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ON GENERALIZED PSEUDO-PROJECTIVE CURVATURE TENSOR OF PARA-KENMOTSU MANIFOLDS

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Abstract

The object of the present paper is to generalize pseudo-projective curvature tensor of para-Kenmotsu manifold with the help of a new generalized (0,2) symmetric tensor $\mathcal{Z}$ introduced by Mantica and Suh. Various geometric properties of generalized pseudo-projective curvature tensor of para-Kenmotsu manifold have been studied. It is shown that a generalized pseudo-projectively ϕ-symmetric para-Kenmotsu manifold is an Einstein manifold.

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1 Introduction

The projective tensor is one of the major curvature tensors. The study of pseudo-projective curvature tensor has been a very attractive field for investigations in the past decades. A tensor field $\overline{P}$ was defined and studied in 2002 by Bhagwat Prasad [18] on a Riemannian manifold of dimension n, which includes projective curvature tensor P. This tensor field $\overline{P}$ referred to as pseudo-projective curvature tensor. In 2011, H.G. Nagaraja and G. Somashekhara [14] extended pseudo-projective curvature tensor in Sasakian manifolds. After 2012, the pseudo-projective curvature tensor analysis in LP-Sasakian manifolds was resumed by

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Y.B. Maralabhavi and G.S. Shivaprasanna [12]. In 2016, S. Mallick, Y.J. Suh and U.C. De [11] defined and studied a space time with pseudo-projective curvature tensor. Subsequently, several researchers performed a study of pseudo-projective curvature tensor in a number of directions, such as [4, 5, 13, 15, 17, 21, 22]. The pseudo-projective curvature tensor is defined by [18]

$$\begin{aligned} \overline{P}(X,Y,U) =& aR(X,Y,U) + b[S(Y,U)X - S(X,U)Y] \\ & - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,U)X - g(X,U)Y], \end{aligned} \tag{1}$$

where a and b are constants such that a, b $\neq 0$ and R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature tensor.

The notion of an almost para-contact manifold was introduced by I. Sato [19]. Since the publication of [26], paracontact metric manifolds have been studied by many authors in recent years. The importance of para-Kenmotsu geometry, have been pointed out especially in the last years by several papers highlighting the exchanges with the theory of para-Kähler manifolds and its role in semi-Riemannian geometry and mathematical physics [3, 7, 8, 20].

In this paper, we consider the generalized pseudo-projective curvature tensor of para-Kenmotsu manifolds and study some properties of generalized pseudo-projective curvature tensor. The organisation of the paper is as follows: After preliminaries on para-Kenmotsu manifold in Section 2, we describe briefly the generalized pseudo-projective curvature tensor on para-Kenmotsu manifold in Section 3 and also we study some properties of generalized pseudo-projective curvature tensor in para-Kenmotsu manifold. In Section 4, we study a generalized pseudo-projectively semi-symmetric para-Kenmotsu manifold is an η Einstein manifold. Further in the Section 5, we show that a generalized pseudo-projectively Ricci semi-symmetric para-Kenmotsu manifold is either Einstein manifold or $\psi = \frac{an(n-1)+ra+br(n-1)}{bn(n-1)}$ on it. In the last section we show that the generalized pseudo-projectively ϕ-symmetric para-Kenmotsu manifold is an Einstein manifold.

2 Preliminaries

An n-dimensional differentiable manifold M^n is said to have almost para-contact structure (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field known as characteristic vector field and η is a 1-form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi, \tag{2}$$

$$\eta(\phi X) = 0, \tag{3}$$

$$\phi(\xi) = 0, \tag{4}$$

and

$$\eta(\xi) = 1. \tag{5}$$

A differentiable manifold with almost para-contact structure (ϕ, ξ, η) is called an almost para-contact manifold. Further, if the manifold M^n has a semi-Riemannian metric g satisfying

$$\eta(X) = g(X, \xi) \tag{6}$$

and

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{7}$$

Then the structure (ϕ, ξ, η, g) satisfying conditions (2) to (7) is called an almost para-contact Riemannian structure and the manifold M^n with such a structure is called an almost para-contact Riemannian manifold [1, 19].

Now we briefly present an account of an analogue of the Kenmotsu manifold in paracontact geometry which will be called para-Kenmotsu.

