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PSEUDO-VALUATIONS ON COMMUTATIVE BE-ALGEBRAS

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Abstract

In this paper, we study pseudo-valuations on commutative BE-algebras and obtain some related results. Using the pseudo-metrics induced by pseudo valuations we introduce a congruence relation on BE-algebras.

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1 Introduction

The study of BCK/BCI-algebras was initiated by K. Iséki as a generalization of the concept of set-theoretic difference and propositional calculus([3],[4]). In [9], J.Neggeres and H. S. Kim introduced the notion of d-algebras which is a generalization of BCK-algebras. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim [5] introduced a new notion, called BH-algebras, which is a generalization of BCK/BCI-algebras. Recently, as another generalization of BCK-algebras, the notion of BE-algebras was introduced by H. S. Kim and Y. H. Kim [6]. Lee in [7] introduced the notion of pseudo-valutions on BE-algebras and investigated several properties. He provided relations between pseudo-valutions and some subalgebras of BE-algebras. Using the notion of a pseudo-valution he constructed (pseudo)metric space and showed that the binary operation of $*$ in BE-algebras is uniformly continuous. In this paper, we study pseudo-valuations on BE-algebras and investigate its properties. We discuss the relation between pseudo-valuations and filters of a BE-algebra. We obtain some results of pseudo-metrics induced by pseudo-valuations on BE-algebras. Also, we prove that the pseudo-metric induced by a pseudo-valuation φ is a metric on a BE-algebra if and only if φ is a valuation. Finally, using the pseudo-metrics induced by pseudo-valuations we construct the quotient structures and investigate related properties.

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2 Preliminary

2.1 BE-algebras

Definition 1. *[11] A BE-algebra is an algebra* $(X, *, 1)$ *of type* $(2, 0)$ *such that satisfying the following axioms:*
(1) $x * x = 1$ *for all* $x \in X$,
(2) $x * 1 = 1$ *for all* $x \in X$,
(3) $1 * x = x$ *for all* $x \in X$,
(4) $x * (y * z) = y * (x * z)$, *for all* $x, y, z \in X$.

A relation $\leq$ on X is defined by $x \leq y$ if and only if $x * y = 1$. If X is a BE-algebra and $x, y \in X$, then $x * (y * x) = 1$.

Definition 2. *[11] We say that a BE-algebra* X *is commutative if* $(x * y) * y = (y * x) * x$ *for all* $x, y \in X$.

Proposition 1. *[1] Let* X *be a commutative BE-algebra and* $x, y, z \in X$. *Then,*
(5) $x * y = y * x = 1 \Rightarrow x = y$,
(6) $(x * y) * ((y * z) * (x * z)) = 1$.

Definition 3. *[11] We say that a BE-algebra* X *is transitive if* $(y * z) \leq (x * y) * (x * z)$ *for all* $x, y, z \in X$.

Proposition 2. *[11] If* X *is a commutative BE-algebra, then it is transitive.*

Definition 4. *[11] Let* A *be a BE-algebra. A filter is a nonempty set* $F \subseteq X$ *such that for all* $x, y \in A$
(i) $1 \in F$,
(ii) $x \in F$ *and* $x * y \in F$ *imply* $y \in F$.

Let F be a filter in X. If $x \in F$ and $x \leq y$ then $y \in F$.

Definition 5. *[11] A filter* F *of a BE-algebra* X *is said to be normal if it satisfies the following condition:*

$$x * y \in F \Rightarrow [(z * x) * (z * y) \in F \text{ and } (y * z) * (x * z) \in F]$$

for all $x, y, z \in X$.

Proposition 3. *[11] If* X *is a transitive BE-algebra, then every filter of* X *is normal.*

Definition 6. *An equivalence relation* θ *on a BE-algebra* X *is called a congruence relation on* X, *if* $(x, y) \in \theta$ *implies* $(x * z, y * z) \in \theta$ *and* $(z * x, z * y) \in \theta$ *for all* $x, y, z \in X$.

Proposition 4. *[11] Let F be a normal filter of a BE-algebra X. Define*

$$x \equiv^F y \ \Leftrightarrow \ x * y \ , \ y * x \in F$$

Then,
(i) $\equiv^F$ is a congruence relation on X.
(ii) Let $[x]_F = \{y \in x : x \equiv^F y\}$ be an equivalence class of x and $X/F = \{[x]_F : x \in X\}$. Then X/F is a BE-algebra under the binary operations given by:

$$[x]_F \oplus [y]_F = [x * y]_F.$$

Definition 7. *[2] Let X be a BE-algebra. If there exists an element 0 satisfying $0 \leq x$ (or $0 * x = 1$) for all $x \in X$, then X is called a bounded BE-algebra.*

Definition 8. *[12] Let $X = (X, *, 1)$ and $X^{'} = (X^{'}, *^{'}, 1^{'})$ be two BE-algebras. A mapping $f : X \to X^{'}$ is called a BE-algebra homomorphism from X into $X^{'}$ if for any $x, y \in X, f(x * y) = f(x) *^{'} f(y)$.*

For a homomorphism $f : X \to X^{'}$, we have $f(1) = 1^{'}$.

