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MONOTONICITY FORMULAE AND F-STRESS ENERGY

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Abstract

The goal of this work is the application of the f-stress energy of differential forms to study the generalized monotonicity formulae and generalized vanishing theorems. We obtain some generalized monotonicity formulas for p-forms $\omega \in A^p(\xi)$, which satisfy the generalized f-conservation laws, with $f \in C^\infty(M \times \mathbb{R})$ satisfying some conditions.

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1 Introduction

In 1980, Baird and Eells [2] introduced the stress-energy tensor for maps between Riemannian manifolds, which unifies various results on harmonic maps. Following [2], Sealey [13] introduced the stress-energy tensor for p-forms with values in vector bundles and established some vanishing theorems for harmonic p-forms. Since then, the stress-energy tensors have become a useful tool for investigating the energy behavior of vector bundle valued p-forms in various problems. In [7] the authors presented a unified method to establish monotonicity formulae for p-forms with values in vector bundles by means of the stress-energy tensors of various energy functionals in geometry and physics.

Recently in 2010, M. Djaa and all introduced the notion of f-harmonic and f-stress energy [11], [4] and studied by many authors Chiang [3], Y.L. Ou [12], S. Feng [8], W.J. Lu [10] and others.

The goal of this work is the application of the f-stress energy of differential forms to study the generalized monotonocity formulae and generalized vanishing

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theorem (Theorem 3.1, Theorem 3.3 and Theorem 3.4).

Let (M,g) be a Riemannian manifold and $\xi : E \longrightarrow M$ be a smooth Riemannian vector bundle over M. Set:

$$A^p(\xi) = \Lambda^p T^*M \otimes E$$

the space of smooth p-froms on M with values in the vector bundle $\xi : E \longrightarrow M$.

For a linear connection ∇^E on E we define the covariant derivative on $A^p(\xi)$ by

$$(\nabla_X \omega)(X_1, ..., X_p) = \nabla^E_X \omega(X_1, ..., X_p) - \sum_{i=1}^{p} \omega(X_1, .., \nabla_X X_i, ..., X_p) \tag{1}$$

The exterior covariant differentiation $d^\nabla : A^p(\xi) \longrightarrow A^{p+1}(\xi)$ relative to the connection ∇^E is defined by

$$(d^\nabla \omega)(X_1, X_2, .., X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{X_i} \omega)(X_1, X_2, .., \hat{X}_i, .., X_{p+1}),$$

where the symbols covered by $\hat{}$ are omitted and $X_i \in \Gamma(TM)$ for $i = 1, ..., p+1$. The codifferential operator $\delta^\nabla : A^p(\xi) \longrightarrow A^{p-1}(\xi)$ characterized as the adjoint of d^∇ is defined by

$$(\delta^\nabla \omega)(X_1, X_2, .., X_{p-1}) = -\sum_{i=1}^{m} (\nabla_{e_i} \omega)(e_i, X_1, X_2, .., X_{p-1}).$$

Here $\{e_1, e_2, .., e_m\}$ is a local frame field on (M, g).

For $\omega \in A^p(\xi)$, we define the generalized f-energy functional of ω as follows:

$$E_f(\omega) = \int_M f(x, \frac{|\omega|^2}{2}) v_g, \tag{2}$$

where $f : M \times \mathbb{R} \longrightarrow \mathbb{R}$, $(x, s) \longmapsto f(x, s)$ is a smooth function such that $f(x, 0) = 0$, for all $x \in M$ and $f(x, s) > 0$, for all $(x, s) \in M \times \mathbb{R}^*_+$. Here:

$$|\omega|^2 = < \omega, \omega > = \sum_{i_1 < i_2 < .. < i_p} < \omega(e_{i_1}, e_{i_2}, .., e_{i_p}), \omega(e_{i_1}, e_{i_2}, .., e_{i_p}) >_E .$$

If $f(x, s) = \tilde{f}(x)s$, $\forall (x, s) \in M \times \mathbb{R}$ we deduce the energy functional of ω defined in [9], where $\tilde{f} : M \longrightarrow (0, \infty)$ is a smooth function.

