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A NUMERICAL INDIRECT METHOD FOR SOLVING A CLASS OF OPTIMAL CONTROL PROBLEMS

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Abstract

In this paper, a numerical indirect method based on wavelets is proposed for solving the general continuous time-variant linear quadratic optimal control problem. The necessary optimality conditions are applied to convert the main problem into a boundary value problem, as a dynamic system. The new problem, using two discrete schemes, Legendre and Chebyshev wavelets, is changed to a system of algebraic equations. To demonstrate the efficiency of the proposed method two analytical and two numerical examples are given.

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Key words: optimal control problem, wavelet, indirect method, general time-variant system.

1 Introduction

Optimal control problems (OCP) are dynamic optimization problems with many applications in industrial processes such as airplane, robotic arm, bioprocess system, biomedicine, electric power systems and plasma physics, etc. [3].

Using the necessary optimality conditions, the analytical solution of these problems can be achieved. In this approach, the main problem, using the calculus of variation and the Pontryagins minimum principle (PMP), leads to one of the following problems: Hamilton-Jacobi-Belman (HJB) [27], two-boundary value problem [3] and Riccati equation [12, 25]. These methods are named indirect methods [31]. The exact solution of these methods, except special cases,

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is found difficulty. So, the numerical indirect methods, such as shooting method [23], are proposed. These methods require the good initial guesses that must be in the domain of convergence.

Direct methods are other types of numerical algorithms. Based on parametrization(s) of control or/and state, the continuous problem is converted to a finite dimensional optimization problem in these methods. The quality of solution depends on discretization resolution. The pseudospectral approaches [9, 18] are examples as these methods, where the signals are approximated by Lagrange functions, as global basis, and the time nodes are the roots of orthogonal polynomials, e.g. Legendre or Chebyshev polynomials.

Using the orthogonal set of functions, as a basis for $L_2(\mathbb{R})$, is common in direct methods. The unknown functions in the problem are approximated as series of orthogonal functions with unknown coefficients. So, the dynamic equations in OCPs are converted to a system of algebraic equations. As for as we know, there are three classes of the orthogonal functions: piecewise continuous functions (e.g. Walsh [6], Block pulse [34], Haar wavelet [15]), orthogonal polynomials (e.g. Legendre [10], Chebyshev [21], Lagrange [22]), sine and cosine functions (e.g. Fourier series [28]).

The orthogonal functions are supported on the whole interval, so for functions with abrupt variations or functions which are vanishing outside a short interval, wavelets, as local basis, are introduced [7, 29]. Wavelets [8] are a family of functions which constructed from dilation and translation of a single function named mother wavelet. These functions have been extensively used to find an approximation solution for OCPs [5, 14, 16, 24, 26, 29, 30].

In this work, using two classes of wavelets, we apply a numerical indirect method for solving two-boundary value problem (TBVP) arising from the necessary optimality conditions of the general time variant linear quadratic (TVLQ) optimal control problem. The derivative of state and costate signals are expanded as wavelet basis (Legendre and Chebyshev wavelets) with unknown coefficients. Next, using the operational matrix of integration, the TBVP is converted to a system of algebraic equations, which can be solved easily by commercial packages and softwares. Usually, wavelets are used as direct method for solving OCPs. But we here apply them to make a discrete version of the two-boundary value problem.

Two main contributions of this paper are:

1. Introducing an analytical-numerical approach for solving the TVLQ problem as a main class of OCPs.

2. Converting the main problem to a set of algebraic equations instead a set of differential equations (see [29]).

The paper is organized as follows: in Section 2, the problem formulation is given. In Section 3, wavelets are introduced and two classes of them are presented. In Section 4, the proposed method is presented. In Section 5, some numerical examples are discussed. We conclude in Section 6.

2 Problem statement

We consider the general continuous TVLQ optimal control problem given in (1), in which a control function, u, is exerted over the planning horizon $[t_0, t_f]$. The goal of this problem is to find the control input $u(.) \in \mathbb{R}^r$ that minimize the cost functional with quadratic Bolza form with linear time-variant state equation, as follows:

$$(TVLQ) \begin{cases} \min \quad J = \frac{1}{2}x^T(t_f)\bar{F}x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t))dt \\ \dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in [t_0, t_f] \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where, the state matrix, $A(t)$, and the control matrix, $B(t)$, are time-variant matrices with dimension $n \times n$ and $n \times r$, respectively, $R(t)$ is a time-variant $r-$square positive symmetric matrix, $Q(t)$ is an $n-$square semi-positive symmetric matrix, $\bar{F}$ is $n-$square, $x(.) \in \mathbb{R}^n$ denotes the state vector for the system, $x_0 \in \mathbb{R}^n$ is the initial state, t_0 and t_f are constant initial and final times, respectively.
TVLQ optimal control problems have some applications in many areas, such as game theory, quantum mechanics, economy, environment problems, etc., (see [4]), or in engineering models [1].

3 Wavelet

Wavelets are a family of functions which are constructed from dilation and translation of a special function called the mother wavelet. If a and b are the dilation and the translation parameters, then the continuous wavelets are constructed as following [29]:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi(\frac{t-b}{a}), \quad a, b \in \mathbb{R}, \quad a \neq 0, \tag{2}$$

where $\psi(t)$ is the mother wavelet. If $a = a_0^{-k}$ and $b = nb_0a_0^{-k}$ ($a_0 > 1,\ b_0 > 0$ and $n,\ k$ are positive in integer numbers) then we have the following discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\psi(a_0^k t - nb_0). \tag{3}$$

The family of wavelets $\{\psi_{k,n}\}$ constructs a basis for $L^2(\mathbb{R})$. Based on the mother wavelet, we can achieve special wavelets. In the following, using two classes of polynomials, Legendre and Chebyshev, we introduce two settings of wavelets.