2.2 Pseudo-valuations

Definition 9. *[7] A real-valued function φ on a BE-algebra X is called a pseudo-valuation on X if it satisfies the following condition:*
(i) $\varphi(1) = 0$,
(ii) $\varphi(x * z) \leq \varphi(x * (y * z)) + \varphi(y)$, for all $x, y, z \in X$.

A pseudo-valuation φ on a BE-algebra X satisfying the following condition:

$$x \neq 1 \Rightarrow \varphi(x) \neq 0 \quad \forall x \in X \quad (7)$$

is called a valuation on X.
Let φ be a pseudo-valuation on a BE-algebra X. Then for all $x, y, z \in X$,
(8) φ is order reversing,
(9) $\varphi(x * y) \leq \varphi(y)$,
(10)$\varphi((x * (y * z)) * z) \leq \varphi(x) + \varphi(y)$,
(11) $\varphi(x) \geq 0$, for all $x \in X$,
(12) $\varphi(y) \leq \varphi(x * y) + \varphi(x)$,
(13) $x \leq y * z \Rightarrow \varphi(z) \leq \varphi(x) + \varphi(y)$,
(14) $\varphi((x * y) * y) \leq \varphi(x)$.[See,[7]]

Proposition 5. *[7] Let X be a transitive BE-algebra. Every pseudo-valuation φ on X satisfies the following inequality,*

$$\varphi(x * z) \leq \varphi(x * y) + \varphi(y * z) \quad \forall x, y, z \in X. \quad (15)$$

For a real-valued function φ on a BE-algebra X define a mapping $d_\varphi : X \times X \to \mathbb{R}$ by $d_\varphi(x, y) = \varphi(x * y) + \varphi(y * x)$ for all $(x, y) \in X \times X$.

Theorem 1. *[7] Let X be a transitive BE-algebra. If a real-valued function φ on X is a pseudo-valuation on X, then d_φ is a pseudo-metric on X, and so (X, d_φ) is a pseudo-metric space.*

Suppose that (X, d) is a pseudo-metric space. Then
(i) For each $x \in X$ and $\varepsilon > 0$, the set $B_\varepsilon(x) = \{y \in X : d(y, x) < \varepsilon\}$ is called the ball of radius ε with center at x.
(ii) We say that $U \subseteq X$ is open in (X, d) if for each $x \in U$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.
(iii) The topology τ_d induced by d is the collection of all open sets in (X, d).

Theorem 2. *[7] Let X be a transitive BE-algebra such that*

$$x * y = y * x = 1 \Rightarrow x = y \quad \forall x, y \in X$$

If $\varphi : X \to \mathbb{R}$ is a valuation on X, then (X, d_φ) is a metric space.

Proposition 6. *[7] Let X be a transitive BE-algebra satisfying in following condition:*

$$x * y = y * x = 1 \Rightarrow x = y \quad \forall x, y \in X.$$

Then,
(16) $d_\varphi(x, y) \geq d_\varphi(z * x, z * y)$,
(17) $d_\varphi(x, y) \geq d_\varphi(x * z, y * z)$,
(18) $d_\varphi(x * y, z * w) \leq d_\varphi(x * y, z * y) + d_\varphi(z * y, z * w)$,
for all $x, y, z, w \in X$.

notation. From now on, in this paper X is a commutative BE-algebra.

3 Pseudo-valuations on commutative BE-algebras

Proposition 7. *Let $(X, *, 1)$ be a BE-algebra. Then for each filter F of X there exists a pseudo-valuation φ_F on X which is called the pseudo-valuation induced by filter F. Moreover φ_F is a valuation if and only if $F = \{1\}$.*

Proof. Let s be a positive element of $\mathbb{R}$. Define $\varphi_F : X \to \mathbb{R}$ by

$$\varphi_F(x) = \begin{cases} 0 & \text{if } x \in F \\ s & \text{if } x \notin F. \end{cases}$$

We show that φ_F is a pseudo-valuation on X. Since $1 \in F$, $\varphi_F(1) = 0$. Let $x, y, z \in X$. We consider following cases:
Case 1. If $x * z \in F$, then $\varphi_F(x * z) = 0$.
(i) If $y \in F$, then $y * (x * z) \in F$. Hence $\varphi_F(y * (x * z)) = 0$. By (4) we have,

$$\varphi_F(x * z) = 0 \leq \varphi_F(y * (x * z)) + \varphi_F(y) = \varphi_F(x * (y * z)) + \varphi_F(y) = 0.$$

(ii) If $y \notin F$, then $\varphi_F(y) = s$. Hence

$$\varphi_F(x * z) = 0 \leq s = \varphi_F(y) \leq \varphi_F(y * (x * z)) + \varphi_F(y) = \varphi_F(x * (y * z)) + \varphi_F(y).$$