Definition 1. *The almost paracontact metric structure (ϕ, ξ, η, g) is para-Kenmotsu should this relation hold[2, 16], if the Levi-Civita connection ∇ of g satisfies $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any $X, Y \in \mathfrak{X}(M)$.*

On a para-Kenmotsu manifold [2, 20], the following relations hold:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{8}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{9}$$

$$\eta(R(X, Y, Z)) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{10}$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X, \tag{11}$$

$$R(X, \xi, Y) = -R(\xi, X, Y) = g(X, Y)\xi - \eta(Y)X, \tag{12}$$

$$S(\phi X, \phi Y) = -(n-1)g(\phi X, \phi Y), \tag{13}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{14}$$

$$Q\xi = -(n-1)\xi, \tag{15}$$

$$r = -n(n-1), \tag{16}$$

for any vector fields X, Y, Z , where Q is the Ricci operator that is $g(QX, Y) = S(X, Y)$, S is the Ricci tensor and r is the scalar curvature.

In A. M. Blaga [2], gave an example on para-Kenmotsu manifold:

Example 1. *We consider the three dimensional manifold $M^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard co-ordinates in $\mathbb{R}^3$. The vector fields*

$$e_1 := \frac{\partial}{\partial x}, e_2 := \frac{\partial}{\partial y}, e_3 := -\frac{\partial}{\partial z}$$

are linearly independent at each point of the manifold.
Define

$$\phi := \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \xi := -\frac{\partial}{\partial z}, \eta := -dz,$$

$$g := dx \otimes dx - dy \otimes dy + dz \otimes dz.$$

Then it follows that

$$\phi e_1 = e_2, \phi e_2 = e_1, \phi e_3 = 0,$$
$$\eta(e_1) = 0, \eta(e_2) = 0, \eta(e_3) = 1.$$

Let ∇ be the Levi-Civita connetion with respect to metric g. Then, we have

$$[e_1, e_2] = 0, [e_2, e_3] = 0, [e_3, e_1] = 0$$

The Riemannian connection ∇ of the metric g is deduced from Koszul's formula

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).$$

Then Koszul's formula yields

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e_1,$$
$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = e_3, \nabla_{e_2} e_3 = e_2,$$
$$\nabla_{e_3} e_1 = e_1, \nabla_{e_3} e_2 = e_2, \nabla_{e_3} e_3 = 0.$$

These results shows that the manifold satisfies

$$\nabla_X \xi = X - \eta(X)\xi,$$

for $\xi = e_3$. Hence the manifold under consideration is para-Kenmotsu manifold of dimension three.

A para-Kenmotsu manifold is said to be an η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{17}$$

for the vector fields X, Y, where a and b are functions on M^n.

3 Generalized pseudo-projective curvature tensor of para-Kenmotsu manifold

In this section, we give a brief account of generalized pseudo-projective curvature tensor of para-Kenmotsu manifold and study various geometric properties of it.

The pseudo-projective curvature tensor of para-Kenmotsu manifold M^n is given by the following relation:

$$\overline{P}(X,Y,U) = aR(X,Y,U) + b[S(Y,U)X - S(X,U)Y] \\ - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,U)X - g(X,U)Y], \tag{18}$$

Also, the type $(0,4)$ tensor field $'\overline{P}$ is given by

$$\begin{aligned} {}'\overline{P}(X,Y,U,V) &= a'R(X,Y,U,V) + b[S(Y,U)g(X,V) - S(X,U) \\ g(Y,V)] &- \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,U)g(X,V) - g(X,U)g(Y,V)], \end{aligned} \tag{19}$$

where

$$'\overline{P}(X,Y,U,V) = g(\overline{P}(X,Y,U),V)$$

and

$$'R(X,Y,U,V) = g(R(X,Y,U),V)$$

for the arbitrary vector fields X, Y, U, V.

Differentiating covariantly with respect to W in equation (18), we get

$$\begin{aligned} (\nabla_W\overline{P})(X,Y)U) &= a(\nabla_W R)(X,Y)U) + b[(\nabla_W S)(Y,U)X \\ -(\nabla_W S)(X,U)Y] &- \frac{dr(W)}{n}\left(\frac{a}{n-1} + b\right)[g(Y,U)X - g(X,U)Y]. \end{aligned} \tag{20}$$

Divergence of pseudo-projective curvature tensor in equation (18) is given by

$$\begin{aligned} (div\overline{P})(X,Y)U) &= a(divR)(X,Y)U) + b[(\nabla_X S)(Y,U) \\ -(\nabla_Y S)(X,U)] &- (divr)\left[\frac{a+b(n-1)}{n(n-1)}\right][g(Y,U)div(X) \\ &\qquad -g(X,U)div(Y)]. \end{aligned} \tag{21}$$

