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SOME INEQUALITIES OF JENSEN-MERCER TYPE FOR CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract

In this paper we extend Mercer's discrete inequality for univariate functions to the case of convex functions on convex subsets of linear spaces and provide some natural applications for norms.

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1 Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p} = (p_1, \ldots, p_n)$ is a probability sequence and $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i), \tag{1}$$

is well known in the literature as Jensen's inequality.

Recently the author obtained the following refinement of Jensen's inequality (see [8])

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$$f\left(\sum_{j=1}^{n} p_j x_j\right) \leq \min_{k\in\{1,\dots,n\}} \left[(1-p_k) f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + p_k f(x_k)\right] \tag{2}$$
$$\leq \frac{1}{n}\left[\sum_{k=1}^{n} (1-p_k) f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + \sum_{k=1}^{n} p_k f(x_k)\right]$$
$$\leq \max_{k\in\{1,\dots,n\}} \left[(1-p_k) f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + p_k f(x_k)\right]$$
$$\leq \sum_{j=1}^{n} p_j f(x_j),$$

where f, x_k and p_k are as above.

The above result provides a different approach to the one that J. Pečarić and the author obtained in 1989, namely (see [23]):

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\dots p_{i_{k+1}} f\left(\frac{x_{i_1}+\dots+x_{i_{k+1}}}{k+1}\right) \tag{3}$$
$$\leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\frac{x_{i_1}+\dots+x_{i_k}}{k}\right)$$
$$\leq \dots \leq \sum_{i=1}^{n} p_i f(x_i),$$

for $k \geq 1$ and $\mathbf{p}$, $\mathbf{x}$ as above.

If $q_1, \dots, q_k \geq 0$ with $\sum_{j=1}^{k} q_j = 1$, then the following refinement obtained in 1994 by the author [5] also holds:

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\frac{x_{i_1}+\dots+x_{i_k}}{k}\right) \tag{4}$$
$$\leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(q_1 x_{i_1}+\dots+q_k x_{i_k}\right)$$
$$\leq \sum_{i=1}^{n} p_i f(x_i),$$

where $1 \leq k \leq n$ and $\mathbf{p}$, $\mathbf{x}$ are as above.

For other refinements and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalized triangle inequality, the f-divergence measures etc. see [2]-[8].

In 2003, A. McD. Mercer [17] obtained the following inequality for convex functions of a real variable $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ and the finite sequences $x_k \in$

$[m, M]$, and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$,

$$f\left(m + M - \sum_{k=1}^{n} p_k x_k\right) \leq f(m) + f(M) - \sum_{k=1}^{n} p_k f(x_k) \tag{5}$$

and applied it to derive a Ky-Fan's type inequality.

Since then, this result has attracted the attention of many authors that extended it for positive linear functionals and integrals [1], [15], [18], [24], in relation with majorization theory [22], for convex functions of selfadjoint operators in Hilbert spaces [14], [16], [19], [20] and for operator convex functions in Hilbert spaces [21] and [24].

Motivated by the above results, we extend this inequality for convex functions on convex subsets of linear spaces and provide some natural applications for normed spaces.

2 Main Results

We have:

Theorem 1. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex subset C and $x, y \in C$. If $x_k := (1 - t_k) x + t_k y$ with $t_k \in [0, 1]$ and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$\begin{aligned} f\left(x + y - \sum_{k=1}^{n} p_k x_k\right) &\leq \sum_{k=1}^{n} p_k f(x + y - x_k) \\ &\leq \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) + \left(\sum_{k=1}^{n} p_k t_k\right) f(x) \\ &\leq f(y) + f(x) - \sum_{k=1}^{n} p_k f(x_k). \end{aligned} \tag{6}$$

Proof. By Jensen's discrete inequality we have

$$\begin{aligned} f\left(x + y - \sum_{k=1}^{n} p_k x_k\right) &= f\left(\sum_{k=1}^{n} p_k (x + y - x_k)\right) \leq \sum_{k=1}^{n} p_k f(x + y - x_k) \\ &= \sum_{k=1}^{n} p_k f(x + y - (1 - t_k) x - t_k y) \\ &= \sum_{k=1}^{n} p_k f(t_k x + (1 - t_k) y), \end{aligned}$$

which proves the first inequality in (6).