Case 2. If $x * z \notin F$, then $\varphi_F(x * z) = s$.
(i) If $y \in F$, then $\varphi_F(y) = 0$. Since $y \in F$, $y * (x * z) \notin F$ because if $y * (x * z) \in F$, then $x * z \in F$ which is a contradiction. Hence $\varphi_F(y * (x * z)) = s$. Therefore,

$$\varphi_F(x * z) = s \leq s + 0 = \varphi_F(y * (x * z)) + \varphi_F(y) = \varphi_F(x * (y * z)) + \varphi_F(y).$$

(ii) If $y \notin F$, then $\varphi_F(y) = s$. Hence,

$$\varphi_F(x * z) = s \leq \varphi_F(y * (x * z)) + \varphi_F(y) = \varphi_F(x * (y * z)) + \varphi_F(y).$$

Hence φ_F is a pseudo-valuation on X. It is clear that φ_F is a valuation if and only if $F = \{1\}$. □

Example. Let $X = \{a, b, c, 1\}$. If the operation $*$ given by the following table

*	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	1
c	1	a	c	1

then $(X, *, 1)$ is a BE-algebra (see [7]). Let $F = \{a, 1\}$. Then F is a filter of X. Define $\varphi_F : X \to \mathbb{R}$ by

$$\varphi_F(x) = \begin{cases} 0 & \text{if } x \in F \\ 2 & \text{if } x \notin F. \end{cases}$$

Then φ_F is a pseudo-valuation on X.

Proposition 8. *Let φ be a pseudo-valuation on BE-algebra X. Then $F_\varphi = \{x \in X : \varphi(x) = 0\}$ is a filter of X which is called the filter induced by pseudo-valuation φ.*

Proof. Since $\varphi(1) = 0$, $1 \in F_\varphi$. Let $x, x * y \in F_\varphi$. Then $\varphi(x) = \varphi(x * y) = 0$. By (11) and (12) we have,

$$0 \leq \varphi(y) \leq \varphi(x * y) + \varphi(x) = 0.$$

Hence $y \in F_\varphi$. □

Example. By assumptions of the previous example, we have $F_{\varphi_F} = \{a, 1\}$.

Proposition 9. *Let F be a filter of a BE-algebra X. Then $F_{\varphi_F} = F$.*

Proof. We have $F_{\varphi_F} = \{x \in X : \varphi_F(x) = 0\} = \{x \in X : x \in F\} = F$. □

Proposition 10. *Let φ be a pseudo-valuation on a BE-algebra X. Then φ is a valuation if and only if d_φ is a metric on X.*

Proof. If φ is a valuation on X, then by Theorem 2, d_φ is a metric. Conversely, suppose that φ is not a valuation on X. Then there exists $x \in X$ such that $x \neq 1$ and $\varphi(x) = 0$. Hence $x, 1 \in F_\varphi$. Therefore

$$d_\varphi(1, x) = \varphi(1 * x) + \varphi(x * 1) = \varphi(x) + \varphi(1) = 0.$$

Hence $d_\varphi(1, x) = 0$. Since d_φ is a metric on X, $x = 1$ which is a contradiction. □

Theorem 3. *Let φ be a pseudo-valuation on a BE-algebra X. Then φ is continuous.*

Proof. By Theorem 1, (X, d_φ) is a pseudo-metric space. Let $x_0 \in X$ be an arbitrary element. For each $x \in X$ by (12) we have

$$\varphi(x_0) - \varphi(x) \leq \varphi(x * x_0) \quad \text{and} \quad \varphi(x) - \varphi(x_0) \leq \varphi(x_0 * x).$$

Hence

$$\varphi(x_0) - \varphi(x) \leq \varphi(x * x_0) + \varphi(x_0 * x) = d_\varphi(x, x_0)$$
$$\varphi(x) - \varphi(x_0) \leq \varphi(x * x_0) + \varphi(x_0 * x) = d_\varphi(x, x_0).$$

Thus $-d_\varphi(x, x_0) \leq \varphi(x) - \varphi(x_0) \leq d_\varphi(x, x_0)$ and so $\mid \varphi(x) - \varphi(x_0) \mid \leq d_\varphi(x, x_0)$. Therefore φ is continuous. □

Corollary 1. *If φ is a pseudo-valuation on X, then F_φ is a closed subset of X.*

Proof. Since $F_\varphi = \{x \in X : \varphi(x) = 0\} = \varphi^{-1}(\{0\})$, by Theorem 3, the proof is clear. □

Let X be a BE-algebra and τ be a topology on X. $(X, *, \tau)$ is a topological BE-algebra, if $x * y \in U \in \tau$, then there are two open sets V and W containing x and y respectively, such that $V * W \subseteq U$, for all $x, y \in X$.