But

$$(divR)(X,Y)U) = (\nabla_X S)(Y,U) - (\nabla_Y S)(X,U). \tag{22}$$

From equations (21) and (22), we have

$$\begin{aligned} (div\overline{P})(X,Y)U) &= (a+b)[(\nabla_X S)(Y,U) - (\nabla_Y S)(X,U)] - (divr) \\ &\left[\frac{a+b(n-1)}{n(n-1)}\right][g(Y,U)div(X) - g(X,U)div(Y)]. \end{aligned} \tag{23}$$

Definition 2. *An almost paracontact structure (ϕ, ξ, η, g) is said to be locally pseudo-projectively symmetric if*

$$(\nabla_W\overline{P})(X,Y,U) = 0, \tag{24}$$

for all vector fields $X, Y, U, W \in T_pM^n$.

Definition 3. *An almost paracontact structure (ϕ, ξ, η, g) is said to be locally pseudo-projectively ϕ-symmetric if*

$$\phi^2((\nabla_W\overline{P})(X,Y,U)) = 0, \tag{25}$$

for all vector fields X, Y, U, W orthogonal to ξ.

Definition 4. *An almost paracontact structure (ϕ, ξ, η, g) is said to be pseudo-projectively ϕ-recurrent if*

$$\phi^2((\nabla_W \overline{P})(X, Y, U)) = A(W)\overline{P}(X, Y, U), \tag{26}$$

for arbitrary vector fields X, Y, U, W.

If the 1-form A vanishes, then the manifold reduces to a locally pseudo-projectively ϕ-symmetric.

A new generalized $(0, 2)$ symmetric tensor $\mathcal{Z}$, defined by Mantica and Suh [9], is given by the following relation

$$\mathcal{Z}(X, Y) = S(X, Y) + \psi g(X, Y), \tag{27}$$

where ψ is an arbitrary scalar function.
From equation (27), we have

$$\mathcal{Z}(\phi X, \phi Y) = S(\phi X, \phi Y) + \psi g(\phi X, \phi Y), \tag{28}$$

which, on using equations (7) and (13), gives

$$\mathcal{Z}(\phi X, \phi Y) = [\psi - (n-1)][-g(X, Y) + \eta(X)\eta(Y)]. \tag{29}$$

From equation (19), we have

$$\begin{aligned} {}'\overline{P}(X, Y, U, V) &= a'R(X, Y, U, V) + b[S(Y, U)g(X, V) - S(X, U) \\ g(Y, V)] &- \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y, U)g(X, V) - g(X, U)g(Y, V)]. \end{aligned} \tag{30}$$

From equation (27) the above equation reduces to

$$\begin{aligned} {}'\overline{P}(X, Y, U, V) &= a'R(X, Y, U, V) + b[\mathcal{Z}(Y, U)g(X, V) - \mathcal{Z}(X, U) \\ g(Y, V)] &- \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y, U)g(X, V) - g(X, U)g(Y, V)] \\ &+ b\psi[g(Y, V)g(X, U) - g(Y, U)g(X, V)], \end{aligned} \tag{31}$$

If we put

$$\begin{aligned} {}'\overline{\overline{P}}(X, Y, U, V) &= a'R(X, Y, U, V) + b[\mathcal{Z}(Y, U)g(X, V) - \mathcal{Z}(X, U) \\ g(Y, V)] &- \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y, U)g(X, V) - g(X, U)g(Y, V)]. \end{aligned} \tag{32}$$

Then equation (31) reduces to

$$\begin{aligned} {}'\overline{\overline{P}}(X, Y, U, V) =' \overline{P}(X, Y, U, V) - b\psi[g(Y, V)g(X, U) \\ - g(X, V)g(Y, U)]. \end{aligned} \tag{33}$$

We call this new tensor $'\overline{\overline{P}}$ given in equation (32) as generalized pseudo-projective curvature tensor of para-Kenmotsu manifold.

If ψ=0, then from eqauation (33), we have

$$'\overline{\overline{P}}(X,Y,U,V) = '\,\overline{P}(X,Y,U,V). \tag{34}$$

If the scalar function ψ vanishes on para-Kenmotsu manifold, then the pseudo-projective curvature tensor and generalized pseudo-projective curvature tensor are identicle.