2 Generalized f-stress energy tensor

If $(g)_t$ is a smooth 1-parameter family of metrics with $g_0 = g$, then the variation $\delta g = \frac{\partial g}{\partial t}\big|_{t=0}$ is a smooth symmetric tensor on M. We have:

Proposition 1. *Let (M, g) be a Riemannian manifold and $\xi : E \longrightarrow M$ be a smooth Riemannian vector bundle over M with a metric compatible connection ∇^E and $\omega \in A^p(\xi)$, then*

$$\frac{d}{dt}E_f(\omega)\Big|_{t=0} = \frac{1}{2}\int_M < S_f(\omega), \delta g > v_g, \tag{3}$$

where:

$$S_f(\omega) = f_\omega g - f^{'}_\omega \omega \odot \omega, \tag{4}$$

is called generalized f-stress energy tensor for ω, $f_\omega \in C^\infty(M)$ is a smooth function defined by:

$$f_\omega(x) = f(x, \frac{|\omega|^2}{2}), \quad \forall x \in M$$

and $f^{'}_\omega \in C^\infty(M)$ is a smooth function defined by:

$$f^{'}_\omega(x) = \frac{\partial f}{\partial s}(x, \frac{|\omega|^2}{2}), \quad \forall x \in M.$$

Proof. We have:

$$\begin{aligned}
\frac{d}{dt}E_f(\omega)\Big|_{t=0} &= \int_M \frac{\partial}{\partial t}\Big[f(x, \frac{|\omega|^2}{2})v_{g_t}\Big]_{t=0} \\
&= \int_M \Big[\frac{\partial}{\partial t}\Big(f(x, \frac{|\omega|^2}{2})\Big)v_{g_t} + f(x, \frac{|\omega|^2}{2})\frac{\partial}{\partial t}(v_{g_t})\Big]_{t=0} \\
&= \int_M \Big[\frac{1}{2}\frac{\partial}{\partial t}(\frac{|\omega|^2}{2})\frac{\partial f}{\partial s}(x, \frac{|\omega|^2}{2})v_{g_t} + f(x, \frac{|\omega|^2}{2})\frac{\partial}{\partial t}(v_{g_t})\Big]_{t=0}.
\end{aligned}$$

From [1], we know that:

$$\frac{\partial}{\partial t}(\frac{|\omega|^2}{2})\Big|_{t=0} = - < \omega \odot \omega, \delta g >,$$

and:

$$\frac{\partial}{\partial t}(v_{g_t})\Big|_{t=0} = \frac{1}{2} < g, \delta g > v_g,$$

so that

$$\begin{aligned}
\frac{d}{dt}E_f(\omega)\Big|_{t=0} &= \frac{1}{2}\int_M < f_\omega g - f^{'}\omega \odot \omega, \delta g > v_g \\
&= \frac{1}{2}\int_M < S_f(\omega), \delta g > v_g.
\end{aligned}$$

□

Proposition 2. *Let (M, g) be a Riemannian manifold and $\xi : E \longrightarrow M$ be a smooth Riemannian vector bundle over M with a metric compatible connection ∇^E, $f : M \times \mathbb{R} \longrightarrow (0, \infty)$, $(x, s) \longmapsto f(x, s)$ a smooth function and $\omega \in A^p(\xi)$, then:*

$$\begin{aligned}(\operatorname{div} S_f(\omega))(X) &= X(f)_\omega - < i_{\operatorname{grad} f'_\omega}\omega, i_X\omega > \\ &\quad + f'_\omega\big[< \delta^\nabla \omega, i_X\omega > + < i_X d^\nabla \omega, \omega > \big],\end{aligned} \tag{5}$$

for all $X \in \Gamma(TM)$, where $X(f)_\omega \in C^\infty(M)$ is a smooth function defined by:

$$X(f)_\omega(x) = X(f)(x, \frac{|\omega|^2}{2}), \quad \forall x \in M.$$

Proof. Let $\{e_1, .., e_m\}$ be an orthonormal frame on (M, g), such that at $x \in M$, $\nabla_{e_i} e_j = 0$. By using (4), we have:

$$\begin{aligned}(\operatorname{div} S_f(\omega))(X) &= \sum_{i=1}^m \Big[\nabla_{e_i} S_f(\omega))(e_i, X) - S_f(\omega))(e_i, \nabla_{e_i} X)\Big] \\ &= \sum_{i=1}^m \nabla_{e_i}\Big[f_\omega g(e_i, X) - f'_\omega \omega \odot \omega(e_i, X)\Big] \\ &\quad - \sum_{i=1}^m \Big[f_\omega g(\nabla_{e_i} X, e_i) + f'_\omega . \omega \odot \omega(\nabla_{e_i} X, e_i)\Big] \\ &= df_\omega(X) - < i_{\operatorname{grad} f'_\omega}\omega, i_X\omega > \\ &\quad - \sum_{i=1}^m f'_\omega[\nabla_{e_i} < i_{e_i}\omega, i_X\omega > - < i_{e_i}\omega, i_{\nabla_{e_i} X}\omega >].\end{aligned} \tag{6}$$