Theorem 4. *Let τ_{d_φ} be the topology induced by d_φ. Then $(X, *, \tau_{d_\varphi})$ is a topological BE-algebra.*

Proof. Let $x * y \in B_\epsilon(x * y)$. We claim that $B_{\frac{\varepsilon}{2}}(x) * B_{\frac{\varepsilon}{2}}(y) \subseteq B_\varepsilon(x * y)$. Let $z \in B_{\frac{\varepsilon}{2}}(x) * B_{\frac{\varepsilon}{2}}(y)$. Then there exist $p \in B_{\frac{\varepsilon}{2}}(x)$ and $q \in B_{\frac{\varepsilon}{2}}(y)$ such that $z = p*q$. Hence $d_\varphi(x, p) \leq \frac{\varepsilon}{2}$ and $d_\varphi(y, q) \leq \frac{\varepsilon}{2}$. By (16) and (17) we have $d_\varphi(x * y, p * y) \leq d_\varphi(x, p)$ and $d_\varphi(p*y, p*q) \leq d_\varphi(y, q)$. By (18), $d_\varphi(x*y, p*q) \leq d_\varphi(x*y, p*y) + d_\varphi(p*y, p*q) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus $z = p * q \in B_\varepsilon(x * y)$. Therefore $(X, *, \tau)$ is a topological BE-algebra. □

4 Quotient BE-algebras induced by pseudo-valuations

Definition 10. *Let φ be a pseudo-valuation on a BE-algebra X. Define the relation θ_φ by*

$$(x, y) \in \theta_\varphi \Leftrightarrow d_\varphi(x, y) = 0$$

for all $x, y \in X$.

Proposition 11. *Let φ be a pseudo-valuation on a BE-algebra X. Then θ_φ is a congruence relation on X which is called the congruence relation induced by φ.*

Proof. Since θ_φ induced by a pseudo-metric, it is an equivalence relation on X. We show that θ_φ is a congruence relation. Let $(x, y), (z, w) \in \theta_\varphi$. Then $d_\varphi(x, y) = 0 = d_\varphi(z, w)$. By (11) and (17) we have $0 \le d_\varphi(x * z, y * z) \le d_\varphi(x, y) = 0$. Hence $d_\varphi(x * z, y * z) = 0$ and so $(x * z, y * z) \in \theta_\varphi$. Similarly, we can show that $(y * z, y * w) \in \theta_\varphi$. Since θ_φ is a equivalence relation, $(x * z, y * w) \in \theta_\varphi$. Therefore θ_φ is a congruence relation on X. □

Proposition 12. *Let φ be a pseudo-valuation on a BE-algebra and θ_φ be the congruence relation induced by φ. Then for all $x, y \in X$,*
(i) $(x, y) \in \theta_\varphi \Leftrightarrow (1, x * y), (1, y * x) \in \theta_\varphi$.
(ii) $F_{\theta_\varphi} = \{x * y : (x, y) \in \theta_\varphi\}$ is a filter.

Proof. (*i*) We have,

$$\begin{aligned}(1, x * y), (1, y * x) \in \theta_\varphi &\iff d_\varphi(1, x * y) = 0 = d_\varphi(1, y * x),\\ &\iff \varphi(1 * (x * y)) + \varphi((x * y) * 1) = 0\\ &\qquad = \varphi(1 * (y * x)) + \varphi((y * x) * 1),\\ &\iff \varphi(x * y) = 0 = \varphi(y * x),\\ &\iff d_\varphi(x, y) = 0,\\ &\iff (x, y) \in \theta_\varphi.\end{aligned}$$

(*ii*) Let $x * y, x = 1 * x \in F_{\theta_\varphi}$. Then $d_\varphi(1, x) = 0 = d_\varphi(x, y)$. Hence $\varphi(x) = 0 = \varphi(x * y)$. By (11) and (12) we have,

$$0 \le d_\varphi(1, y) = \varphi(y) \le \varphi(x * y) + \varphi(x) = 0.$$

Hence $d_\varphi(1, y) = 0$ and so $y = 1 * y \in F_{\theta_\varphi}$.

□

Let φ be a pseudo-valuation on a BE-algebra X and θ_φ be the congruence relation induced by φ. Let $[x]_\varphi = \{y \in X : (x, y) \in \theta_\varphi\}$ and $X/\varphi = \{[x]_\varphi : x \in X\}$. Define a binary operation $\odot$ on X/φ as follows:

$$[x]_\varphi \odot [y]_\varphi = [x * y]_\varphi.$$

The resulting algebra is denoted by X/φ and is called the quotient algebra of X induced by pseudo-valuation φ.

Theorem 5. *Let φ be a pseudo-valuation on a BE-algebra X. Then $(X/\varphi, \odot, [1]_\varphi)$ is a BE-algebra and $d^*([x]_\varphi, [y]_\varphi) = d_\varphi(x, y)$ is a metric on X/φ.*

Proof. The proof is straightforward. We only show that d^* is well defined. Let x, y, a and b be in X and $[x]_\varphi = [a]_\varphi$ and $[y]_\varphi = [b]_\varphi$. Then $\varphi(x * a) = \varphi(a *$