3.1 Chebyshev wavelets

The Chebyshev polynomials of the first kind can be constructed by the following recursive formula [13]:

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_{m+1} = 2tT_m(t) - T_{m-1}(t), \qquad m = 1, 2, \ldots. \tag{4}$$

These functions are orthogonal with respect to the weight function $w(t) = 1/\sqrt{1-t^2}$, on the interval $[-1,1]$, (see [13]). For the orthonormality, these functions are used by the following coefficients:

$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}} T_m(t), & m > 0. \end{cases} \tag{5}$$

Now, the Chebyshev wavelets (CWs) can be constructed on the time interval $[0,1)$ as follows [36]:

$$\phi^{(1)}_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{T}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & O.W., \end{cases} \tag{6}$$

where, $k \in \mathbb{N}$ is the index of dilation, or level of CWs, $n = 1, 2, \ldots, 2^{k-1}$ is the index of the translation, $m = 0, 1, \ldots, M-1$ is the order for Chebyshev polynomials and t denotes the time. In CWs, the translation and dilation parameters are $b = (2n-1)2^{-k}$ and $a = 2^{-k}$, respectively. Thus, CWs are denoted by $\phi^{(1)}_{nm} = \phi^{(1)}(k, n, m, t)$ and they are orthonormal with the following weight functions:

$$w_n(t) = w(2^k t - 2n + 1), \tag{7}$$

where, $w(t) = 1/\sqrt{1-t^2}$. The sequence $\{\phi^{(1)}_{nm}(t)\}$, in $L^2[0,1]_{w_n}$, will be used as basis for the discretization of TBVP in the next section.

The vector of CWs is defined as following:

$$\phi^{(1)}_{(M)} = [\phi^{(1)}_{10}, \phi^{(1)}_{11}, \ldots, \phi^{(1)}_{1M-1}, \phi^{(1)}_{20}, \phi^{(1)}_{21}, \ldots, \phi^{(1)}_{2M-1}, \ldots, \phi^{(1)}_{2^{k-1}0}, \ldots, \phi^{(1)}_{2^{k-1}M-1}]^T. \tag{8}$$

For computational experiment, we also need the operational matrix of integration of the vector of CWs, i.e. the matrix $P^{(1)}$ in following approximation [20]:

$$\int_0^t \phi^{(1)}_{(M)}(t)dt \simeq P^{(1)} \phi^{(1)}_{(M)}(t), \tag{9}$$

where, $P^{(1)}$ as the operational matrix is as $2^{k-1}M \times 2^{k-1}M$ square matrix (for more details see [20]).

3.2 Legendre wavelets

Legendre wavelets (LWs) are constructed by Legendre polynomials, which are defined by the following recessive formula [19]:

$$L_0(t) = 1,\ L_1(t) = t,\ L_{m+1}(t) = \frac{2m+1}{m+1} t L_m(t) - \frac{m}{m+1} L_{m-1}(t),\ m = 1, 2, 3, \ldots \tag{10}$$

where, $L_m(t),\ t = 0, 1, 2, \dots$ is a Legendre polynomial with order m. These functions are orthogonal with respect the weight function $w(x) = 1$ on the interval $[-1, 1]$. The Legendre wavelets are constructed on $[0, 1]$ as follows [29]:

$$\phi^{(2)}_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{\frac{k}{2}} L_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t \le \frac{n}{2^{k-1}} \\ 0, & O.W., \end{cases} \tag{11}$$

where, $n = 1, 2, \dots, 2^{k-1}$, k is any positive integer number and m is the order of Legendre polynomial. In (11), the coefficient $\sqrt{m+\frac{1}{2}}$ is applied to have the orthogonality and the translation and dilation parameters are $b = (2n-1)2^{-k}$ and $a = 2^{-k}$, respectively. The sequence $\{\phi^{(2)}_{nm}(t)\}$ in space $L^2[0, 1]_w$ is used to discrete TBVP in the next section. The vector of LWs is defined as follows:

$$\phi^{(2)}_{(M)} = [\phi^{(2)}_{10}, \phi^{(2)}_{11}, \dots, \phi^{(2)}_{1M-1}, \phi^{(2)}_{20}, \phi^{(2)}_{21}, \dots, \phi^{(2)}_{2M-1}, \dots, \phi^{(2)}_{2^{k-1}0}, \dots, \phi^{(2)}_{2^{k-1}M-1}]^T. \tag{12}$$

The integrate of LW vector, $\phi^{(2)}_{(M)}(t)$, can be approximate as following [29]:

$$\int_0^t \phi^{(2)}_{(M)}(t)dt \simeq P^{(2)}\phi^{(2)}_{(M)}(t), \tag{13}$$

where, $P^{(2)}$ as the operational matrix is as $2^{k-1}M \times 2^{k-1}M$ square matrix (for more details see [29]).

3.3 Function approximation

Each integrable function as $f(t) \in L^2[0, 1)$ can be approximated by LWs or CWs. Let $\psi_{ij}(t)$ be the wavelet basis, in which for CW basis $\psi_{ij}(t) = \phi^{(1)}_{ij}(t)$ (see (6)) or for LW basis $\psi_{ij}(t) = \phi^{(2)}_{ij}(t)$ (see (11)), for $i = 1, 2, \dots, 2^{k-1}$, $j = 0, 1, \dots, M-1$. Now, let:

$$c_{ij} = \frac{\langle f(t), \psi_{i,j}(t)\rangle_w}{\langle \psi_{i,j}(t), \psi_{i,j}(t)\rangle_w},\ i = 1, 2, \dots, 2^{k-1},\ j = 0, 1, \dots, M-1, \quad t \in [0, 1], \tag{14}$$

where, the notation $\langle , \rangle_w$, denotes the inner product and defined as following:

$$\langle f(t), g(t)\rangle_w = \int_0^1 w(t)f(t)g(t)dt. \tag{15}$$

Now, let:

$$C = [c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, c_{21}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}]^T. \tag{16}$$

The function $f(t)$ can be approximated by wavelet basis [36]:

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t). \tag{17}$$

The eqn (17) can be represented as following:

$$f(t) \simeq C^T \psi_{(M)}(t). \tag{18}$$

where $\psi_{(M)}(t) = \phi^{(1)}_{(M)}(t)$, for CW and $\psi_{(M)}(t) = \phi^{(2)}_{(M)}(t)$, for LW, and C is defined in (16).

Remark 1. *Every constant scaler, $x \in \mathbb{R}$, can be approximated by wavelets (LW or CW). Let:*

$$\bar{X} = [x, \overbrace{0, \ldots, 0}^{(M-1)times}, x, \overbrace{0, \ldots, 0}^{(M-1)times}, \ldots, x, \overbrace{0, \ldots, 0}^{(M-1)times}]^T_{M2^{k-1}}, \tag{19}$$

then, $x \simeq \bar{X}^T \psi_{(M)}(t)$. Moreover, for $x \in \mathbb{R}^n$ we have $x \simeq X\psi_{(M)}(t)$, where $X_{n\times M2^{k-1}}$ is the coefficient matrix, when each row of X is constructed by (19).