The first term in the right side of (6) is

$$\begin{aligned} df_\omega(X) &= X(f_\omega) \\ &= X(f(x, \frac{|\omega|^2}{2})) \\ &= X(f)_\omega + X(\frac{|\omega|^2}{2}) f'_\omega.\end{aligned} \tag{7}$$

From the Lemma (1.2) in ([9]), we deduce that:

$$\begin{aligned} X(\frac{|\omega|^2}{2}) = &\sum_{j_1 < .. < j_{p-1}, i} < \omega(e_i, e_{j_1}, .., e_{j_{p-1}}), (\nabla_{e_i}\omega)(X, e_{j_1}, .., e_{j_{p-1}}) > \\ &+ < i_X d^\nabla \omega, \omega >,\end{aligned}$$

so that:

$$\begin{aligned} df_\omega(X) &= X(f)_\omega + f'_\omega < i_X d^\nabla \omega, \omega > \\ &\quad + f'_\omega \sum_{j_1 < j_2 < .. < j_{p-1}, i} < \omega(e_i, e_{j_1}, .., e_{j_{p-1}}), (\nabla_{e_i}\omega)(X, e_{j_1}, .., e_{j_{p-1}}) > .\end{aligned} \tag{8}$$

for the third term in the right side of (6), we have:

$$
\begin{aligned}
< \delta^{\nabla}\omega, i_X\omega > &=< \delta^{\nabla}\omega(e_{i_1},..,e_{i_{p-1}}), \omega(X, e_{i_1},..,e_{i_{p-1}}) > \\
&= - < \sum_{i=1}^{m} (\nabla_{e_i}\omega)(e_i, e_{j_1},..,e_{j_{p-1}}), \omega(X, e_{j_1},..,e_{j_{p-1}}) > \\
&= - < \sum_{i=1}^{m} \nabla_{e_i}\omega(e_i, e_{j_1},..,e_{j_{p-1}}), \omega(X, e_{j_1},..,e_{j_{p-1}}) > \\
&= - \sum_{i=1}^{m} \nabla_{e_i} < \omega(e_i, e_{j_1},..,e_{j_{p-1}}), \omega(X, e_{j_1},..,e_{j_{p-1}}) > \\
&\quad + \sum_{j_1<j_2<..<j_{p-1}, i} < \omega(e_i, e_{j_1},..,e_{j_{p-1}}), \nabla_{e_i}\omega(X, e_{j_1},..,e_{j_{p-1}}) > \\
&= - \sum_{i=1}^{m} \nabla_{e_i} < i_{e_i}\omega, i_X\omega > \\
&\quad + \sum_{j_1<..<j_{p-1}, i} < i_{e_i}\omega, (\nabla_{e_i}\omega)(X, e_{j_1},..,e_{j_{p-1}}) + \omega(\nabla_{e_i}X, e_{j_1},..,e_{j_{p-1}}) > \\
&= - \sum_{i=1}^{m} [\nabla_{e_i} < i_{e_i}\omega, i_X\omega > - < i_{e_i}\omega, i_{\nabla_{e_i}X}\omega) >] \\
&\quad + \sum_{j_1<j_2<..<j_{p-1}, i} < \omega(e_i, e_{j_1},..,e_{j_{p-1}}), \nabla_{e_i}\omega(X, e_{j_1},..,e_{j_{p-1}}) >,
\end{aligned}
$$

that is:

$$
\sum_{i=1}^{m} [\nabla_{e_i} < i_{e_i}\omega, i_X\omega > - < i_{e_i}\omega, i_{\nabla_{e_i}X}\omega) >] =
$$

$$
\sum_{j_1<..<j_{p-1}, i} < \omega(e_i, e_{j_1},..,e_{j_{p-1}}), \nabla_{e_i}\omega(X, e_{j_1},..,e_{j_{p-1}}) > - < \delta^{\nabla}\omega, i_X\omega > . \tag{9}
$$

Finally, by replacing (8) and (9) in (6), we obtain the result of the proposition (2). □

Definition 1. $\omega \in A^p(\xi)$ $(p \geq 1)$ *is said to satisfy a generalized f-conservation law if it satisfies the following equation:*

$$
(\operatorname{div} S_f(\omega))(X) = X(f)_\omega.
$$

By applying $T = S_{f,\omega}$ in (7) (see [9]), we have:

$$
\int_{\partial D} S_{f,\omega}(X, \nu) ds_g = \int_D \Big[\langle S_{f,\omega}, \frac{1}{2} L_X g \rangle + X(f)_\omega \Big] dv_g. \tag{10}
$$

Let $\Gamma_0(TM)$ be a subset of $\Gamma(TM)$ consisting of all elements with compact supports contained in the interior of M.