Theorem 1. *Generalized pseudo-projective curvature tensor $'\overline{\overline{P}}$ of para-Kenmotsu manifold is*

(a) skew symmetric in first two slots.

(b) skew symmetric in last two slots.

(c) symmetric in pair of slots.

Proof. (a) From equation (33), we have

$$\begin{aligned} '\overline{\overline{P}}(Y,X,U,V) = '\,\overline{P}(Y,X,U,V) - b\psi[g(X,V)g(Y,U) \\ -g(Y,V)g(X,U)]. \end{aligned} \tag{35}$$

Now adding equations (33) and (35) and using the following

$$'\overline{P}(X,Y,U,V) +'\,\overline{P}(Y,X,U,V) = 0,$$

we get

$$'\overline{\overline{P}}(X,Y,U,V) = -'\overline{\overline{P}}(Y,X,U,V),$$

which shows that generalized pseudo-projective curvature tensor $'\overline{\overline{P}}$ is skew symmetric in first two slots.

(b) Again from equation (33), we have

$$\begin{aligned} '\overline{\overline{P}}(X,Y,V,U) = '\,\overline{P}(X,Y,V,U) - b\psi[g(X,V)g(Y,U) \\ -g(Y,V)g(X,U)]. \end{aligned} \tag{36}$$

Now, adding (33) and (36) and using the following

$$'\overline{P}(X,Y,U,V) +'\,\overline{P}(X,Y,V,U) = 0,$$

we obtain

$$'\overline{\overline{P}}(X,Y,U,V) = -'\overline{\overline{P}}(X,Y,V,U),$$

which shows that generalized pseudo-projective curvature tensor $'\overline{\overline{P}}$ is skew symmetric in last two slots.

(c) From equation (33), interchanging pair of slots X by U and Y by V, we have

$$'\overline{\overline{P}}(U,V,X,Y) = '\overline{P}(U,V,X,Y) - b\psi[g(V,Y)g(U,X) - g(U,Y)g(V,X)]. \tag{37}$$

Now, using equations (33) and (37) and using the following

$$'\overline{P}(X,Y,U,V) = '\overline{P}(U,V,X,Y),$$

we get

$$'\overline{\overline{P}}(X,Y,U,V) = '\overline{\overline{P}}(U,V,X,Y),$$

which shows that generalized pseudo-projective curvature tensor $'\overline{\overline{P}}$ is symmetric in pair of slots. □

Theorem 2. *Generalized pseudo-projective curvature tensor of para-Kenmotsu manifold satisfies Bianchi's first identity.*

Proof. From equation (33), we have

$$\overline{\overline{P}}(X,Y,U) = \overline{P}(X,Y,U) - b\psi[g(X,U)Y - g(Y,U)X]. \tag{38}$$

Writing two more equations by the cyclic permutations of X, Y and U in the above equation, we get

$$\overline{\overline{P}}(Y,U,X) = \overline{P}(Y,U,X) - b\psi[g(Y,X)U - g(U,X)Y] \tag{39}$$

and

$$\overline{\overline{P}}(U,X,Y) = \overline{P}(U,X,Y) - b\psi[g(U,Y)X - g(X,Y)U]. \tag{40}$$

Adding equations (38), (39) and (40) with the fact that

$$\overline{P}(X,Y,U) + \overline{P}(Y,U,X) + \overline{P}(U,X,Y) = 0,$$

we get

$$\overline{\overline{P}}(X,Y,U) + \overline{\overline{P}}(Y,U,X) + \overline{\overline{P}}(U,X,Y) = 0,$$

which shows that generalized pseudo-projective curvature tensor of para-Kenmotsu manifold satisfies Bianchi's first identity. □

Theorem 3. *Generalized pseudo-projective curvature tensor of para-Kenmotsu manifold satisfies the following identites:*

$$
\begin{aligned}
(a)\overline{\overline{P}}(\xi,Y,U) = -\overline{\overline{P}}(Y,\xi,U) = g(Y,U)\left[-a-\frac{r}{n}\left(\frac{a}{n-1}+b\right)+b\psi\right] \\
\xi + \left[a+b(n-1)+\frac{r}{n}\left(\frac{a}{n-1}+b\right)-b\psi\right]\eta(U)Y \\
+bS(Y,U)\xi,
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
(b)\overline{\overline{P}}(X,Y,\xi) = \left[a+b(n-1)+\frac{r}{n}\left(\frac{a}{n-1}+b\right)-b\psi\right][\eta(X)Y \\
-\eta(Y)X],
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
(c)\eta(\overline{\overline{P}}(U,V,Y)) = \left[a+\frac{r}{n}\left(\frac{a}{n-1}+b\right)-b\psi\right][g(U,Y)\eta(V) \\
-g(V,Y)\eta(U)] + b[S(V,Y)\eta(U)-S(U,Y)\eta(V)].
\end{aligned} \tag{43}
$$

