)}))$
$= \mathrm{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})} \backslash (\mathrm{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)}))$.

Proof. (a). Notice that $(\exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \cap \mathcal{L}_s(^2\mathbb{R}^2_{o(w)})) \subseteq \exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ for all $0 < w, w \neq 1$. By Theorems 4 and 10, it suffices to show that

$$T = \Big(1, 1, \frac{w^2 + 2w - 1}{2}, \frac{w^2 + 2w - 1}{2}\Big) \in \exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$$

for $\sqrt{2} - 1 < w < 1$.

Let $f \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})^*$ be such that $\alpha = 1 - \frac{(1+w)^2}{2n}, \beta = \frac{1}{n}, \gamma = \delta = \frac{1}{2n}$, where $n \in \mathbb{N}$ is big such that $\|f\| = 1$. By Theorem 7, $1 = \|f\| = f(T)$ and $|f(S)| < 1$ for every $S \in \mathrm{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} \backslash \{\pm T\}$. By Theorem 9, $T \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$. Hence, $T \in \exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$.

(b) follows from (a) and Theorem 10. □

6 The smooth points of the unit balls of $\mathcal{L}(^2\mathbb{R}^2_{o(w)})$ and $\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$

Theorem 12. *Let* $0 < w < 1$ *and* $T = (a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ *be such that* $\|T\| = 1$. *Then,* $T \in \mathrm{sm}\, B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ *if and only if there are* $i_1, j_1 \in \{1, 2, 3, 4\}$ *such that*

$$|T(X_{i_1}, X_{j_1})| = 1 \text{ and } |T(X_i, X_j)| < 1$$

for every $i, j \in \{1, 2, 3, 4\}$ *with* $(i, j) \neq (i_1, j_1)$, *where* $X_1 = (1, 0), X_2 = (0, 1), X_3 = ((1 + w)^{-1}, (1 + w)^{-1}), X_4 = ((1 + w)^{-1}, -(1 + w)^{-1})$.

Proof. $(\Rightarrow)$. Assume the assertion is not true.

Suppose that $|T(X_1, X_2)| = 1$, $|T(X_3, x_4)| = 1$. Let $f_1 = \mathrm{sign}(T(X_1, X_2))\delta_{X_1,X_2}$ and $f_2 = \mathrm{sign}(T(X_3, X_4))\delta_{X_3,X_4}$ be elements of $\mathcal{L}(^2\mathbb{R}^2_{o(w)})^*$, where $\delta_{X_1,X_2}(S) = S(X_1, X_2)$ for $S \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$. Notice that

$$f_1 \neq f_2, \ \|f_j\| = 1 = f_j(T) \text{ for } j = 1, 2.$$

Hence, T is not a smooth point. This is a contradiction. Similarly, we conclude that the other cases reach a contradiction. Therefore, the assertion is true.

$(\Leftarrow)$. Let $f \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})^*$ be such that $f(T) = 1 = \|f\|$. Let $\alpha = f(x_1y_1), \beta = f(x_2y_2), \gamma = f(x_1y_2), \rho = f(x_2y_1)$.

Case 1. $|T(X_1, X_1)| = 1$ and $|T(X_i, X_j)| < 1$ for every $i, j \in \{1, 2, 3, 4\}$ with $(i, j) \neq (1, 1)$.

Without loss of generality, we may assume that $a = T(X_1, X_1) = 1$. By Theorem 2, there is $N \in \mathbb{N}$ such that

$$1 = \Big\|T \pm \Big(0, \frac{1}{N}, 0, 0\Big)\Big\| = \Big\|T \pm \Big(0, 0, \frac{1}{N}, 0\Big)\Big\| = \Big\|T \pm \Big(0, 0, 0, \frac{1}{N}\Big)\Big\|.$$