Definition 2. *$\omega \in A^p(\xi)$ $(p \geq 1)$ is said to satisfy a generalized integral f-conservation law if it satisfies the following equation:*

$$\int_M (\operatorname{div} S_{f,\omega})(X) dv_g = \int_M X(f)_\omega dv_g,$$

for any $X \in \Gamma_0(TM)$.

From the equation (7) in [9], we have:

$$\int_M \Big[\Big\langle T, \frac{1}{2} L_X g \Big\rangle + (\operatorname{div} T)(X)\Big] dv_g = 0, \tag{11}$$

for any $X \in \Gamma_0(TM)$. From Definition (2) and equation (11), we have:

$$\int_M \Big[\Big\langle S_{f,\omega}, \frac{1}{2} L_X g \Big\rangle + X(f)_\omega\Big] dv_g = 0, \tag{12}$$

for any $X \in \Gamma_0(TM)$ and $\omega \in A^p(\xi)$ $(p \geq 1)$ satisfies the generalized integral f-conservation law.

3 Monotonicity Formulas and Vanishing Theorems

Let (M, g_0) be a complete Riemannian manifold with a pole x_0. Denote by $r(x)$ the g_0 distance function relative to the pole x_0, that is $r(x) = \operatorname{dis}_{g_0}(x, x_0)$. Set:

$$B(r) = \{x \in M^m : r(x) \leq r\}.$$

It is known that $\frac{\partial}{\partial r}$ is always an eigenvector of $Hess_{g_0}(r^2)$ associated to eigenvalue 2. Denote by λ_{max} (resp. λ_{min}) the maximum (resp. minimal) eigenvalues of $\operatorname{Hess}_{g_0}(r^2) - 2dr \otimes dr$ at each point of $M - \{x_0\}$.

From now on, we suppose that $\omega \in A^p(\xi)$ satisfies the generalized f-conservation law and $\xi : E \longrightarrow (M, g)$ is a smooth Riemannian vector bundle over (M, g) where $g = \varphi^2 g_0$, $0 < \varphi \in \mathcal{C}^\infty(M)$. Clearly, the vector field $\nu = \varphi^{-1}\frac{\partial}{\partial r}$ is an outer normal vector field along $\partial B(r) \subset (M, g)$. We assume the following conditions for φ:

(φ_1) $\frac{\partial \log \varphi}{\partial r} \geq 0$;

(φ_2) there is a constant $C_0 > 0$ such that:

$$(m - 2p) r \frac{\partial \log \varphi}{\partial r} + \frac{m-1}{2} \lambda_{\min} + 1 - p \max(2, \lambda_{\max}) \geq C_0,$$

now we set $\mu = \sup_{M \times \mathbb{R}_+^*} r |\frac{\partial \log f}{\partial r}| < \infty$.

Theorem 1. *Suppose that $\omega \in A^p(\xi)$ $(p \geq 1)$ satisfies the generalized f-conservaion law and $\xi : E \longrightarrow (M, g)$ is a smooth Riemannian vector bundle over $(M, \varphi^2 g_0)$. If $C_0 - \mu > 0$, φ satisfies (φ_1) , (φ_2) with $s\frac{\partial f}{\partial s} \leq f$, and $\frac{\partial f}{\partial s} \geq 0$, then*

$$\frac{\int_{B(\rho_1)} f_\omega dv_g}{\rho_1^{C_0-\mu}} \leq \frac{\int_{B(\rho_2)} f_\omega dv_g}{\rho_2^{C_0-\mu}},$$

for any $0 < \rho_1 < \rho_2$

Proof. We take $D = B(r)$ and $X = r\frac{\partial}{\partial r} = \frac{1}{2}\nabla^0 r^2$, where ∇^0 denotes the covariant derivative determined by g_0. By a direct computation, we have :