Remark 2. *The product of two wavelet vectors can be approximated by wavelet. Let c be $M2^{k-1} \times 1$ constant vector. Then we have:*

$$\psi_{(M)}(t)\psi^T_{(M)}(t)c \simeq \tilde{C}\psi_{(M)}(t), \tag{20}$$

where, $\tilde{C}_{M2^{k-1}\times M2^{k-1}}$ is the product operational matrix (see [20] for CW and [29] for LW).

4 The proposed approach

In this section, we give a numerical indirect method for solving TVLQ optimal control problem, based on CW and LW approximations. Without loss of the generality, we can assume that $[t_0, t_f] = [0, 1]$ by using a simple transformation $\tau = (t - t_0)/(t_f - t_0)$, the time interval $[t_0, t_f]$ can be embedded in $[0, 1]$. Firstly, the problem is converted into a TBVP, using necessary optimality conditions, PMP principle. Next, the new problem is converted into a system of the matrix algebraic equations, using CW and LW.

4.1 Necessary optimality conditions

For the TVLQ problem given in (1), the necessary optimality conditions, which can be achieved by variational approach, are as follows [27]:

$$\frac{\partial H}{\partial u} = R(t)u(t) + \lambda^T(t)B(t) = 0, \tag{21}$$

$$\dot{x} = \frac{\partial H}{\partial \lambda} = A(t)x(t) + B(t)u(t), \tag{22}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -Q(t)x(t) - A^T(t)\lambda(t), \tag{23}$$

$$\lambda(t_f) = \bar{F}x(t_f), \quad x(t_0) = x_0., \tag{24}$$

where, the scaler function H, called Hamiltonian, is defined as follows:

$$
\begin{aligned}
H(x(t),u(t),\lambda(t)) \quad &= \tfrac{1}{2}(x^T(t)Q(t)x(t)+u^T(t)R(t)u(t))+ \\
&\quad \lambda^T(t)(A(t)x(t)+B(t)u(t)),
\end{aligned}
\tag{25}
$$

where, $\lambda(t)_{n\times 1}$ is the Lagrange multiplier that is also known as the costate or adjoint variable. From (21), the control signals can be represented by the costate vector as $u(t) = -R(t)^{-1}B^T(t)\lambda(t)$. By replacing this equation in eqn (22), we have:

$$
\dot{x} = A(t)x(t) - \bar{B}(t)\lambda(t), \tag{26}
$$

$$
\dot{\lambda} = -A^T(t)\lambda(t) - Q(t)x(t), \tag{27}
$$

$$
\lambda(t_f) = \bar{F}x(t_f), \quad x(t_0) = x_0. \tag{28}
$$

where, $\bar{B}(t) = B(t)R^{-1}(t)B(t)^T$. The eqns (26)-(28), are constructed a TBVP of $2n$ dynamic equations with $2n$ unknowns. Here, we propose a numerical method based on CWs and LWs to solve it.

4.2 Solving TBVP by a system of algebraic equations

In this section, we use wavelets (CWs (see section 3.1) or LWs (see section 3.2)), to solve TBVP, approximately. For this purpose, we approximate each component of TBVP in system (26)-(28), which contains coefficient matrices, state and costate signals.

The control matrix, $A(t)$, in eqn (26), could be approximated by wavelets. Let

$$
A(t) = \begin{bmatrix} \alpha_1(t) & \dots & \alpha_n(t) \\ \alpha_{n+1}(t) & \dots & \alpha_{2n}(t) \\ \vdots & \ddots & \vdots \\ \alpha_{n^2-n+1}(t) & \dots & \alpha_{n^2}(t) \end{bmatrix} = \sum_{i=1}^{n^2} A_i\alpha_i(t), \tag{29}
$$

where $\alpha_i(t)$, for $i = 1,2,\dots,n^2$ are continuous scaler functions, in $L^2[0,1]$ and $A_i,\ i = 1,2,\dots,n^2$, is constructed as follows:

$$
(A_i)_{kj} = \begin{cases} 1 & ,k = [\frac{i}{n}]+1, j = i-(k-1)n, \\ 0 & ,O.W. \end{cases} \tag{30}
$$

Now, similar to (18), $\alpha_i(t) \simeq d_i^T\psi_{(M)}(t)$, where d_i are defined in (16). Therefore, we have

$$
A(t) \simeq \sum_{i=1}^{n^2} A_i d_i^T \psi_{(M)}(t). \tag{31}
$$

By a very similar manner, the matrices $Q(t)$ and $\bar{B}(t)$ replaced by following:

$$
Q(t) \simeq \sum_{i=1}^{n^2} Q_i h_i^T \psi_{(M)}(t), \qquad \bar{B}(t) \simeq \sum_{i=1}^{n^2} \bar{B}_i e_i^T \psi_{(M)}(t), \tag{32}
$$

where, Q_i and $\bar{B}_i$, $i = 1, 2, \dots, n^2$, have a similar definition given in (30).

The derivative of the control and costate variables can be approximated by unknown wavelet coefficients, as follows (see Remark 1):

$$\dot{x}(\tau) \simeq F\psi_{(M)}(\tau), \tag{33}$$

$$\dot{\lambda}(\tau) \simeq G\psi_{(M)}(\tau), \tag{34}$$

where, F and G are unknowns $n \times m$ matrices ($m = M2^{k-1}$) . Now, by the wavelet operational matrix, $P_{(m\times m)}$ (for CWs, $P = P^{(1)}$ in eqn (9) and for LWs, $P = P^{(2)}$ in eqn (13)) and integrating of both sides of eqns (33) and (34), the state and costate signals can be calculated as follows:

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)d\tau \simeq x_0 + F\int_0^t \psi_{(M)}(t) \simeq (X_0 + FP)\psi_{(M)}(t), \tag{35}$$

$$\lambda(t) = \lambda(0) + \int_0^t \dot{\lambda}(\tau)d\tau \simeq \lambda_0 + G\int_0^t \psi_{(M)}(t) \simeq (\Lambda_0 + GP)\psi_{(M)}(t). \tag{36}$$