Proof. (a) Putting $X = \xi$ in equation (38), we have

$$\overline{\overline{P}}(\xi,Y,U) = \overline{P}(\xi,Y,U) - b\psi[g(\xi,U)Y - g(Y,U)\xi],$$

which on using equations (6), (12), (14), (18), gives the desired result.

(b) Again putting $U = \xi$ in equation (38), we have

$$\overline{\overline{P}}(X,Y,\xi) = \overline{P}(X,Y,\xi) - b\psi[g(X,\xi)Y - g(Y,\xi)X].$$

With the use of equations (6), (11), (14), (18) in the above equation, we obtain the required result.

(c) Taking innner product with ξ of equation (38), we have

$$\eta(\overline{\overline{P}}(U,V,Y)) = \eta(\overline{P}(U,V,Y)) - b\psi[g(U,Y)\eta(V) - g(V,Y)\eta(U)],$$

which on using equations (6), (10), (18), gives the desired result. □

4 Generalized pseudo-projectively semi-symmetric para-Kenmotsu manifold

Definition 5. *A Para-Kenmotsu manifold is said to be semi-symmetric [23] if it satisfies the condition*

$$R(X,Y)\cdot R = 0, \tag{44}$$

where $R(X,Y)$ is considered as the derivation of the tensor algebra at each point of the manifold.

Definition 6. *A para-Kenmotsu manifold is said to be generalized pseudo-projectively semi-symmetric if it satisfies the condition*

$$R(X,Y)\cdot \overline{\overline{P}} = 0, \tag{45}$$

where $\overline{\overline{P}}$ *is generalized pseudo-projective curvature tensor and* $R(X,Y)$ *is considered as the derivation of the tensor algebra at each point of the manifold.*

Theorem 4. *A generalized pseudo-projectively semi-symmetric para-Kenmotsu manifold is an* η*-Einstein manifold.*

Proof. Consider

$$(R(\xi,X)\cdot \overline{\overline{P}})(U,V,Y) = 0,$$

for any $X,Y,U,V \in T_PM$, where $\overline{\overline{P}}$ is generalized Pseudo-projective curvature tensor.
Then we have

$$\begin{aligned} R(\xi,X,\overline{\overline{P}}(U,V,Y)) - \overline{\overline{P}}(R(\xi,X,U),V,Y) \\ -\overline{\overline{P}}(U,R(\xi,X,V),Y) - \overline{\overline{P}}(U,V,R(\xi,X,Y) = 0. \end{aligned} \tag{46}$$

In view of equation (12) the above equation takes the form

$$\begin{aligned} \eta(\overline{\overline{P}}(U,V,Y))X -' \overline{\overline{P}}(U,V,Y,X)\xi - \eta(U)\overline{\overline{P}}(X,V,Y) + g(X,U) \\ \overline{\overline{P}}(\xi,V,Y) - \eta(V)\overline{\overline{P}}(U,X,Y) + g(X,V)\overline{\overline{P}}(U,\xi,Y) - \eta(Y)\overline{\overline{P}}(U,V,X) \\ +g(X,Y)\overline{\overline{P}}(U,V,\xi) = 0. \end{aligned}$$

Taking inner product of above eqquation with ξ and using equations (5), (33), (41), (42), (43), we get

$$\begin{aligned} -'\overline{P}(U,V,Y,X) + b\psi[g(X,V)g(Y,U) - g(X,U)g(Y,V)] - bg(X,V) \\ S(Y,U) - \left[a + \frac{r}{n}\left(\frac{a}{n-1} + b\right) - b\psi\right][g(X,U)\eta(Y)\eta(V) - g(X,V) \\ \eta(Y)\eta(U)] - \left[a + b(n-1) + \frac{r}{n}\left(\frac{a}{n-1} + b\right) - b\psi\right] g(X,V)\eta(Y)\eta(U) \\ +bg(X,U)S(Y,V) - b[S(X,V)\eta(U)\eta(Y) - S(X,U)\eta(V)\eta(Y)] \\ +g(X,U)\eta(Y)\eta(V)\left[a + b(n-1) + \frac{r}{n}\left(\frac{a}{n-1} + b\right) - b\psi\right] \\ -g(X,V)g(Y,U)\left[-a - \frac{r}{n}\left(\frac{a}{n-1} + b\right) + b\psi\right] + g(X,U)g(Y,V) \\ \left[-a - \frac{r}{n}\left(\frac{a}{n-1} + b\right) + b\psi\right] = 0. \end{aligned}$$