$$\begin{aligned} \Big\langle S_{f,\omega}, \frac{1}{2}L_X(g)\Big\rangle &= \Big\langle S_{f,\omega}, r\frac{\partial \log\varphi}{\partial r}g + \frac{1}{2}\varphi^2 L_X(g_0)\Big\rangle \\ &= r\frac{\partial \log\varphi}{\partial r}\Big\langle S_{f,\omega}, g\Big\rangle + \frac{1}{2}\varphi^2\Big\langle S_{f,\omega}, Hess_{g_0}(r^2)\Big\rangle. \end{aligned} \tag{13}$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal basis with respect to g_0 and $e_m = \frac{\partial}{\partial r}$. We may assume that $Hess_{g_0}(r^2)$ becomes a diagonal matrix with respect to $\{e_i\}$, then $\{\tilde{e_i} = \varphi^{-1}e_i\}$ is an orthonormal basis with respect to g. For the first term of (13):

$$\begin{aligned} \Big\langle S_{f,\omega}, g\Big\rangle &= \sum_{i,j=1}^m S_{f,\omega}(\tilde{e_i},\tilde{e_j})g(\tilde{e_i},\tilde{e_j}) \\ &= \sum_{i,j=1}^m [f_\omega g(\tilde{e_i},\tilde{e_j})g(\tilde{e_i},\tilde{e_j}) - f'_\omega(\omega\odot\omega)(\tilde{e_i},\tilde{e_j})g(\tilde{e_i},\tilde{e_j})] \\ &= mf_\omega - f'_\omega\sum_{i=1}^m < i_{\tilde{e_i}}\omega, i_{\tilde{e_i}}\omega > \\ &= mf_\omega - f'_\omega p|\omega|^2. \end{aligned} \tag{14}$$

For the second term of (13), we have:

$$\begin{aligned} \frac{1}{2}\varphi^2\Big\langle S_{f,\omega}, Hess_{g_0}(r^2)\Big\rangle &= \frac{1}{2}\varphi^2\sum_{i,j=1}^m S_{f,\omega}(\tilde{e_i},\tilde{e_j})Hess_{g_0}(r^2)(\tilde{e_i},\tilde{e_j}) \\ &= \frac{1}{2}\varphi^2\sum_{i,j=1}^m\Big[f_\omega g(\tilde{e_i},\tilde{e_j})Hess_{g_0}(r^2)(\tilde{e_i},\tilde{e_j}) \\ &\quad - f'_\omega(\omega\odot\omega)(\tilde{e_i},\tilde{e_j})Hess_{g_0}(r^2)(\tilde{e_i},\tilde{e_j})\Big] \\ &= \frac{1}{2}f_\omega\sum_{i=1}^m Hess_{g_0}(r^2)(e_i,e_i) \\ &\quad - \frac{1}{2}f'_\omega\sum_{i=1}^m < i_{\tilde{e_i}}\omega, i_{\tilde{e_i}}\omega > Hess_{g_0}(r^2)(e_i,e_i) \\ &\geq \frac{1}{2}f_\omega[(m-1)\lambda_{min}+2] - \frac{1}{2}max(2,\lambda_{max})f'_\omega p|\omega|^2. \end{aligned} \tag{15}$$

From (13), (14), (15), (φ_1) and (φ_2), we have:

$$\begin{aligned} \Big\langle S_{f,\omega}, \frac{1}{2}L_X(g)\Big\rangle &\geq r\frac{\partial\log\varphi}{\partial r}\Big[mf_\omega - f'_\omega p|\omega|^2\Big] + \frac{1}{2}f_\omega[(m-1)\lambda_{\min}+2] \\ &\quad - \frac{1}{2}\max(2,\lambda_{\max})f'_\omega p|\omega|^2 \\ &\geq \Big(mr\frac{\partial\log\varphi}{\partial r} + \frac{m-1}{2}\lambda_{\min} + 1\Big)f_\omega \\ &\quad - \Big(2pr\frac{\partial\log\varphi}{\partial r} + p\max(2,\lambda_{\max})\Big)f'_\omega\frac{|\omega|^2}{2}, \end{aligned} \tag{16}$$

by using the condition $s\frac{\partial f}{\partial s} \leq f$, we have $f'_\omega \frac{|\omega|^2}{2} \leq f_\omega$, then:

$$\begin{aligned}\Big\langle S_{f,\omega}, \frac{1}{2}L_X(g)\Big\rangle &\geq \Big[(m-2p)r\frac{\partial \log\varphi}{\partial r} + \frac{m-1}{2}\lambda_{\min} + 1 - p\max(2,\lambda_{\max})\Big]f_\omega \\ &\geq C_0 f_\omega.\end{aligned} \tag{17}$$