In (36), $X_0 = [x_0, 0, \dots, 0]$ and $\Lambda_0 = [\lambda_0, 0, \dots, 0]$ are $n \times m$ matrices, where $\lambda(0) \simeq \Lambda_0\psi_{(M)}(t)$ and $x(0) \simeq X_0\psi_{(M)}(t)$ (see Remark 1). For TBVP, given in (26)-(28), the initial condition of the costate vector, $\lambda_0 = \lambda(t_0)$, is not available. From eqn (36), let $t = 1$, we have:

$$\lambda(1) = \lambda(0) + \int_0^1 \dot{\lambda}(\tau)d\tau \simeq \lambda(0) + G\int_0^1 \psi_{(M)}(\tau)d\tau. \tag{37}$$

But, from Remark 1, we have $\int_0^1 \psi_{(M)}(\tau)d\tau \simeq O\psi_{(M)}(t)$, where O is an $m \times m$ zero matrix which the first entire is one. Therefore, we have:

$$\lambda(1) \simeq \lambda_0 + GO\psi_{(M)}(t) \simeq (\Lambda_0 + GO)\psi_{(M)}(t). \tag{38}$$

Using a very similar manner, we have:

$$x(1) \simeq (X_0 + GO)\psi_{(M)}(t). \tag{39}$$

But, from the boundary conditions given in eqn (28), we have $\lambda(1) = \bar{F}x(1)$. So, from (38) and (39), the initial condition of the costate vector is approximated as follows:

$$\Lambda_0 \simeq \bar{F}(X_0 + FO) - GO. \tag{40}$$

4.2.1 TBVP approximation

Using the wavelet approximation of matrices, eqns (31)-(32), state and costate signals, eqns (35)-(36), we now can approximate TBVP, (26)-(28), based on CWs or LWs. For this purpose, by replacing eqns (31), (32) and (33)-(36) in dynamic system (26)-(28), we have:

$$\begin{cases} F\psi_{(M)}(t) &= \sum\limits_{i=1}^{n^2} A_i d_i^T \psi_{(M)}(t)(X_0+FP)\psi_{(M)}(t) \\ & - \sum\limits_{i=1}^{n^2} \bar{B}_i e_i^T \psi_{(M)}(t)(\Lambda_0+GP)\psi_{(M)}(t), \\ G\psi_{(M)}(t) &= -\sum\limits_{i=1}^{n^2} \psi_{(M)}^T(t) d_i A_i^T (\Lambda_0+GP)\psi_{(M)}(t) \\ & - \sum\limits_{i=1}^{n^2} Q_i h_i^T \psi_{(M)}(t)(X_0+FP)\psi_{(M)}(t). \end{cases} \tag{41}$$

By rearranging each component of the system (41), we have:

$$\begin{cases} F\psi_{(M)}(t) &= \sum\limits_{i=1}^{n^2} A_i(X_0+FP)\psi_{(M)}(t)\psi_{(M)}^T(t) d_i \\ & - \sum\limits_{i=1}^{n^2} \bar{B}_i(\Lambda_0+GP)\psi_{(M)}(t)\psi_{(M)}^T(t) e_i, \\ G\psi_{(M)}(t) &= -\sum\limits_{i=1}^{n^2} A_i^T(\Lambda_0+GP)\psi_{(M)}(t)\psi_{(M)}^T(t) d_i \\ & - \sum\limits_{i=1}^{n^2} Q_i(X_0+FP)\psi_{(M)}(t)\psi_{(M)}^T(t) h_i. \end{cases} \tag{42}$$

From (20), there exist m-square matrices D_i, E_i and H_i, for $i = 1, 2, \dots, n^2$, such that:

$$\psi_{(M)}(t)\psi_{(M)}^T(t) d_i \simeq D_i \psi_{(M)}(t), \tag{43}$$

$$\psi_{(M)}(t)\psi_{(M)}^T(t) e_i \simeq E_i \psi_{(M)}(t), \tag{44}$$

$$\psi_{(M)}(t)\psi_{(M)}^T(t) h_i \simeq H_i \psi_{(M)}(t). \tag{45}$$

From (43)-(45) and (42), we have:

$$\begin{cases} F\psi_{(M)}(t) = \sum\limits_{i=1}^{n^2} A_i(X_0+FP)D_i\psi_{(M)}(t) - \sum\limits_{i=1}^{n^2} \bar{B}_i(\Lambda_0+GP)E_i\psi_{(M)}(t), \\ G\psi_{(M)}(t) = -\sum\limits_{i=1}^{n^2} A_i^T(\Lambda_0+GP)D_i\psi_{(M)}(t) - \sum\limits_{i=1}^{n^2} Q_i(X_0+FP)H_i\psi_{(M)}(t). \end{cases} \tag{46}$$

By removing the wavelet vector $\psi_{(M)}(t)$ from both sides of system (46), and replacing Λ_0 from eqn (40), we have:

$$\begin{cases} F = \sum\limits_{i=1}^{n^2} A_i(X_0+FP)D_i - \sum\limits_{i=1}^{n^2} \bar{B}_i(\bar{F}X_0 + \bar{F}FO + G(P-O))E_i, \\ G = -\sum\limits_{i=1}^{n^2} A_i^T(\bar{F}X_0 + \bar{F}FO + G(P-O))D_i - \sum\limits_{i=1}^{n^2} Q_i(X_0+FP)H_i. \end{cases} \tag{47}$$

Let:

$$\Delta_1 = \sum_{i=1}^{n^2} A_i X_0 D_i - \sum_{i=1}^{n^2} \bar{B}_i \bar{F} X_0 E_i, \qquad \Delta_2 = -\sum_{i=1}^{n^2} A_i^T \bar{F} X_0 D_i - \sum_{i=1}^{n^2} Q_i X_0 H_i. \tag{48}$$

Now, from eqn (48) and rearranging the system (47), we have:

$$\begin{cases} (F - \sum_{i=1}^{n^2} A_i F P D_i + \sum_{i=1}^{n^2} \bar{B}_i \bar{F} F O E_i) + \sum_{i=1}^{n^2} \bar{B}_i G(P-O)E_i = \Delta_1, \\ (\sum_{i=1}^{n^2} A_i^T \bar{F} F O D_i + \sum_{i=1}^{n^2} Q_i F P H_i) + (G + \sum_{i=1}^{n^2} A_i^T G(P-O)D_i) = \Delta_2. \end{cases} \tag{49}$$

To convert the system (49) to a linear system, we use the following definition and lemma given by [17].