By virtue of equation (19), the above equation reduces to

$$\begin{aligned}
-a'R(U,V,Y,X)+\frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,V)g(U,X)-g(Y,U)\\
g(V,X)]-\left[a+\frac{r}{n}\left(\frac{a}{n-1}+b\right)-b\psi\right][g(U,X)\eta(Y)\eta(V)\\
-g(V,X)\eta(Y)\eta(U)]+\left[a+b(n-1)+\frac{r}{n}\left(\frac{a}{n-1}+b\right)-b\psi\right]\\
g(X,U)\eta(Y)\eta(V)-\left[a+b(n-1)+\frac{r}{n}\left(\frac{a}{n-1}+b\right)-b\psi\right]\\
g(X,V)\eta(Y)\eta(U)-g(X,V)g(Y,U)\left[-a-\frac{r}{n}\left(\frac{a}{n-1}+b\right)+b\psi\right]\\
+g(X,U)g(Y,V)\left[-a-\frac{r}{n}\left(\frac{a}{n-1}+b\right)+b\psi\right]\\
-b[S(X,V)\eta(U)\eta(Y)-S(X,U)\eta(V)\eta(Y)]\\
+b\psi[g(Y,U)g(X,V)-g(Y,V)g(X,U)]=0.
\end{aligned}$$

Let $\{e_i : i = 1, 2......n\}$ be an orthonormal basis. Putting $X = U = e_i$ in the above equation and taking summation over i, we get

$$S(Y,V)=-(n-1)g(Y,V)+\frac{2nb}{a}\eta(Y)\eta(V).$$

This shows that generalized pseudo-projectively semi-symmetric para-Kenmotsu manifold is an η-Einstein manifold. □

5 Generalized pseudo-projectively Ricci semi-symmetric para-Kenmotsu manifold

Definition 7. *Para-Kenmotsu manifold M is said to be Ricci semi-symmetric [10] if the condition*

$$R(X,Y)\cdot S=0, \tag{47}$$

holds for all $X, Y \in T_pM$.

Definition 8. *Para-Kenmotsu manifold is said to be generalized pseudo-projectively Ricci semi-symmetric if the condition*

$$\overline{\overline{P}}(X,Y)\cdot S=0, \tag{48}$$

holds for all X, Y, where $\overline{\overline{P}}$ is generalized pseudo-projective curvature tensor of para-Kenmotsu manifold.

Theorem 5. *A generalized pseudo-projectively Ricci semi-symmetric para-Kenmotsu manifold is either Einstein manifold or $\psi = \frac{an(n-1)+ra+br(n-1)}{bn(n-1)}$ on it .*

Proof. Consider

$$(\overline{\overline{P}}(\xi, X) \cdot S)(U, V) = 0,$$

which gives

$$S(\overline{\overline{P}}(\xi, X, U), V) + S(U, \overline{\overline{P}}(\xi, X, V)) = 0.$$

Using equations (14) and (41) in the above equation, we get

$$\begin{aligned} 0 = & \left[a + \frac{r}{n}\left(\frac{a}{n-1} + b\right) - b\psi\right][S(X, V)\eta(U) + S(X, U)\eta(V)] \\ & - (n-1)\left[-a - \frac{r}{n}\left(\frac{a}{n-1} + b\right) + b\psi\right][g(X, U)\eta(V) + g(X, V)\eta(U)]. \end{aligned}$$

Putting $U = \xi$ in the above equation and using (5), (6) and (14), we get

$$\left[a + \frac{r}{n}\left(\frac{a}{n-1} + b\right) - b\psi\right][S(X, V) + (n-1)g(X, V)] = 0,$$

which gives either

$$\psi = \frac{an(n-1) + ra + br(n-1)}{bn(n-1)}$$

or

$$S(X, V) = -(n-1)g(X, V).$$

This shows that generalized pseudo-projectively Ricci semi-symmetric para-Kenmotsu manifold is an Einstein manifold. □

6 Generalized pseudo-projectively ϕ-symmetric para-Kenmotsu manifold

Definition 9. *A para-Kenmotsu manifold M^n is said to be locally ϕ-symmetric if*

$$\phi^2((\nabla_W R)(X, Y, U)) = 0, \tag{49}$$

for all vector fields X, Y, U, W orthogonal to ξ.