On the other hand, by the Coarea formula and $|\nabla r|_g = \varphi^{-1}$, we have:

$$\begin{aligned}\int_{\partial B(r)} S_{f,\omega}(X,\nu)ds_g &= \int_{\partial B(r)} \Big[f_\omega g(X,\nu) - f'_\omega(\omega\odot\omega)(X,\nu)\Big]ds_g \\ &= \int_{\partial B(r)} \Big[f_\omega g(r\frac{\partial}{\partial r}, \varphi^{-1}\partial r) - f'_\omega(\omega\odot\omega)(r\frac{\partial}{\partial r}, \varphi^{-1}\partial r)\Big]ds_g \\ &= r\int_{\partial B(r)} f_\omega \varphi ds_g - \int_{\partial B(r)} f'_\omega r\varphi^{-1}(\omega\odot\omega)(\partial r,\partial r)ds_g \\ &\leq r\int_{\partial B(r)} f_\omega\varphi ds_g \\ &\leq r\frac{d}{dr}\int_0^r \Big(\int_{\partial B(r)} \frac{f_\omega}{|\nabla r|}ds_g\Big)dt \\ &\leq r\frac{d}{dr}\int_{\partial B(r)} f_\omega dv_g.\end{aligned} \tag{18}$$

If ω satisfies a generalized f-conservation law, then from (10), (18) and (17), we obtain:

$$C_0\int_{B(r)} f_\omega dv_g + \int_{B(r)} r(df(\frac{\partial}{\partial r}))_\omega dv_g \leq r\frac{d}{dr}\int_{B(r)} f_\omega dv_g \tag{19}$$

On the other hand, we have

$$\begin{aligned}\mu &= \sup_{M\times\mathbb{R}_+^*} r|\frac{\partial \log f}{\partial r}| \\ &= \sup_{M\times\mathbb{R}_+^*} \frac{r}{f}|\frac{\partial f}{\partial r}|,\end{aligned}$$

then $\mu \geq -\frac{r}{f_\omega}(\frac{\partial f}{\partial r})_\omega$, so that $\mu\frac{f_\omega}{r} \geq -(\frac{\partial f}{\partial r})_\omega$, and:

$$\mu\int_{B(r)} f_\omega dv_g \geq -\int_{B(r)} (r\frac{\partial f}{\partial r})_\omega dv_g,$$

we get:

$$C_0\int_{B(r)} f_\omega dv_g - \mu\int_{B(r)} f_\omega dv_g \leq r\frac{d}{dr}\int_{B(r)} f_\omega dv_g,$$

that is:

$$(C_0-\mu)\int_{B(r)} f_\omega dv_g \leq r\frac{d}{dr}\int_{B(r)} f_\omega dv_g,$$

so that:

$$\frac{d}{dr}\frac{\int_{B(r)} f_\omega dv_g}{r^{C_0-\mu}} \geq 0.$$

Therefore,

$$\frac{\int_{B(\rho_1)} f_\omega dv_g}{\rho_1^{C_0-\mu}} \leq \frac{\int_{B(\rho_2)} f_\omega dv_g}{\rho_2^{C_0-\mu}},$$

for any $0 < \rho_1 < \rho_2$. This proves Theorem (1). □

Let $u : M \longrightarrow N$ be an f-harmonic map. Then its differential du can be viewed as a 1-form with values in the induced bundle $u^{-1}TN$. Since $\omega = du$ satisfies the generalized conservation law (see [11]), we obtain the following Liouville-type result:

Corollary 1. *Suppose that* $u : (M, \varphi^2 g_0) \longrightarrow N$ *is an* f*-harmonic map. If* $C_0 - \mu > 0$, φ *satisfies* (φ_1), (φ_2) *with* $s\frac{\partial f}{\partial s} \leq f$ *and* $\frac{\partial f}{\partial s} \geq 0$, *then*

$$\frac{\int_{B(\rho_1)} f_u dv_g}{\rho_1^{C_0-\mu}} \leq \frac{\int_{B(\rho_2)} f_u dv_g}{\rho_2^{C_0-\mu}}$$

for any $0 < \rho_1 < \rho_2$, *where* $f_u \in C^\infty(M)$ *defined by:*

$$f_u(x) = f(x, \tfrac{|du|^2}{2}), \ \forall x \in M.$$

Proof. This follows at once from Theorem 1 in which $p = 1$ and $\omega = du$. □

From (19), we immediately get the following:

Corollary 2. *Suppose that* $\omega \in A^p(\xi)$, $(p \geq 1)$ *satisfies the generalized* f*-conservation law, and* $\xi : E \longrightarrow (M, g)$ *is a smooth Riemannian vector bundle over* $(M, \varphi^2 g_0)$. *If* φ *satisfies* (φ_1), (φ_2) *and* f *satisfies* $s\frac{\partial f}{\partial s} \leq f$, $\frac{\partial f}{\partial s} \geq 0$, *and* $\frac{\partial f}{\partial r} \geq 0$, *then*

$$\frac{\int_{B(\rho_1)} f_u dv_g}{\rho_1^{C_0}} \leq \frac{\int_{B(\rho_2)} f_u dv_g}{\rho_2^{C_0}},$$

for any $0 < \rho_1 < \rho_2$.

Lemma 1. ([9]) *Let* (M, g) *be a complete Riemannian manifold with a pole* x_0. *Denote by* K_r *the radial curvature of* M.

i) *If* $-\alpha^2 \leq K_r \leq -\beta^2$ *with* $\alpha \geq \beta > 0$ *and* $(m-1)\beta - 2p\alpha \geq 0$, *then:*

$$[(m-1)\lambda_{\min} + 2p\max(2, \lambda_{\max})] \geq 2(m - \frac{2p\alpha}{\beta}).$$

ii) *If* $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ *with* $\epsilon > 0$, $A > 0$ *and* $0 \leq B \leq 2\epsilon$, *then:*

$$[(m-1)\lambda_{\min} + 2p\max(2, \lambda_{\max})] \geq 2[1 + (m-1)(1 - \frac{B}{2\epsilon}) - 2pe^{\frac{A}{2\epsilon}}].$$

iii) *If* $-\frac{a^2}{c^2+r^2} \leq K_r \leq \frac{b^2}{c^2+r^2}$ *with* $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ *and* $c^2 \geq 0$, *then:*

$$[(m-1)\lambda_{\min}+2p\max(2,\lambda_{\max})] \geq 2\Big[1+(m-1)\frac{1+\sqrt{1-4b^2}}{2}-2p\frac{1+\sqrt{1+4a^2}}{2}\Big].$$

Theorem 2. *Let* (M, g) *be an* m*-dimensional complete manifold with a pole* x_0. *Assume that the radial curvature* K_r *of* M *satisfies one of the following three conditions:*

i) $-\alpha^2 \leq K_r \leq -\beta^2$ *with* $\alpha \geq \beta > 0$ *and* $(m-1)\beta - 2p\alpha \geq 0$*;*

ii) $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ *with* $\epsilon > 0$, $A > 0$ *and* $0 \leq B \leq 2\epsilon$*;*

iii) $-\frac{a^2}{c^2+r^2} \leq K_r \leq \frac{b^2}{c^2+r^2}$ *with* $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ *and* $c^2 \geq 0$.

If $\omega \in A^p(\xi)$, $(p \geq 1)$ *satisfies the generalized* f*-conservation law with* $s\frac{\partial f}{\partial s} \leq f$, $\frac{\partial f}{\partial s} \geq 0$ *and* $\Lambda - \mu > 0$, *then*

$$\frac{\int_{B(\rho_1)} f_\omega dv_g}{\rho_1^{\Lambda-\mu}} \leq \frac{\int_{B(\rho_2)} f_\omega dv_g}{\rho_2^{\Lambda-\mu}},$$

for any $0 < \rho_1 < \rho_2$, *where:*

$$\Lambda = \begin{cases} m - \frac{2p\alpha}{\beta}, \text{ if } K_r \text{ satisfies } \textbf{\textit{i)}} \\ 1 + (m-1)(1 - \frac{\beta}{2\epsilon}) - 2pe^{\frac{A}{2\epsilon}}, \text{ if } K_r \text{ satisfies } \textbf{\textit{ii)}} \\ 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 2p\frac{1+\sqrt{1+4a^2}}{2}, \text{ if } K_r \text{ satisfies } \textbf{\textit{iii)}}. \end{cases}$$