$x) = \varphi(y*b) = \varphi(b*y) = 0$. By (6) we have $x*y \leq (y*b)*(x*b)$ and $x*b \leq (b*a)*(x*a)$. By (13), $\varphi(x*b) \leq \varphi(x*y)+\varphi(y*b) = \varphi(x*y)$ and $\varphi(b*a) \leq \varphi(x*a)+\varphi(x*b) = \varphi(x*b)$. Hence $\varphi(b*a) \leq \varphi(x*y)$. By a similar argument we have $\varphi(x*y) \leq \varphi(b*a)$. Thus $\varphi(x*y) = \varphi(b*a)$. Similarly, we have $\varphi(y*x) = \varphi(a*b)$. Hence $d_\varphi(x,y) = d_\varphi(a,b)$ and so d^* is well defined. □

Example. Let $X = \{a, b, c, d, 1\}$. Define a binary operation $*$ on X as follows:

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	1	1	c	1

Then $(X, *, 1)$ is a BE-algebra. If $\varphi : X \to \mathbb{R}$ is defined by

$$\varphi = \begin{pmatrix} 1 & a & b & c & d \\ 0 & 3 & 3 & 0 & 3 \end{pmatrix}$$

then by Proposition7, φ is a pseudo-valuation on X. It is easy to verify that

$$\theta_\varphi = \{(1,1),(a,a),(b,b),(c,c),(d,d),(0,c),(c,0)\}$$

and $X/\varphi = \{[a]_\varphi, [b]_\varphi, [d]_\varphi, [1]_\varphi\}$ where $[a]_\varphi = \{a\}$, $[b]_\varphi = \{b\}$, $[d]_\varphi = \{d\}$ and $[1]_\varphi = \{c, 1\}$.

Proposition 13. *Let φ be a pseudo-valuation on a BE-algebra X. Then for all $x, y, z \in X$,*
(i) If $[x]_\varphi \odot [y]_\varphi = [y]_\varphi \odot [x]_\varphi = [1]_\varphi$, then $[x]_\varphi = [y]_\varphi$,
(ii) $([x]_\varphi \odot [y]_\varphi) \odot (([y]_\varphi \odot [z]_\varphi) \odot ([x]_\varphi \odot [z]_\varphi)) = [1]_\varphi$,

Proof. (i) Let $[x]_\varphi \odot [y]_\varphi = [y]_\varphi \odot [x]_\varphi = [1]_\varphi$ for some $x, y \in X$. Then $[x*y]_\varphi = [y*x]_\varphi = [1]_\varphi$ and so $(1, x*y), (1, y*x) \in \theta_\varphi$. By Proposition 12 part (i), we have $(x, y) \in \theta_\varphi$, that is $[x]_\varphi = [y]_\varphi$.
(ii) The proof is straightforward. □

Proposition 14. *Let φ be a pseudo-valuation on a BE-algebra X. Then $F_\varphi = [1]_\varphi$.*

Proof. We have,

$$\begin{aligned} x \in [1]_\varphi &\iff (1,x) \in \theta_\varphi, \\ &\iff d_\varphi(1,x) = 0, \\ &\iff \varphi(1*x)+\varphi(x*1) = 0, \\ &\iff \varphi(x) = 0, \\ &\iff x \in F_\varphi. \end{aligned}$$

□

Proposition 15. *Let φ be a pseudo-valuation on a BE-algebra X. Then $\overline{\varphi}([x]_\varphi) = inf\{\varphi(z) : z \in [x]_\varphi\}$ is a pseudo-valuation on X/φ.*

Proof. Since for each $x \in X$, $\varphi(x) \geq 0$, $\overline{\varphi}([x]_\varphi)$ is well defined. By Proposition 14, we have

$$\overline{\varphi}([1]_\varphi) = inf\{\varphi(z) : z \in [1]_\varphi\} = inf\{\varphi(z) : z \in F_\varphi\} = \varphi(1) = 0.$$

Now, we show that for each $x, y, z \in X$, $\overline{\varphi}([x]_\varphi \odot [z]_\varphi) \leq \overline{\varphi}([x]_\varphi \odot ([y]_\varphi \odot [z]_\varphi)) + \overline{\varphi}([y]_\varphi)$. Let $a \in [x]_\varphi, b \in [y]_\varphi$ and $c \in [z]_\varphi$. Then $a * c \in [x * z]_\varphi = [x]_\varphi \odot [z]_\varphi$. By (11), we have

$$\overline{\varphi}([x]_\varphi \odot [z]_\varphi) = \overline{\varphi}([x * z]_\varphi) \leq \varphi(a * b) \leq \varphi(a * (b * c)) + \varphi(b).$$

Hence

$$\begin{aligned}
\overline{\varphi}([x]_\varphi \odot [z]_\varphi) &\leq inf\{\varphi(a * (b * c) : a \in [x]_\varphi, b \in [y]_\varphi, c \in [z]_\varphi\} \\
&\quad + inf\{\varphi(b) : b \in [y]_\varphi\}, \\
&= \overline{\varphi}([x]_\varphi \odot ([y]_\varphi \odot [z]_\varphi)) + \overline{\varphi}([y]_\varphi).
\end{aligned}$$

□

Proposition 16. *Let φ be a valuation on a BE-algebra X. Then $\overline{\varphi}([x]_\varphi) = inf\{\varphi(z) : z \in [x]_\varphi\}$ is a valuation on X/φ if and only if for each $x \in X$ the set $[x]_\varphi$ have a maximum.*