By making use of the convexity of f on C, we derive

$$\begin{aligned}\sum_{k=1}^{n} p_k f\left(t_k x + (1-t_k) y\right) &\le \sum_{k=1}^{n} p_k \left[t_k f(x) + (1-t_k) f(y)\right] \\ &= \left(\sum_{k=1}^{n} p_k t_k\right) f(x) + \left(1 - \sum_{k=1}^{n} p_k t_k\right) f(y),\end{aligned}$$

which proves the second inequality in (6).

Now, by the convexity of f we also have

$$f\left((1-t_k) x + t_k y\right) \le (1-t_k) f(x) + t_k f(y)$$

for $k \in \{1, ..., n\}$. If we multiply this inequality by $p_k \ge 0$, $k \in \{1, ..., n\}$ and sum over k from 1 to n, then we deduce

$$\begin{aligned}\sum_{k=1}^{n} p_k f(x_k) &\le \sum_{k=1}^{n} p_k \left[(1-t_k) f(x) + t_k f(y)\right] \\ &= \left(1 - \sum_{k=1}^{n} p_k t_k\right) f(x) + \left(\sum_{k=1}^{n} p_k t_k\right) f(y).\end{aligned} \tag{7}$$

Therefore, by (7)

$$\begin{aligned}&\left(\sum_{k=1}^{n} p_k t_k\right) f(x) + \left[1 - \left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y) \\ &= f(y) + f(x) - \left[\left(1 - \sum_{k=1}^{n} p_k t_k\right) f(x) + \left(\sum_{k=1}^{n} p_k t_k\right) f(y)\right] \\ &\le f(y) + f(x) - \sum_{k=1}^{n} p_k f(x_k),\end{aligned}$$

which proves the last part of (6). □

Remark 1. *We observe that the inequality (6) is equivalent to the inequality*

$$\begin{aligned}\sum_{k=1}^{n} p_k f(x_k) &\le \left[\left(1 - \sum_{k=1}^{n} p_k t_k\right) f(x) + \left(\sum_{k=1}^{n} p_k t_k\right) f(y)\right] \\ &\le f(x) + f(y) - \sum_{k=1}^{n} p_k f(x + y - x_k) \\ &\le f(x) + f(y) - f\left(x + y - \sum_{k=1}^{n} p_k x_k\right),\end{aligned} \tag{8}$$

where $f : C \subset X \to \mathbb{R}$ is a convex function on the convex subset C, $x, y \in C$, $x_k := (1-t_k) x + t_k y$ with $t_k \in [0,1]$ and $p_k \ge 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$.

Since the distance between the extreme terms is greater than the distance between the internal ones, we can state the following corollary as well:

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\begin{aligned} 0 &\le \left[1-\left(\sum_{k=1}^{n} p_k t_k\right)\right] f(y)+\left(\sum_{k=1}^{n} p_k t_k\right) f(x)-\sum_{k=1}^{n} p_k f(x+y-x_k) \qquad (9)\\ &\le f(y)+f(x)-\sum_{k=1}^{n} p_k f(x_k)-f\left(x+y-\sum_{k=1}^{n} p_k x_k\right). \end{aligned}$$

Further on, we recall the following result obtained by the author in [10] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n \min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right] \qquad (10)\\ &\le \frac{1}{P_n}\sum_{i=1}^{n}p_i\Phi(x_i)-\Phi\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right)\\ &\le n \max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right], \end{aligned}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

We also have:

Theorem 2. *With the assumptions of Theorem 1 we also have*

$$\begin{aligned} & f\left(x+y-\sum_{k=1}^{n}p_k x_k\right) \qquad (11)\\ &\le \min_{i\in\{1,\dots,n\}}\{p_i\}\left[\sum_{k=1}^{n} f(x+y-x_k)-nf\left(x+y-\frac{1}{n}\sum_{k=1}^{n}x_k\right)\right]\\ &+f\left(x+y-\sum_{k=1}^{n}p_k x_k\right)\\ &\le \sum_{k=1}^{n}p_k f(x+y-x_k)\le f(x)+f(y)-\sum_{k=1}^{n}p_k f(x_k). \end{aligned}$$

Proof. By the convexity of f we have

$$f(x+y-x_k)+f(x_k)\le f(x)+f(y)$$

for $k \in \{1,\dots,n\}$.

Indeed, since $x_k = (1 - t_k)\, x + t_k y$, then

$$\begin{aligned} f(x + y - x_k) + f(x_k) &= f(x + y - (1 - t_k)\, x - t_k y) + f((1 - t_k)\, x + t_k y) \\ &= f(t_k x + (1 - t_k)\, y) + f((1 - t_k)\, x + t_k y) \\ &\le t_k f(x) + (1 - t_k)\, f(y) + (1 - t_k)\, f(x) + t_k f(y) \\ &= f(x) + f(y), \end{aligned}$$

for $k \in \{1, ..., n\}$.