We claim that $\alpha = 1, \beta = \gamma = \rho = 0$. It follows that

$$\begin{aligned} 1 \ &\geq \ \max\Big\{\Big|f\Big(T \pm \Big(0, \frac{1}{N}, 0, 0\Big)\Big)\Big|, \Big|f\Big(T \pm \Big(0, 0, \frac{1}{N}, 0\Big)\Big)\Big|, \Big|f\Big(T \pm \Big(0, 0, 0, \frac{1}{N}\Big)\Big)\Big|\Big\} \\ &= \ \max\Big\{1 + \Big|f\Big(\Big(0, \frac{1}{N}, 0, 0\Big)\Big)\Big|, 1 + \Big|f\Big(\Big(0, 0, \frac{1}{N}, 0\Big)\Big)\Big|, 1 + \Big|f\Big(\Big(0, 0, 0, \frac{1}{N}\Big)\Big)\Big|\Big\}, \end{aligned}$$

which shows that

$$0 = f\Big(\Big(0, \frac{1}{N}, 0, 0\Big)\Big) = f\Big(\Big(0, 0, \frac{1}{N}, 0\Big)\Big) = f\Big(\Big(0, 0, 0, \frac{1}{N}\Big)\Big).$$

Hence, $\beta = \gamma = \rho = 0$. Since

$$a = 1 = f(T) = a\alpha + b\beta + c\gamma + d\rho = a\alpha,$$

$\alpha = 1$. Hence, f is unique. Hence, $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$.

Case 2. $|T(X_1, X_3)| = 1$ and $|T(X_i, X_j)| < 1$ for every $i, j \in \{1, 2, 3, 4\}$ with $(i, j) \neq (1, 3)$.

Without loss of generality, we may assume that $\frac{1}{1+w}(a + c) = T(X_1, X_3) = 1$. By Theorem 2, there is $N \in \mathbb{N}$ such that

$$1 = \Big\|T \pm \Big(\frac{1}{N}, 0, -\frac{1}{N}, 0\Big)\Big\| = \Big\|T \pm \Big(0, \frac{1}{N}, 0, \frac{1}{N}\Big)\Big\| = \Big\|T \pm \Big(0, \frac{1}{N}, 0, -\frac{1}{N}\Big)\Big\|.$$

We claim that $\alpha = \gamma = \frac{1}{1+w}, \beta = \rho = 0$. It follows that

$$\begin{aligned} 1 \ &\geq \ \max\Big\{\Big|f\Big(T \pm \Big(\frac{1}{N}, 0, -\frac{1}{N}, 0\Big)\Big)\Big|, \Big|f\Big(T \pm \Big(0, \frac{1}{N}, 0, \frac{1}{N}\Big)\Big)\Big|, \\ &\quad \Big|f\Big(T \pm \Big(0, \frac{1}{N}, 0, -\frac{1}{N}\Big)\Big)\Big|\Big\} \\ &= \ \max\Big\{1 + \Big|f\Big(\Big(\frac{1}{N}, 0, -\frac{1}{N}, 0\Big)\Big)\Big|, 1 + \Big|f\Big(\Big(0, \frac{1}{N}, 0, \frac{1}{N}\Big)\Big)\Big|, \\ &\quad 1 + \Big|f\Big(\Big(0, \frac{1}{N}, 0, -\frac{1}{N}\Big)\Big)\Big|\Big\}, \end{aligned}$$

which shows that

$$0 = f\Big(\Big(\frac{1}{N}, 0, -\frac{1}{N}, 0\Big)\Big) = f\Big(\Big(0, \frac{1}{N}, 0, \frac{1}{N}\Big)\Big) = f\Big(\Big(0, \frac{1}{N}, 0, -\frac{1}{N}\Big)\Big).$$

Hence, $\alpha = \gamma, \beta = \rho = 0$. Since

$$\frac{1}{1+w}(a + c) = 1 = f(T) = \alpha(a + c),$$

$\alpha = \gamma = \frac{1}{1+w}$. Hence, f is unique. Hence, $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$.

Case 3. $|T(X_3, X_3)| = 1$ and $|T(X_i, X_j)| < 1$ for every $i, j \in \{1, 2, 3, 4\}$ with $(i, j) \neq (3, 3)$.