Proof. By Proposition 15, φ is a pseudo-valuation on X/φ. Let $\overline{\varphi}([x]_\varphi) = 0$ for some $x \in X$. By assumption, there exists $a \in X$ such that $a = max[x]_\varphi$. Since for each $z \in [x]_\varphi$, $z \leq a$, we get that $\varphi(a) \leq \varphi(z) \leq \overline{\varphi}([z]_\varphi) = \overline{\varphi}([x]_\varphi)$ and so $\varphi(a) = 0$. Since φ is a valuation, $a = 1$. Hence $[x]_\varphi = [1]_\varphi$. Conversely, let $1 \neq x \in X$. If $[x]_\varphi$ does not have a maximum, then for each $y \in [x]_\varphi$ there exists $z \in [x]_\varphi$ such that $y \leq z$. Hence $\varphi(z) \leq \varphi(y)$. Therefore $\overline{\varphi}([x]_\varphi) = 0$. Since $\overline{\varphi}$ is a valuation, $[x]_\varphi = [1]_\varphi$ and hence $\varphi(x) = 0$. Since φ is a valuation we get that $x = 1$ which is a contradiction. □

Proposition 17. *Let φ be a pseudo-valuation on a BE-algebra X and F_φ be the filter induced by φ. Then $\equiv^{F_\varphi} = \theta_\varphi$.*

Proof. We have,

$$\begin{aligned}
(x, y) \in \theta_\varphi &\iff d_\varphi(x, y) = 0, \\
&\iff \varphi(x * y) + \varphi(y * x) = 0, \\
&\iff \varphi(x * y) = \varphi(y * x) = 0, \\
&\iff x * y, y * x \in F_\varphi \\
&\iff (x, y) \in \equiv^{F_\varphi} .
\end{aligned}$$

□

Theorem 6. *Let φ_1 and φ_2 be two different pseudo-valuations on a BE-algebra X such that $[1]_{\varphi_1} = [1]_{\varphi_2}$. Then $X/\varphi_1 = X/\varphi_2$.*

Proof. Let $(x, y) \in \theta_{\varphi_1}$. By Proposition 12, $x * y, y * x \in [1]_{\varphi_1}$. Since $[1]_{\varphi_1} = [1]_{\varphi_2}$, $x*y, y*x \in [1]_{\varphi_2}$. Hence $[x]_{\varphi_2} \odot [y]_{\varphi_2} = [x*y]_{\varphi_2} \in [1]_{\varphi_2}$ and $[y]_{\varphi_2} \odot [x]_{\varphi_2} = [y*x]_{\varphi_2} \in [1]_{\varphi_2}$. By Proposition 13, $[x]_{\varphi_2} = [y]_{\varphi_2}$. Therefore $(x, y) \in \theta_{\varphi_2}$ and then $\theta_{\varphi_1} \subseteq \theta_{\varphi_2}$. Similarly, we have $\theta_{\varphi_2} \subseteq \theta_{\varphi_1}$. Hence $\theta_{\varphi_1} = \theta_{\varphi_2}$ and so $X/\varphi_1 = X/\varphi_2$. □

Lemma 1. *Let φ be a pseudo-valuation on a BE-algebra X and F be a filter of X such that $[1]_\varphi \subseteq F$. Denote $F/\varphi = \{[x]_\varphi : x \in F\}$. Then,*
(i) $x \in F$ if and only if $[x]_\varphi \in F/\varphi$ for all $x \in X$.
(ii) F/φ is a filter of X/φ.

Proof. (i) Let $[x]_\varphi \in F/\varphi$. There exists $y \in F$ such that $[x]_\varphi = [y]_\varphi$. Hence $(x, y) \in \theta_\varphi$. By Proposition 12, $(1, y * x) \in \theta_\varphi$ and thus $y * x \in [1]_\varphi$. Since $[1]_\varphi \subseteq F$, $y * x, y \in F$. Hence $x \in F$. It is easy to show the Converse.
(ii) Since $1 \in F$, $[1]_\varphi \in F/\varphi$ by part (i). Let $[x]_\varphi \odot [y]_\varphi, [x]_{\varphi_1} \in F/\varphi$. Then $[x]_\varphi \odot [y]_\varphi = [x * y]_\varphi$. By part (i) we have $x * y, x \in F$. Since F is a filter, $y \in F$. Hence $[y]_\varphi \in F/\varphi$. Thus F/φ is a filter of X/φ. □

Proposition 18. *Let φ be a valuation on a BE-algebra X and F_φ be the filter induced by φ. Then $F/\varphi = \{[1]_\varphi\}$ if and only if $F = F_\varphi$.*