Definition 1. *[17] Let $A = [a_1, a_2, \ldots, a_m] \in \mathbb{R}^{n\times m}$ be a matrix. The linear operator $vec(.)$ is defined as follows:*

$$vec(A) = [a_1^T, a_2^T, \ldots, a_m^T]^T.$$

Lemma 1. *Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{p\times q}$ and $X \in \mathbb{R}^{m\times p}$. We have:*

$$vec(AXB) = (B^T \otimes A)vec(X),$$

where, $\otimes$ is the Kronecker product of matrices (see [17]).

Now, from system (49), Definition 1 and Lemma 1, we have:

$$\begin{cases} \{I - \sum_{i=1}^{n^2} (PD_i)^T \otimes A_i + \sum_{i=1}^{n^2} (OE_i)^T \otimes (\bar{B}_i \bar{F})\} vec(F) \\ \qquad + \{\sum_{i=1}^{n^2} ((P-O)E_i)^T \otimes \bar{B}_i\} vec(G) = vec(\Delta_1), \\ \{\sum_{i=1}^{n^2} (OD_i)^T \otimes (A_i^T \bar{F}) + \sum_{i=1}^{n^2} (PH_i)^T \otimes Q_i\} vec(F) \\ \qquad + \{I + \sum_{i=1}^{n^2} ((P-O)D_i)^T \otimes A_i^T\} vec(G) = vec(\Delta_2). \end{cases} \tag{50}$$

To have a convenient form of (50), let:

$$\bar{A} = \begin{bmatrix} I - \sum_{i=1}^{n^2} (PD_i)^T \otimes A_i + \sum_{i=1}^{n^2} (OE_i)^T \otimes (\bar{B}_i \bar{F}) & \sum_{i=1}^{n^2} ((P-O)E_i)^T \otimes \bar{B}_i \\ \sum_{i=1}^{n^2} (OD_i)^T \otimes (A_i^T \bar{F}) + \sum_{i=1}^{n^2} (PH_i)^T \otimes Q_i & I + \sum_{i=1}^{n^2} ((P-O)D_i)^T \otimes A_i^T \end{bmatrix},$$

$$\bar{b} = \begin{bmatrix} vec(\Delta_1) \\ vec(\Delta_2) \end{bmatrix} = \begin{bmatrix} vec(\sum_{i=1}^{n^2} A_i X_0 D_i - \sum_{i=1}^{n^2} \bar{B}_i \bar{F} X_0 E_i) \\ vec(-\sum_{i=1}^{n^2} A_i^T \bar{F} X_0 D_i - \sum_{i=1}^{n^2} Q_i^T X_0 H_i) \end{bmatrix}, \quad x = \begin{bmatrix} vec(F) \\ vec(G) \end{bmatrix}.$$

So, the system (50) can be shown by:

$$\bar{A}x = \bar{b}. \tag{51}$$

Therefore, to have an approximation solution of TVLQ, we can solve only the linear system (51).

Remark 3. *The linear system* (51) *contains* $nM2^{k-1}$ *equations and* $nM2^{k-1}$ *unknowns.*

Remark 4. *For a linear quadratic time-variant system with time-invariant system, i.e. when* $A(t) = A$, $B(t) = B$, $Q(t) = Q$ *and* $R(t) = R$ *are constant matrices, the linear system* (51) *can be simplified as follows:*

$$\begin{bmatrix} I - P^T \otimes A + O^T \otimes (\bar{B}\bar{F}) & (P-O)^T \otimes \bar{B} \\ P^T \otimes Q + O^T \otimes (A^T\bar{F}) & I + (P-O)^T \otimes A^T \end{bmatrix} x = \begin{bmatrix} vec(AX_0 - \bar{B}\bar{F}X_0) \\ vec(-QX_0 - A^T\bar{F}X_0) \end{bmatrix}. \tag{52}$$

5 Numerical experiments

In this section, to investigate the efficiency and simplicity of the proposed method, four numerical examples are considered. The numerical results are obtained with two classes of wavelet basis, LW and CW, with different parameters. Also, the results are compared with the results of *bvp4c* as a standard package provided in Matlab, or the exact solution (if that exists).

For the first two examples with time variant system, which don't have analytical solutions, the results are compared with *bvp4c*. For the second two examples with time-invariant system, the numerical results are compared with the exact solution. In this case, the exact error of approximation for the control and state signals are calculated as follows:

$$e_2(f) = \sqrt{\int_0^1 (\tilde{f}(t) - f^*(t))^2 dt.}, \tag{53}$$

where, $f^*(t)$ is the exact signal and $\tilde{f}(t)$ is the approximated signal. Also, we use absolute error for the performance index, $E_J = |J - J^*|$. Moreover, the CW and LW basis are compared in the proposed method.

The method is implemented in Matlab R2011a environment on a Notebook under Windows 7 Ultimate, CPU 2.53 GHz and 4.00 GB of RAM.

5.1 Example 1 [2]

Consider the following TVLQ with single-input scaler system [2]:

$$\begin{aligned} J &= \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \\ \dot{x}(t) &= tx(t) + u(t), \\ x(0) &= 1. \end{aligned}$$

The numerical solution of the corresponding Riccati equation with four different orthogonal polynomials was reported in [2]. Table 1 shows the error of optimal feedback gain in time nods, $t_j = j/10$, $j = 0, 1, \ldots, 9$, for the proposed method (CW and LW), with two classes of parameters, $M = 2$, and $k = 3, 4$, and *bvp4c*.

Table 1: The obtained optimal feedback gain for LW and CW and *bvp4c* methods, for Example 1.

			CW $(M=2)$		LW $(M=2)$	
t	Exact[2]	*bvp4c*	$k=3$	$k=4$	$k=3$	$k=4$
0	0.9689	4.88×10^{-4}	1.74×10^{-4}	7.32×10^{-4}	1.02×10^{-3}	1.35×10^{-3}
0.1	0.9518	2.52×10^{-3}	1.51×10^{-3}	9.03×10^{-4}	1.05×10^{-3}	1.53×10^{-3}
0.2	0.9109	3.81×10^{-4}	7.4×10^{-5}	1.38×10^{-3}	8.4×10^{-5}	8.90×10^{-3}
0.3	0.8444	1.15×10^{-3}	9.12×10^{-4}	9.22×10^{-4}	1.50×10^{-4}	6.22×10^{-4}
0.4	0.7526	5.67×10^{-4}	2.31×10^{-3}	6.97×10^{-4}	1.8×10^{-3}	1.66×10^{-4}
0.5	0.6387	1.54×10^{-3}	6.87×10^{-3}	5.07×10^{-4}	8.62×10^{-3}	1.58×10^{-3}
0.6	0.5088	7.13×10^{-4}	4.41×10^{-3}	6.11×10^{-4}	3.1×10^{-3}	4.9×10^{-5}
0.7	0.3713	7.46×10^{-4}	7.58×10^{-4}	1.23×10^{-3}	9.0×10^{-5}	8.16×10^{-4}
0.8	0.2354	4.08×10^{-4}	1.91×10^{-3}	2.34×10^{-3}	6.2×10^{-4}	9.72×10^{-4}
0.9	0.1095	4.12×10^{-4}	5.97×10^{-3}	1.01×10^{-3}	3.81×10^{-3}	1.55×10^{-4}
	e_2	3.49×10^{-2}	1.07×10^{-2}	3.36×10^{-3}	1.02×10^{-2}	9.37×10^{-3}

Table 2: The obtained performance index for LW and CW methods and *bvp4c* for Example 1.