This notion was introduced by Takahashi for Sasakian manifolds [24].

Definition 10. *A para-Kenmotsu manifold is said to be ϕ-symmetric if*

$$\phi^2((\nabla_W R)(X, Y, U)) = 0, \tag{50}$$

for arbitrary vector fields X, Y, U, W.

This notion was also introduced by Takahashi for Sasakian manifold [25]. Also analogous to these definitons, we define

Definition 11. *A para-Kenmotsu manifold M^n is said to be generalized pseudo-projective locally ϕ-symmetric para-Kenmotsu manifold if*

$$\phi^2((\nabla_W \overline{\overline{P}})(X,Y,U)) = 0, \tag{51}$$

for all vector fields X, Y, U, W orthogonal to ξ.

And also

Definition 12. *A para-Kenmotsu manifold M^n is said to be generalized pseudo-projectively ϕ-symmetric para-Kenmotsu manifold if*

$$\phi^2((\nabla_W \overline{\overline{P}})(X,Y,U)) = 0, \tag{52}$$

for arbitary vector fields X, Y, U, W.

Theorem 6. *A generalized pseudo projectively ϕ-symmetric para Kenmotsu manifold is an Einstein manifold.*

Proof. Taking covariant derivative of equation (38) with respect to vector field W, we obtain

$$(\nabla_W \overline{\overline{P}})(X,Y,U) = (\nabla_W \overline{P})(X,Y,U) - bdr(\psi)[g(X,U)Y - g(Y,U)X]. \tag{53}$$

Using equation (20) in the above equation, we get

$$\begin{aligned}(\nabla_W \overline{\overline{P}})(X,Y,U) &= a(\nabla_W R)(X,Y,U) - bdr(\psi)[g(X,U)Y \\ -g(Y,U)X] &+ b[(\nabla_W S)(Y,U)X - (\nabla_W S)(X,U)Y] - \frac{dr(W)}{n} \\ &\left(\frac{a}{n-1} + b\right)[g(Y,U)X - g(X,U)Y],\end{aligned} \tag{54}$$

Assume that the manifold is generalized pseudo-projectively ϕ-symmetric, then from equation (52), we have

$$\phi^2((\nabla_W \overline{\overline{P}})(X,Y,U)) = 0,$$

which on using equation (2), gives

$$(\nabla_W \overline{\overline{P}})(X,Y,U) = \eta((\nabla_W \overline{\overline{P}})(X,Y,U))\xi. \tag{55}$$

Using equation (54) in above equation, we get

$$\begin{aligned}&a(\nabla_W R)(X,Y,U) - bdr(\psi)[g(X,U)Y - g(Y,U)X] + b \\ &[(\nabla_W S)(Y,U)X - (\nabla_W S)(X,U)Y] - \frac{dr(W)}{n}\left(\frac{a}{n-1} + b\right) \\ &[g(Y,U)X - g(X,U)Y] = a\eta((\nabla_W R)(X,Y,U))\xi - bdr(\psi) \\ &[g(X,U)\eta(Y) - g(Y,U)\eta(X)]\xi + b[(\nabla_W S)(Y,U)\eta(X) \\ &-(\nabla_W S)(X,U)\eta(Y)]\xi - \frac{dr(W)}{n}\left(\frac{a}{n-1} + b\right) \\ &[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\xi,\end{aligned} \tag{56}$$