Without loss of generality, we may assume that $\frac{1}{(1+w)^2}(a + b + c + d) = T(X_3, X_3) = 1$. By Theorem 2, there is $N \in \mathbb{N}$ such that

$$1=\Big\|T\pm\Big(\frac{1}{N},-\frac{1}{N},\frac{1}{N},-\frac{1}{N}\Big)\Big\|=\Big\|T\pm\Big(\frac{1}{N},\frac{1}{N},-\frac{1}{N},-\frac{1}{N}\Big)\Big\|=\Big\|T\pm\Big(\frac{1}{N},0,-\frac{1}{N},0\Big)\Big\|.$$

We claim that $\alpha = \gamma = \beta = \rho = \frac{1}{(1+w)^2}$. It follows that

$$\begin{aligned} 1 &\geq \max\Big\{\Big|f\Big(T \pm \Big(\frac{1}{N}, -\frac{1}{N}, \frac{1}{N}, -\frac{1}{N}\Big)\Big)\Big|, \Big|f\Big(T \pm \Big(\frac{1}{N}, \frac{1}{N}, -\frac{1}{N}, -\frac{1}{N}\Big)\Big)\Big|, \\ &\qquad \Big|f\Big(T \pm \Big(\frac{1}{N}, 0, -\frac{1}{N}, 0\Big)\Big)\Big|\Big\} \\ &= \max\Big\{1 + \Big|f\Big(\Big(\frac{1}{N}, -\frac{1}{N}, \frac{1}{N}, -\frac{1}{N}\Big)\Big)\Big|, 1 + \Big|f\Big(\Big(\frac{1}{N}, \frac{1}{N}, -\frac{1}{N}, -\frac{1}{N}\Big)\Big)\Big|, \\ &\qquad 1 + \Big|f\Big(\Big(\frac{1}{N}, 0, -\frac{1}{N}, 0\Big)\Big)\Big|\Big\}, \end{aligned}$$

which shows that

$$0 = f\Big(\Big(\frac{1}{N}, -\frac{1}{N}, \frac{1}{N}, -\frac{1}{N}\Big)\Big) = f\Big(\Big(\frac{1}{N}, \frac{1}{N}, -\frac{1}{N}, -\frac{1}{N}\Big)\Big) = f\Big(\Big(\frac{1}{N}, 0, -\frac{1}{N}, 0\Big)\Big).$$

Hence, $\alpha = \gamma = \beta = \rho$. Since

$$\frac{1}{(1+w)^2}(a+b+c+d) = 1 = f(T) = \alpha(a+b+c+d),$$

$\alpha = \gamma = \beta = \rho = \frac{1}{(1+w)^2}$. Hence, f is unique. Hence, $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$.

By analogous arguments in cases 1-3, in the other cases we may conclude that $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$. We omit the proofs. Therefore, we complete the proof. □

Theorem 13. *Let $w > 1$ and $T = (a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ be such that $\|T\| = 1$. Then, $T \in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$ if and only if there are $i_1, j_1 \in \{1, 2, 3, 4\}$ such that*

$$|T(Y_{i_1}, Y_{j_1})| = 1 \text{ and } |T(Y_i, Y_j)| < 1$$

for every $i, j \in \{1, 2, 3, 4\}$ with $(i, j) \neq (i_1, j_1)$, where $Y_1 = (w^{-1}, 0), Y_2 = (0, w^{-1})$, $Y_3 = ((1 + w)^{-1}, (1 + w)^{-1}), Y_4 = ((1 + w)^{-1}, -(1 + w)^{-1})$.

Proof. It follows from analogous arguments in the proof of Theorem 12. □

Theorem 14. *Let $0 < w < 1$ and $T = (a, b, c, c) \in \mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$ be such that $\|T\| = 1$. Then, $T \in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ if and only if there are $i_1, j_1 \in \{1, 2, 3, 4\}$ such that*

$$|T(X_{i_1}, X_{j_1})| = |T(X_{j_1}, X_{i_1})| = 1 \text{ and } |T(X_i, X_j)| < 1$$

for every $i, j \in \{1, 2, 3, 4\}$ with $(i, j) \neq (i_1, j_1), (j_1, i_1)$.