Proof. By definition $F_\varphi = [1]_\varphi \subseteq F$. We show that $F \subseteq F_\varphi$. Suppose that $F \nsubseteq F_\varphi$. Then there exists $x \in F$ such that $\varphi(x) \neq 0$. Since $x \in F$, $[x]_\varphi \in F/\varphi$ by Proposition 1. By asumption $[1]_\varphi = [x]_\varphi$ and so $d_\varphi(1, x) = 0$. Therefore $\varphi(x) = 0$ which is a contradiction. Conversely, If $F = F_\varphi$, then

$$F_\varphi/\varphi = \{[x]_\varphi : x \in F_\varphi\} = \{[x]_\varphi : \varphi(x) = 0\} = \{[x]_\varphi : x = 1\} = \{[1]_\varphi\}.$$

□

Lemma 2. *Let φ be a pseudo-valuation on a BE-algebra X and G be a filter of X/φ. Then $F = \{x \in X : [x]_\varphi \in G\}$ is a filter of X such that $[1]_\varphi \subseteq F$.*

Proof. Since $[1]_\varphi \in G$, $1 \in F$ by definition of F. Suppose that $x * y, x \in F$. Then $[x]_\varphi, [x * y]_\varphi = [x]_\varphi \odot [y]_\varphi \in G$. Since G is a filter of X/φ, $[y]_\varphi \in G$. By definition of F, we have $y \in F$. Thus F is a filter of X. Let $x \in [1]_\varphi$. Then $[x]_\varphi = [1]_\varphi$. Hence $[x]_\varphi \in G$ and by definition of F we have $x \in F$. Thus $[1]_\varphi \subseteq F$. □

Theorem 7. *Let φ be a pseudo-valuation on a BE-algebra X, $F(X, \varphi)$ the collection of all filters of X containing $[1]_\varphi$, and $F(X/\varphi)$ the collection of all filters of X/φ. Then $\psi : F(X, \varphi) \to F(X/\varphi)$, $F \to F/\varphi$ is a bijection.*

Proof. Let $F, G \in F(X, \varphi)$ and $\psi(F) = \psi(G)$. Then $F/\varphi = G/\varphi$. Let $x \in F$. Then $[x]_\varphi \in F/\varphi = G/\varphi$. By Lemma 1 part (i), $x \in G$. Thus $F \subseteq G$. Similarly, we have $G \subseteq F$ and so $F = G$. Therefore ψ is injective. Now, suppose that $G \in F(X/\varphi)$. Put $F = \{x \in X : [x]_\varphi \in G\}$. By Lemma 2, F is a filter of X containing $[1]_\varphi$. We have $\psi(F) = F/\varphi = G$ because $[x]_\varphi \in F/\varphi$, if and only if $x \in F$ if and only if $[x]_\varphi \in G$. Hence ψ is surjective. □

Lemma 3. *Let X and Y be Be-algebras, $f : X \to Y$ a homomorphism and φ a pseudo-valuation on Y. Then $\varphi \circ f : X \to \mathbb{R}$ defined by $\varphi \circ f(x) = \varphi(f(x))$ for all $x \in X$ is a pseudo-valuation on X.*

Proof. The proof is clear. □

Theorem 8. *Let X and Y be Be-algebras, $f : X \to Y$ a epimorphism and φ a pseudo-valuation on Y. Then $X/\varphi \circ f \cong Y/\varphi$.*

Proof. By Lemma 3 and Theorem 5, $X/\varphi \circ f$ and Y/φ are Be-algebras. Define $\psi : X/\varphi \circ f \to Y/\varphi$ by $\psi([x]_{\varphi \circ f}) = [f(x)]_{\varphi}$ for all $x \in X$. Suppose that $[x]_{\varphi \circ f} = [y]_{\varphi \circ f}$. Then $(x, y) \in \theta_{\varphi \circ f}$ and so $\varphi \circ f(x * y) + \varphi \circ f(y * x) = 0$. Since f is a homomorphism, $\varphi(f(x) * f(y)) + \varphi(f(y) * f(x)) = 0$. Hence $(f(x), f(y)) \in \theta_{\varphi}$ and so $[f(x)]_{\varphi} = [f(y)]_{\varphi}$. Thus $\psi([x]_{\varphi \circ f}) = \psi([y]_{\varphi \circ f})$, that is ψ is well defined. Now, we show that ψ is a homomorphism.
(i) $\psi([1]_{\varphi \circ f}) = [f(1)]_{\varphi} = [1]_{\varphi}$.
(ii) $\psi([x]_{\varphi \circ f} \odot [y]_{\varphi \circ f}) = \psi([x * y]_{\varphi \circ f}) = [f(x * y)]_{\varphi} = [f(x) * f(y)]_{\varphi} = [f(x)]_{\varphi} \odot [f(y)]_{\varphi} = \psi([x]_{\varphi \circ f}) \odot \psi([x]_{\varphi \circ f})$.
Finally, we show that ψ is a bijection. Let $[y]_{\varphi} \in Y/\varphi$. Since f is surjective, there exists $x \in X$ such that $y = f(x)$. Hence $\psi([x]_{\varphi \circ f}) = [f(x)]_{\varphi} = [y]_{\varphi}$ and ψ is surjective. Suppose that $\psi([x]_{\varphi \circ f}) = \psi([y]_{\varphi \circ f})$. Then $[f(x)]_{\varphi} = [f(y)]_{\varphi}$. Thus $\varphi(f(x) * f(y)) + \varphi(f(y) * f(x)) = 0$. Since f is a homomorphism, $\varphi \circ f(x * y) + \varphi \circ f(y * x) = 0$. Hence $[x]_{\varphi \circ f} = [x]_{\varphi \circ f}$ and ψ is injective. □