If we multiply this inequality by $p_k \ge 0$ and summing over k from 1 to n, then we get

$$\sum_{k=1}^{n} p_k f(x + y - x_k) + \sum_{k=1}^{n} p_k f(x_k) \le f(x) + f(y),$$

namely

$$\sum_{k=1}^{n} p_k f(x + y - x_k) \le f(x) + f(y) - \sum_{k=1}^{n} p_k f(x_k).$$

This implies that

$$\begin{aligned} &\sum_{k=1}^{n} p_k f(x + y - x_k) - f\left(x + y - \sum_{k=1}^{n} p_k x_k\right) \\ &+ f\left(x + y - \sum_{k=1}^{n} p_k x_k\right) \\ &\le f(x) + f(y) - \sum_{k=1}^{n} p_k f(x_k). \end{aligned} \tag{12}$$

If we apply the first inequality in (10) for the convex function $\Phi(t) = f(x + y - t)$, $t \in [x, y]$ then we have

$$\begin{aligned} 0 &\le \min_{i \in \{1,...,n\}} \{p_i\} \left[\sum_{k=1}^{n} f(x + y - x_k) - n f\left(x + y - \frac{1}{n} \sum_{k=1}^{n} x_k\right)\right] \\ &\le \sum_{k=1}^{n} p_k f(x + y - x_k) - f\left(x + y - \sum_{k=1}^{n} p_k x_k\right) \end{aligned} \tag{13}$$

By making use of (12) and (13) we get the desired result (11). □

We also have:

Corollary 2. *With the assumptions of Theorem 2, we have*

$$0 \leq \sum_{k=1}^{n} p_k f(x+y-x_k) - f\left(x+y-\sum_{k=1}^{n} p_k x_k\right) \tag{14}$$
$$- \min_{i\in\{1,\ldots,n\}} \{p_i\} \left[\sum_{k=1}^{n} f(x+y-x_k) - nf\left(x+y-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]$$
$$\leq f(x)+f(y)-\sum_{k=1}^{n} p_k f(x_k) - f\left(x+y-\sum_{k=1}^{n} p_k x_k\right).$$

The proof follows by Theorem 2 observing that the difference between the extreme terms is greater than the difference between the internal ones.

Remark 2. *Now, observe that by (3),*

$$f\left(x+y-\sum_{i=1}^{n} p_i x_i\right) \tag{15}$$
$$\leq \sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\ldots p_{i_{k+1}} f\left(x+y-\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)$$
$$\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(x+y-\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)$$
$$\leq \cdots \leq \sum_{i=1}^{n} p_i f(x+y-x_i),$$

and therefore by (14) we get the following chain of inequalities

$$0 \leq \sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\ldots p_{i_{k+1}} f\left(x+y-\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \tag{16}$$
$$- f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)$$
$$- \min_{i\in\{1,\ldots,n\}} \{p_i\} \left[\sum_{k=1}^{n} f(x+y-x_k) - nf\left(x+y-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]$$

$$\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(x+y-\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) - f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)$$
$$- \min_{i\in\{1,\ldots,n\}} \{p_i\} \left[\sum_{k=1}^{n} f(x+y-x_k) - nf\left(x+y-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]$$

...

$$\le \sum_{k=1}^{n} p_k f(x+y-x_k) - f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)$$
$$- \min_{i\in\{1,...,n\}}\{p_i\}\left[\sum_{k=1}^{n} f(x+y-x_k) - nf\left(x+y-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]$$
$$\le f(x)+f(y)-\sum_{k=1}^{n} p_k f(x_k) - f\left(x+y-\sum_{k=1}^{n} p_k x_k\right),$$

provided that $f : C \subset X \to \mathbb{R}$ *is a convex function on the convex subset* C, x, $y \in C$, $x_k := (1-t_k)x + t_k y$ *with* $t_k \in [0,1]$ *and* $p_k \ge 0$, $k \in \{1,...,n\}$ *with* $\sum_{k=1}^{n} p_k = 1$.