	CW $(M=2)$		LW $(M=2)$	
bvp4c	$k=3$	$k=4$	$k=3$	$k=4$
0.5948	0.4866	0.4848	0.4843	0.4843

From Table 1, it is evident that the error of the optimal feedback gain, e_2, for CW $(M=2,\ k=4)$ is less than other methods. Table 2 shows the obtained performance index for LW and CW methods $(M=2,\ k=3,4)$ and *bvp4c*. From this table, the performance index for the LW and CW methods are less than *bvp4c* and LW is better than CW. Note that in [2], there aren't any report of the exact and approximated value of the performance index.

To have a graphical comparison, the graph of the optimal feedback gains for the proposed method (LW and CW), with parameters $M=2,\ k=4$ and *bvp4c*, is shown in Fig1. Also, in Fig 2, the state and control signals for these methods, with the parameters $M=2,\ k=4$, are shown.

5.2 Example 2 [16]

Let the following three-dimensional TVLQ problem [16]:

$$
\begin{aligned}
&\min \quad J=\int_0^5 (x^T(t)Q(t)x(t)+u^T(t)R(t)u(t))dt,\\
&\dot{x}=\begin{bmatrix}0.01t & 0.01 & 0\\ 0 & 0.01t & 0\\ 0 & 0 & 0.01t^2\end{bmatrix}x(t)+\begin{bmatrix}0\\0.01\\0.01\end{bmatrix}u(t),\\
&x(0)=[111]^T.
\end{aligned}
$$

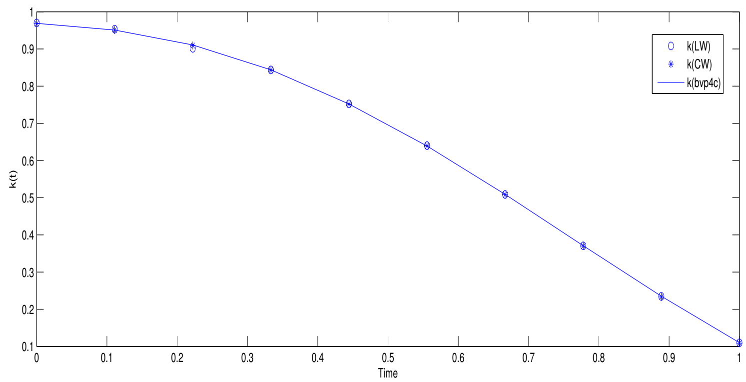

Figure 1: Graphical results of the optimal feedback gain for CW, LW ($M = 2, k = 4$) and bvp4c methods, for Example 1.

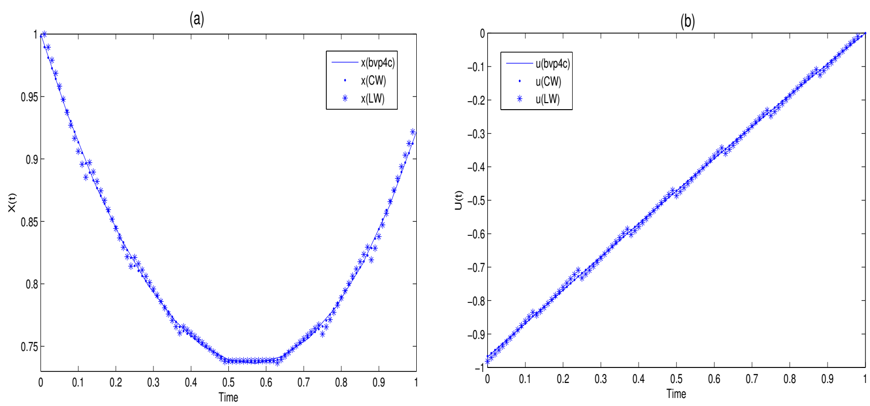

Figure 2: Graphical results of the state (a) and control (b) signals for CW, LW ($M = 2, k = 4$) and bvp4c methods, for Example 1.

where,

$$Q(t) = \begin{bmatrix} 0.01t & 0 & 0 \\ 0 & 0.01t & 0 \\ 0 & 0 & 0.01t \end{bmatrix}, \qquad R(t) = 0.01 \exp(-0.01t).$$

The problem was solved by the proposed method (LW and CW) and the *bvp4c*. The value of state signals, $x_i(t),\ i = 1, 2, 3$, and control signal, $u(t)$, in time nodes $t_j = j/2,\ j = 0, 1, \ldots, 10$ are reported in Table 3.

Table 4 compares the values of the performance index for LW, CW and *bvp4c* methods. From Table 4, the value of performance index, for CW ($M = 2, k = 4$) is the best among other methods.

Fig 3 shows the sate signals, $x_i(t),\ i = 1, 2, 3$, for LW (a), CW (b), and *bvp4c* methods. Also, the graphical comparison of the control signal for these methods is given in Fig 4.

Table 3: The values of the control, u and state signals x_1, x_2, x_3, by LW and CW ($M = 2,\ k = 4$), in Example 2.