Lemma 4. *Let φ be a pseudo-valuatin on a BE-algebra X and X/φ be the corresponding quotient algebra. Then the map $\pi_{\varphi} : X \to X/\varphi$ defined by $\pi_{\varphi}(x) = [x]_{\varphi}$ for all $x \in X$ is an epimorphism.*

Corollary 2. *Let φ be a pseudo-valuatin on a BE-algebra X and X/φ be the corresponding quotient algebra. For each pseudo-valuation $\overline{\varphi_1}$ on BE-algebra X/φ, there exists a pseudo-valuation φ_1 on BE-algebra X such that $\varphi_1 = \overline{\varphi_1} \circ \pi_{\varphi}$.*

Proof. It follows from Lemma 4 and Lemma 3. □

If τ is a topology on X, then the set $\{U \subseteq X/\varphi : \pi_{\varphi}^{-1}(U) \in \tau\}$ is a topology on X/φ and is called quotient topology on X/φ.

Theorem 9. *Let φ be a pseudo-valuatin on a BE-algebra X. Then the metric topology induced by $d^*([x]_{\varphi}, [y]_{\varphi}) = d_{\varphi}(x, y)$ coincides with quotient topology on X/φ.*

Proof. Let τ_{d^*} be the topology induced by d^* and τ be the quotient topology on X/φ. We have to show that $\tau_{d^*} = \tau$. It is clear that the map $\pi_{\varphi} : X \to X/\varphi$ is continuous, because $d^*([x]_{\varphi}, [y]_{\varphi}) = d^*(\pi_{\varphi}(x), \pi_{\varphi}(y)) = d_{\varphi}(x, y)$. If $U \in \tau_{d^*}$, then $\pi_{\varphi}^{-1}(U) \in \tau_{d_{\varphi}}$ and by definition of quotient topology, $U \in \tau$. Conversely, Let $W \in \tau$. Then $\pi_{\varphi}^{-1}(W) \in \tau_{d_{\varphi}}$. Hence $\pi_{\varphi}^{-1}(W) = \cup_{x \in \pi_{\varphi}^{-1}(W)} B_{\varepsilon}(x)$. Since π_{φ} is an epimorphism, $\pi_{\varphi}(\cup_{x \in \pi_{\varphi}^{-1}(W)} B_{\varepsilon}(x)) = W$. It is easy to proof that $\pi_{\varphi}(\cup_{x \in \pi_{\varphi}^{-1}(W)} B_{\varepsilon}(x)) = \cup_{x \in \pi_{\varphi}^{-1}(W)} B^*_{\varepsilon}([x]_{\varphi}) = \cup_{x \in \pi_{\varphi}^{-1}(W)} \{[y]_{\varphi} \in X/\varphi : d^*([y]_{\varphi}, [x]_{\varphi}) < \varepsilon\} \in \tau_{d^*}$. Thus $W \in \tau_{d^*}$. Therefore $\tau_{d^*} = \tau$. □

Theorem 10. *Let φ be a pseudo-valuatin on a BE-algebra X. Then $\overline{\varphi} : X/\varphi \to \mathbb{R}$ defined by $\overline{\varphi}([x]_\varphi) = \varphi(x)$ is a pseudo-valuation.*

Proof. It is enough to show that $\overline{\varphi}$ is well defined. Let $[x]_\varphi = [x]_\varphi$. Then $\varphi(x * y) + \varphi(y * x) = 0$. Thus $\varphi(x * y) = \varphi(y * x) = 0$. Hence

$$\varphi(x) \leq \varphi(y * x) + \varphi(y) = \varphi(y) \quad \text{and} \quad \varphi(y) \leq \varphi(x * y) + \varphi(x) = \varphi(x).$$

Thus $\varphi(x) = \varphi(y)$ and so $\overline{\varphi}$ is well defined. Now, we have $\overline{\varphi}([1]_\varphi) = \varphi(1) = 0$ and

$$\begin{aligned}
\overline{\varphi}([x]_\varphi \odot [z]_\varphi) &= \overline{\varphi}([x * z]_\varphi), \\
&= \varphi(x * z), \\
&\leq \varphi(x * (y * z)) + \varphi(y), \\
&= \overline{\varphi}([x * (y * z)]_\varphi) + \overline{\varphi}([y]_\varphi) \\
&= \overline{\varphi}([x]_\varphi \odot ([y]_\varphi \odot [z]_\varphi)) + \overline{\varphi}([y]_\varphi).
\end{aligned}$$

For all $x, y, z \in X$. Therefore $\overline{\varphi}$ is a pseudo-valuation on X.

□

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