Also, by (4),

$$f\left(x+y-\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i_1,...,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(x+y-\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \tag{17}$$
$$\le \sum_{i_1,...,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(x+y-(q_1 x_{i_1}+\cdots+q_k x_{i_k})\right)$$
$$\le \sum_{i=1}^{n} p_i f(x+y-x_i),$$

which by (14) produce the following sequence of inequalities

$$0 \le \sum_{i_1,...,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(x+y-\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \tag{18}$$
$$- f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)$$
$$- \min_{i\in\{1,...,n\}}\{p_i\}\left[\sum_{k=1}^{n} f(x+y-x_k) - nf\left(x+y-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]$$

$$\le \sum_{i_1,...,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(x+y-(q_1 x_{i_1}+\cdots+q_k x_{i_k})\right)$$
$$- f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)$$
$$- \min_{i\in\{1,...,n\}}\{p_i\}\left[\sum_{k=1}^{n} f(x+y-x_k) - nf\left(x+y-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]$$

$$\leq \sum_{k=1}^{n} p_k f(x+y-x_k) - f\left(x+y-\sum_{k=1}^{n} p_k x_k\right)$$
$$- \min_{i\in\{1,\dots,n\}}\{p_i\}\left[\sum_{k=1}^{n} f(x+y-x_k) - nf\left(x+y-\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]$$
$$\leq f(x)+f(y)-\sum_{k=1}^{n} p_k f(x_k) - f\left(x+y-\sum_{k=1}^{n} p_k x_k\right),$$

where $q_1,\dots,q_k \geq 0$ *with* $\sum_{j=1}^{k} q_j = 1$.

3 Norm Inequalities

If we write the inequality (6) for the convex function $f(x) = \|x\|^p$, $p \geq 1$, where $\|\cdot\|$ is a norm on X, then we get

$$\left\|x+y-\sum_{k=1}^{n} p_k x_k\right\|^p \leq \sum_{k=1}^{n} p_k \|x+y-x_k\|^p \tag{19}$$
$$\leq \left[1-\left(\sum_{k=1}^{n} p_k t_k\right)\right]\|y\|^p + \left(\sum_{k=1}^{n} p_k t_k\right)\|x\|^p$$
$$\leq \|y\|^p + \|x\|^p - \sum_{k=1}^{n} p_k \|x_k\|^p,$$

where x, $y \in X$, $x_k := (1-t_k)x + t_k y$ with $t_k \in [0,1]$ and $p_k \geq 0$, $k \in \{1,...,n\}$ with $\sum_{k=1}^{n} p_k = 1$. This implies that

$$0 \leq \left[1-\left(\sum_{k=1}^{n} p_k t_k\right)\right]\|y\|^p + \left(\sum_{k=1}^{n} p_k t_k\right)\|x\|^p - \sum_{k=1}^{n} p_k \|x+y-x_k\|^p \tag{20}$$
$$\leq \|y\|^p + \|x\|^p - \sum_{k=1}^{n} p_k \|x_k\|^p - \left\|x+y-\sum_{k=1}^{n} p_k x_k\right\|^p.$$

Also, with the same assumptions for x, $y \in X$, x_k and $p_k \geq 0$, $k \in \{1,...,n\}$, we have

$$\left\|x+y-\sum_{k=1}^{n} p_k x_k\right\|^p \tag{21}$$
$$\leq \min_{i\in\{1,\dots,n\}}\{p_i\}\left[\sum_{k=1}^{n} \|x+y-x_k\|^p - n\left\|x+y-\frac{1}{n}\sum_{k=1}^{n} x_k\right\|^p\right]$$
$$+ \left\|x+y-\sum_{k=1}^{n} p_k x_k\right\|^p$$
$$\leq \sum_{k=1}^{n} p_k \|x+y-x_k\|^p \leq \|y\|^p + \|x\|^p - \sum_{k=1}^{n} p_k \|x_k\|^p.$$

This implies that

$$
\begin{aligned}
0 &\le \sum_{k=1}^{n} p_k \|x+y-x_k\|^p - \left\| x+y-\sum_{k=1}^{n} p_k x_k \right\|^p \qquad (22)\\
&- \min_{i\in\{1,\dots,n\}} \{p_i\} \left[\sum_{k=1}^{n} \|x+y-x_k\|^p - n \left\| x+y-\frac{1}{n}\sum_{k=1}^{n} x_k \right\|^p \right]\\
&\le \|y\|^p + \|x\|^p - \sum_{k=1}^{n} p_k \|x_k\|^p - \left\| x+y-\sum_{k=1}^{n} p_k x_k \right\|^p .
\end{aligned}
$$

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