	$x_1(t_j)$		$x_2(t_j)$		$x_3(t_j)$		$u(t_j)$	
t	LW	CW	LW	CW	LW	CW	LW	CW
0	0.9989	0.9996	0.9998	0.9996	1	0.9997	−0.3261	−0.3246
0.5	1.0069	1.0063	0.9997	0.9997	0.9988	0.9989	−0.3212	−0.3207
1	1.0158	1.0152	1.0023	1.002	1.0007	1.0007	−0.3115	−0.3113
1.5	1.0267	1.0265	1.0069	1.0067	1.0071	1.0074	−0.297	−0.2964
2	1.0396	1.0403	1.0135	1.0138	1.0192	1.0212	−0.2775	−0.275
2.5	1.0548	1.0568	1.0223	1.0235	1.0386	1.0443	−0.2527	−0.2473
3	1.0794	1.077	1.0386	1.0369	1.0892	1.0858	−0.208	−0.211
3.5	1.1015	1.1002	1.0542	1.0532	1.1474	1.1457	−0.1668	−0.1681
4	1.1263	1.1264	1.0723	1.0724	1.2263	1.2295	−0.1201	−0.1184
4.5	1.1538	1.1559	1.0931	1.0948	1.3302	1.3436	−0.0679	−0.0626
5	1.8993	1.8932	1.1231	1.1245	1.5023	1.5017	0.0003	0.0004

Table 4: The obtained performance index for LW, CW and *bvp4c* methods for Example 2.

	CW ($M = 2$)		LW ($M = 2$)	
bvp4c	$k = 3$	$k = 4$	$k = 3$	$k = 4$
0.4978	0.4737	0.4716	0.4743	0.4721

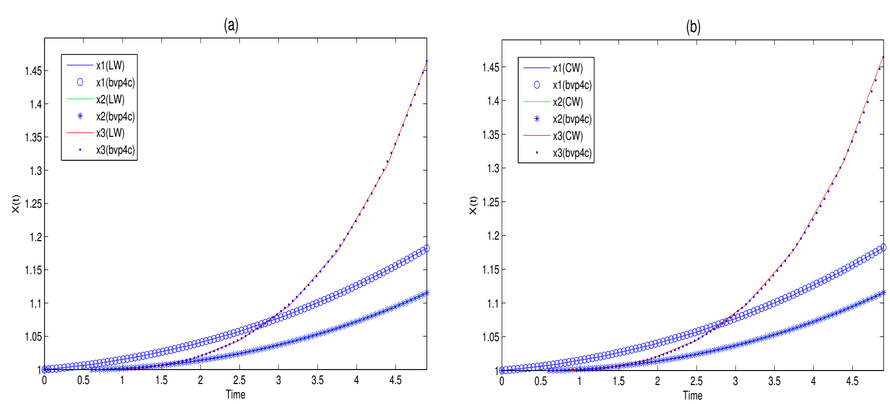

Figure 3: The sate signals, $x_i(t),\ i = 1, 2, 3$, of the LW (a), CW (b) and *bvp4c* methods, for Example 2.

5.3 Example 3 [11]

Consider the following one-dimensional time-invariant linear quadratic OCP [11]:

$$\begin{aligned} &\min \quad J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \\ &\dot{x} = -2x(t) + u(t), \\ &x(0) = 1. \end{aligned}$$

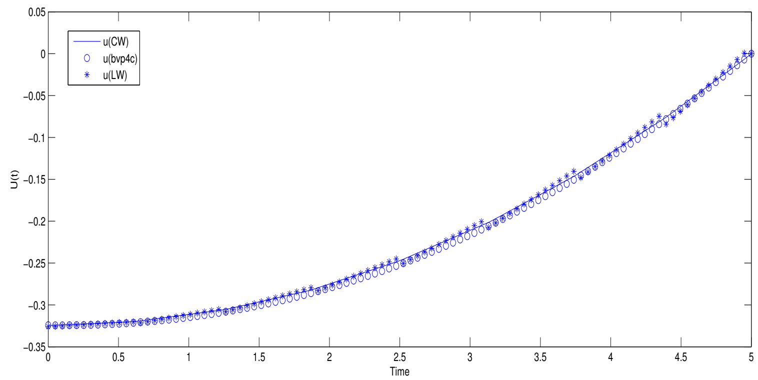

Figure 4: The control signal of the LW, CW and *bvp4c* methods, for Example 2.

The exact solution can be easily achieved by PMP. The exact solution is [11]:

$$\begin{aligned}
x^*(t) &= c_1 e^{\sqrt{5}t} + c_2 e^{-\sqrt{5}t},\\
u^*(t) &= (2+\sqrt{5}t)c_1 e^{-\sqrt{5}t} + (2-\sqrt{5}t)c_2 e^{-\sqrt{5}t},\\
k^*(t) &= \frac{\sqrt{5}\cosh\sqrt{5}(1-t) - \sinh\sqrt{5}(1-t)}{\sqrt{5}\cosh\sqrt{5}(1-t) + 3\sinh\sqrt{5}(1-t)},
\end{aligned}$$

where, $c_1 = -0.0017$ and $c_2 = 1.0017$. The error of the approximate signals, including the state error, $e_2(x)$, the control error, $e_2(u)$ and feedback gain error, $e_2(k)$, for the proposed method (CW and LW), with two classes of parameters, $(M = 2,\ k = 3, 4)$, are reported in Table 5. The exact performance index in this problem is $J^* = 0.121767$. The absolute error of performance index, $e(J)$, for the the proposed method is denoted in the last row of the Table 5.

Table 5: The error of the approximate signals and the performance index for LW and CW for Example 3.

	CW $(M=2)$		LW $(M=2)$	
Error	$k=3$	$k=4$	$k=3$	$k=4$
$e_2(x)$	7.1×10^{-3}	3.8×10^{-3}	6.8×10^{-3}	3.7×10^{-3}
$e_2(u)$	3.8×10^{-3}	3.5×10^{-3}	3.7×10^{-3}	3.5×10^{-3}
$e_2(k)$	1.2368	1.55×10^{-2}	1.55×10^{-3}	1.55×10^{-3}
$e(J)$	3.43×10^{-3}	4.01×10^{-3}	4.18×10^{-3}	4.18×10^{-3}

From Table 5, it is obvious that the error for state, control and feedback gain of CW method with the parameters $(M = 2,\ k = 4)$ is less than the other methods. Though, the performance index for CW with the parameter $(M = 2,\ k = 3)$ is less than others.

The graphical comparison of the exact and approximation signals, with CW and LW $(M = 2,\ k = 4)$, for the state and control signals are shown in Fig 5.