Taking inner product of the above equation with V, we get

$$\begin{aligned}
&ag((\nabla_W R)(X,Y,U),V) - bdr(\psi)[g(X,U)g(Y,V) - g(Y,U)\\
&g(X,V)] + b[(\nabla_W S)(Y,U)g(X,V) - (\nabla_W S)(X,U)g(Y,V)]\\
&-\frac{dr(W)}{n}\left(\frac{a}{n-1}+b\right)[g(Y,U)g(X,V) - g(X,U)g(Y,V)]\\
&= a\eta((\nabla_W R)(X,Y,U))\eta(V) - bdr(\psi)[g(X,U)\eta(Y)\eta(V)\\
&-g(Y,U)\eta(X)\eta(V)] + b[(\nabla_W S)(Y,U)\eta(X)\eta(V)\\
&-(\nabla_W S)(X,U)\eta(Y)\eta(V)] - \frac{dr(W)}{n}\left(\frac{a}{n-1}+b\right)\\
&[g(Y,U)\eta(X)\eta(V) - g(X,U)\eta(Y)\eta(V)].
\end{aligned} \tag{57}$$

Putting $X = V = e_i$ and taking summation over i, we obtaion

$$\begin{aligned}
&a(\nabla_W S)(Y,U) + b[n(\nabla_W S)(Y,U) - (\nabla_W S)(Y,U)]\\
&-\frac{dr(W)}{n}\left(\frac{a}{n-1}+b\right)[ng(Y,U) - g(Y,U)]\\
&-bdr(\psi)[g(Y,U) - ng(Y,U)] - a\eta((\nabla_W R)(e_i,Y,U))\eta(e_i)\\
&-b[(\nabla_W S)(Y,U) - (\nabla_W S)(e_i,U)\eta(Y)\eta(e_i) + \frac{dr(W)}{n}\left(\frac{a}{n-1}+b\right)\\
&[g(Y,U) - \eta(Y)\eta(U)] + bdr(\psi)[\eta(U)\eta(Y) - g(Y,U)] = 0,
\end{aligned} \tag{58}$$

Taking $U = \xi$ in the above equation, we have

$$\begin{aligned}
&a(\nabla_W S)(Y,\xi) + b(n-1)(\nabla_W S)(Y,\xi) - \frac{dr(W)}{n}\left(\frac{a}{n-1}+b\right)\\
&(n-1)\eta(Y) - a\eta((\nabla_W R)(e_i,Y,\xi))\eta(e_i) + bdr(\psi)(n-1)\eta(Y)\\
&-b[(\nabla_W S)(Y,\xi) - (\nabla_W S)(e_i,\xi)\eta(e_i)\eta(Y)] = 0.
\end{aligned} \tag{59}$$

Now

$$\eta((\nabla_W R)(e_i,Y,\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y,\xi),\xi)g(e_i,\xi). \tag{60}$$

Also

$$\begin{aligned}
g((\nabla_W R)(e_i,Y,\xi),\xi) &= g(\nabla_W R(e_i,Y,\xi),\xi) - g(R(\nabla_W e_i,Y,\xi),\xi)\\
&- g(R(e_i,\nabla_W Y,\xi),\xi) - g(R(e_i,Y,\nabla_W \xi),\xi).
\end{aligned} \tag{61}$$

Since $\{e_i\}$ is an orthonormal basis, so $\nabla_X e_i = 0$ and using equation (11), we get

$$g(R(e_i,\nabla_W Y,\xi),\xi) = 0.$$

Since

$$g(R(e_i,Y,\xi),\xi) + g(R(\xi,\xi,Y),e_i) = 0,$$

Therefore, we have

$$g(\nabla_W R(e_i,Y,\xi),\xi) + g(R(e_i,Y,\xi),\nabla_W \xi) = 0,$$

Using this fact in equation (61), we get

$$g((\nabla_W R)(e_i, Y, \xi), \xi) = 0. \tag{62}$$

Using equation (62) in (59), we have

$$\begin{aligned} \left[\frac{dr(W)}{n}\left(\frac{a}{n-1}+b\right)(n-1)\eta(Y) - bdr(\psi)(n-1)\eta(Y)\right] \\ \left[\frac{1}{a+b(n-1)-b}\right] = (\nabla_W S)(Y,\xi), \end{aligned} \tag{63}$$

Taking $Y = \xi$ in above equation and using equations (5) and (14), we get

$$dr(\psi) = \frac{dr(W)}{bn}\left(\frac{a}{n-1}+b\right), \tag{64}$$

which shows that r is constant. Now we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi),$$

Then by using (8), (9), (14) in the above equation, it follows that

$$(\nabla_W S)(Y,\xi) = -S(Y,W) - (n-1)g(Y,W). \tag{65}$$

Thus from equations (63), (64) and (65), we obtain

$$S(Y,W) = -(n-1)g(Y,W), \tag{66}$$

which shows that M^n is an Einstein manifold. □

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