Proof. We follow analogous arguments in the proof of Theorem 12.

($\Rightarrow$) follows by the same argument in the proof ($\Rightarrow$) of Theorem 12.

($\Leftarrow$). Let $g \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})^*$ be such that $g(T) = 1 = \|g\|$ and $\alpha = g(x_1x_2),\ \beta = g(y_1y_2),\ \gamma = g(x_1y_2 + x_2y_1)$.

Case 1. $|T(X_1, X_1)| = 1$ and $|T(X_i, X_j)| < 1$ for every $i, j \in \{1, 2, 3, 4\}$ with $(i, j) \neq (1, 1)$.

Without loss of generality, we may assume that $a = T(X_1, X_1) = 1$. By Theorem 2, there is $N \in \mathbb{N}$ such that

$$1 = \Big\|T \pm \Big(0, \frac{1}{N}, 0, 0\Big)\Big\| = \Big\|T \pm \Big(0, 0, \frac{1}{N}, \frac{1}{N}\Big)\Big\|.$$

We claim that $\alpha = 1, \beta = \gamma = 0$. It follows that

$$\begin{aligned} 1 &\geq \max\Big\{\Big|g\Big(T \pm \Big(0, \frac{1}{N}, 0, 0\Big)\Big)\Big|, \Big|g\Big(T \pm \Big(0, 0, \frac{1}{N}, \frac{1}{N}\Big)\Big)\Big|\Big\} \\ &= \max\Big\{1 + \Big|g\Big(\Big(0, \frac{1}{N}, 0, 0\Big)\Big)\Big|, 1 + \Big|g\Big(\Big(0, 0, \frac{1}{N}, \frac{1}{N}\Big)\Big)\Big|\Big\}, \end{aligned}$$

which shows that

$$0 = g\Big(\Big(0, \frac{1}{N}, 0, 0\Big)\Big) = g\Big(\Big(0, 0, \frac{1}{N}, \frac{1}{N}\Big)\Big).$$

Hence, $\beta = \gamma = 0$. Since

$$a = 1 = g(T) = a\alpha + b\beta + c\gamma = a\alpha,$$

$\alpha = 1$. Hence, g is unique. Hence, $T \in \operatorname{sm} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})}$.

Case 2. $|T(X_1, X_3)| = 1$ and $|T(X_i, X_j)| < 1$ for every $i, j \in \{1, 2, 3, 4\}$ with $(i, j) \neq (1, 3)$.

Without loss of generality, we may assume that $\frac{1}{1+w}(a + c) = T(X_1, X_3) = 1$. By Theorem 2, there is $N \in \mathbb{N}$ such that

$$1 = \Big\|T \pm \Big(-\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}\Big)\Big\| = \Big\|T \pm \Big(0, \frac{1}{N}, 0, 0\Big)\Big\|.$$

We claim that $\alpha = \gamma = \frac{1}{1+w}, \beta = 0$. It follows that

$$\begin{aligned} 1 &\geq \max\Big\{\Big|g\Big(T \pm \Big(-\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}\Big)\Big)\Big|, \Big|g\Big(T \pm \Big(0, \frac{1}{N}, 0, 0\Big)\Big)\Big|\Big\} \\ &= \max\Big\{1 + \Big|g\Big(\Big(-\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}\Big)\Big)\Big|, 1 + \Big|g\Big(\Big(0, \frac{1}{N}, 0, 0\Big)\Big)\Big|\Big\}, \end{aligned}$$

which shows that

$$0 = g\Big(\Big(-\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}\Big)\Big) = g\Big(\Big(0, \frac{1}{N}, 0, 0\Big)\Big).$$

Hence, $\alpha=\gamma, \beta=0$. Since

$$\frac{1}{1+w}(a+c)=1=g(T)=\alpha(a+c),$$

$\alpha=\gamma=\frac{1}{1+w}$. Hence, g is unique. Hence, $T\in \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}$.