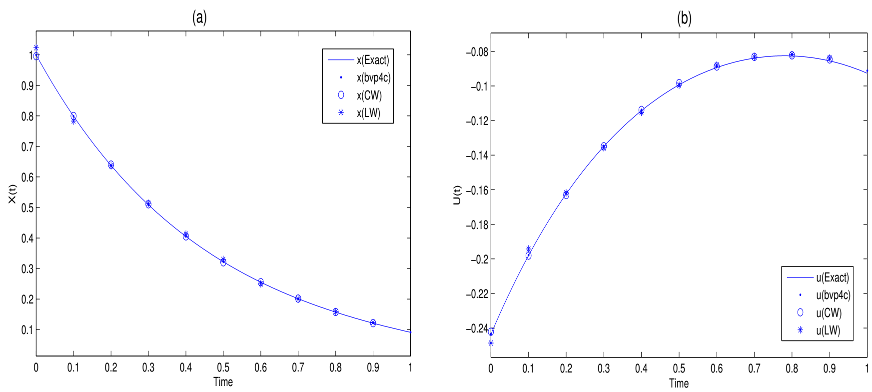

Figure 5: Comparison of the state (a) and control (b) signals for *bvp4c*, LW and CW ($M = 2,\ k = 4$) methods with exact signals, for Example 3.

5.4 Example 4 [32]

Consider the following two dimensional time invariant linear quadratic OCP [32]:

$$\begin{aligned}
&\min \quad J = \int_0^{\frac{\pi}{2}} (x^T(t) \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} x(t) + u^2(t))dt, \\
&\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\
&x(0) = [1, 1]^T.
\end{aligned}$$

The exact feedback gain for this problem is [32]:

$$K^*(t) = \left(\frac{\sinh(\pi - 2t) - \sin(\pi - 2t)}{\cosh^2(\frac{\pi}{2} - t) + \cos^2(\frac{\pi}{2} - t)}, \frac{\cosh(\pi - 2t) - \cos(\pi - 2t)}{\cosh^2(\frac{\pi}{2} - t) + \cos^2(\frac{\pi}{2} - t)} \right).$$

The exact signals, states and control, can be calculated by the $K^*(t)$. The errors of the proposed methods (CW and LW), with two classes of parameters, ($M = 2,\ k = 3, 4$), are reported in Table 6. The errors of state signals, $x_1(t)$ and $x_2(t)$, are shown by $e_2(x_1)$ and $e_2(x_2)$, respectively. Also the errors of the control signal and the feedback gains are shown with $e_2(u)$, $e_2(k_1)$ and $e_2(k_2)$, respectively. The exact performance index in this problem is $J^* = 3.2611$. The results given in Table 6 are the errors of signals. Also, LW is more accurate than CW.

The graphical results of the proposed method, LW and CW, are compared with the exact solution and *bvp4c* in Fig 6 and Fig 7.

These four examples show that the method proposed in this paper is proper and simple.

Table 6: The error of optimal signals, feedback gains and performance index for LW and CW for Example 4.

	CW $(M=2)$		LW $(M=2)$	
Error	$k=3$	$k=4$	$k=3$	$k=4$
$e_2(x_1)$	2.14×10^{-2}	7.0×10^{-3}	1.68×10^{-2}	6.1×10^{-3}
$e_2(x_2)$	1.04×10^{-2}	4.8×10^{-3}	1.0×10^{-2}	4.5×10^{-3}
$e_2(u)$	2.55×10^{-2}	7.1×10^{-3}	2.08×10^{-2}	6.4×10^{-3}
$e_2(k_1)$	0.3646	2.31×10^{-2}	0.5899	1.36×10^{-2}
$e_2(k_2)$	0.5552	4.23×10^{-2}	0.9143	3.11×10^{-2}
$e(J)$	7.51×10^{-2}	1.16×10^{-2}	8.03×10^{-4}	2.29×10^{-4}

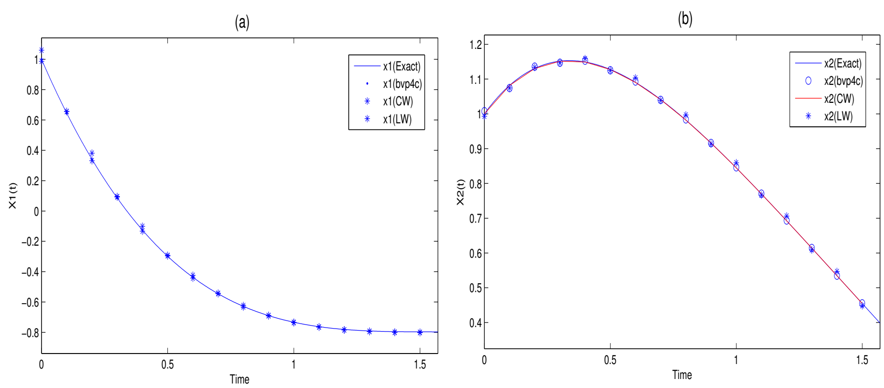

Figure 6: Comparison of the state signals, $x_1(t)$ (a), $x_2(t)$ (b), for *bvp4c*, LW and CW $(M=2,\ k=4)$ methods, for Example 4.

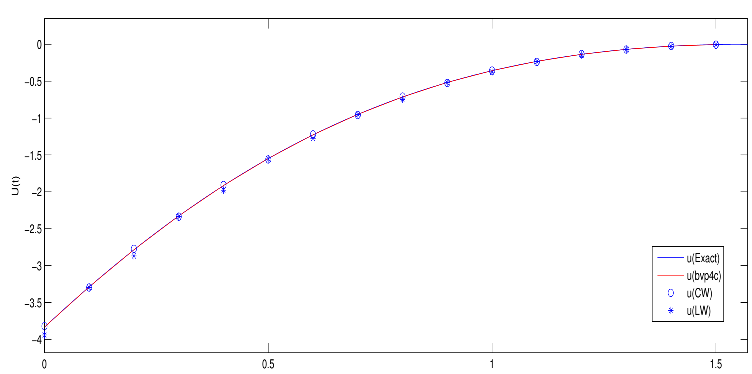

Figure 7: Comparison of the control signal for *bvp4c*, LW and CW $(M=2,\ k=4)$ methods, for Example 4.

6 Conclusions

In this paper, a numerical indirect method based on wavelets was proposed for solving the general continuous time-variant linear quadratic optimal control problem. The necessary optimality conditions were applied to convert the main problem into a boundary value problem, as a dynamic system. The new problem, using two discrete schemes, Legendre and Chebyshev wavelets, was changed to a system of algebraic equations. To demonstrate the efficiency of the proposed method two analytical and two numerical examples were given.

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