Case 3. $|T(X_3,X_3)|=1$ and $|T(X_i,X_j)|<1$ for every $i,j\in\{1,2,3,4\}$ with $(i,j)\neq(3,3)$.

Without loss of generality, we may assume that $\frac{1}{(1+w)^2}(a+b+2c)=T(X_3,X_3)=1$. By Theorem 2, there is $N\in\mathbb{N}$ such that

$$1=\Big\|T\pm\Big(\frac{2}{N},0,-\frac{1}{N},-\frac{1}{N}\Big)\Big\|=\Big\|T\pm\Big(-\frac{1}{N},\frac{1}{N},0,0\Big)\Big\|.$$

We claim that $\alpha=\beta=\frac{\gamma}{2}=\frac{1}{(1+w)^2}$. It follows that

$$\begin{aligned} 1 &\geq \max\Big\{\Big|g\Big(T\pm\Big(\frac{2}{N},0,-\frac{1}{N},-\frac{1}{N}\Big)\Big)\Big|,\Big|g\Big(T\pm\Big(-\frac{1}{N},\frac{1}{N},0,0\Big)\Big)\Big|\Big\}\\ &= \max\Big\{1+\Big|g\Big(\Big(\frac{2}{N},0,-\frac{1}{N},-\frac{1}{N}\Big)\Big)\Big|,1+\Big|g\Big(\Big(-\frac{1}{N},\frac{1}{N},0,0\Big)\Big)\Big|\Big\},\end{aligned}$$

which shows that

$$0=g\Big(\Big(\frac{2}{N},0,-\frac{1}{N},-\frac{1}{N}\Big)\Big)=g\Big(\Big(-\frac{1}{N},\frac{1}{N},0,0\Big)\Big).$$

Hence, $\alpha=\beta=\frac{\gamma}{2}$. Since

$$\frac{1}{(1+w)^2}(a+b+2c)=1=g(T)=\alpha(a+b+2c),$$

$\alpha=\beta=\frac{\gamma}{2}=\frac{1}{(1+w)^2}$. Hence, g is unique. Hence, $T\in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$.

By analogous arguments in cases 1-3, in the other cases we may conclude that $T\in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$. We omit the proofs. Therefore, we complete the proof. □

Theorem 15. *Let $w>1$ and $T=(a,b,c,c)\in\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$ be such that $\|T\|=1$. Then, $T\in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ if and only if there are $i_1,j_1\in\{1,2,3,4\}$ such that*

$$|T(Y_{i_1},Y_{j_1})|=|T(Y_{j_1},Y_{i_1})|=1 \text{ and } |T(Y_i,Y_j)|<1$$

for every $i,j\in\{1,2,3,4\}$ with $(i,j)\neq(i_1,j_1),(j_1,i_1)$.

Proof. It follows from analogous arguments in the proof of Theorem 14. □

Theorem 16. *Let $0<w, w\neq 1$. Then, $\operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}\bigcap\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})\subsetneq \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$.*

Proof. By Theorems 12–14, $\operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}\bigcap\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$ is a subset of $\operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$. Let $0<w<1$. Let $T_0\in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ be such that

$$|T_0(X_1,X_2)|=1 \text{ and } |T_0(X_i,X_j)|<1$$

for every $i,j\in\{1,2,3,4\}$ with $(i,j)\neq(1,2)$. Since $|T_0(X_2,X_1)|=1$, by Theorem 4.1, $T_0\notin \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}\bigcap\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$. If $w>1$, we may choose $T_1\in \operatorname{sm} B_{\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})}$ such that $T_1\notin \operatorname{sm} B_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}\bigcap\mathcal{L}_s(^2\mathbb{R}^2_{o(w)})$. We complete the proof. □

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