Proof. From the proof of Theorem 1 for $\varphi = 1$ and Lemma 1, we have

$$\frac{d}{dr}\frac{\int_{B(r)} f_\omega dv_g}{r^{\Lambda-\mu}} \geq 0,$$

therefore, we get the monotonicity formula:

$$\frac{\int_{B(\rho_1)} f_\omega dv_g}{\rho_1^{\Lambda-\mu}} \leq \frac{\int_{B(\rho_2)} f_\omega dv_g}{\rho_2^{\Lambda-\mu}},$$

for any $0 < \rho_1 < \rho_2$. □

Corollary 3. *Let* M , K_r *and* Λ *be as in Theorem 2. If* $\omega \in A^p(\xi)$, $(p \geq 1)$ *satisfies the generalized* f*-conservation law, with* $s\frac{\partial f}{\partial s} \leq f$, $\frac{\partial f}{\partial s} \geq 0$ *and* $\frac{\partial f}{\partial r} \geq 0$, *then*

$$\frac{\int_{B(\rho_1)} f_\omega dv_g}{\rho_1^{\Lambda}} \leq \frac{\int_{B(\rho_2)} f_\omega dv_g}{\rho_2^{\Lambda}}, \tag{20}$$

for any $0 < \rho_1 < \rho_2$.

Proof. from Corollary (2) for $\varphi = 1$, we know that formula (20) is true. □

From [9], we can say that the functional $E_f(\omega)$ of $\omega \in A^p(\xi)$ is slowly divergent if there exists a positive function $\psi(r)$ with $\int_{R_0}^{\infty} \frac{dr}{r\psi(r)} = \infty$ $(R_0 > 0)$, such that:

$$\lim_{R\to\infty} \int_{B(R)} \frac{f_\omega}{\psi(r(x))} dv_g < \infty. \tag{21}$$

Theorem 3. *Suppose that $\omega \in A^p(\xi)$ $(p \geq 1)$ satisfies the following equation:*

$$\int_M (\operatorname{div} S_{f,\omega})(X) dv_g = \int_M X(f)_\omega dv_g \tag{22}$$

for any $X \in \Gamma(TM)$, and $\xi : E \longrightarrow (M, g)$ is a smooth Riemannian vector bundle over $(M, \varphi^2 g_0)$. If $C_0 - \mu > 0$, φ satisfies (φ_1), (φ_2) and $E_f(\omega)$ is slowly divergent with $s\frac{\partial f}{\partial s} \leq f$ and $\frac{\partial f}{\partial s} \geq 0$, then $\omega = 0$.

Proof. From (17), we have:

$$\left\langle S_{f,\omega}, \frac{1}{2} L_X g \right\rangle + (X(f))_\omega \geq (C_0 - \mu) f_\omega. \tag{23}$$

On the other hand, taking $D = B(r)$ and $T = S_{f,\omega}$ in (7) of [9], we have

$$\int_{\partial D} S_{f,\omega}(X, \nu) ds_g = \int_D \left[\left\langle S_{f,\omega}, \frac{1}{2} L_X g \right\rangle + (\operatorname{div} S_{f,\omega})(X) \right] dv_g,$$

so that:

$$\begin{aligned}
&\int_{B(r)} \left\langle S_{f,\omega}, \frac{1}{2} L_X g \right\rangle dv_g + \int_{B(r)} (\operatorname{div} S_{f,\omega})(X) dv_g \\
&= \int_{\partial B(r)} S_{f,\omega}(X, \nu) ds_g \\
&= \int_{\partial B(r)} f_\omega g(X, \nu) ds_g - \int_{\partial B(r)} f'_\omega (\omega \odot \omega)(X, \nu) ds_g \\
&= r \int_{\partial B(r)} f_\omega \varphi ds_g - \int_{\partial B(r)} f'_\omega \varphi^{-1} r \sum_{i=1}^{m} < i_{\frac{\partial}{\partial r}} \omega, i_{\frac{\partial}{\partial r}} \omega) ds_g \\
&\leq r \int_{\partial B(r)} f_\omega \varphi ds_g.
\end{aligned} \tag{24}$$

Here $f'_\omega \geq 0$ because $\frac{\partial f}{\partial s} \geq 0$. Now suppose that ω is not always zero, so there exists a constant $R_1 > 0$ such that for $R \geq R_1$,

$$\int_{B(R)} f_\omega dv_g \geq C_3, \ C_3 > 0. \tag{25}$$

From (22):

$$\lim_{R\to\infty} \int_{B(R)} (\operatorname{div} S_{f,\omega})(X) dv_g = \lim_{R\to\infty} \int_{B(R)} (X(f))_\omega dv_g,$$

then, $\exists R_2 > R_1$ such that $\forall R \geq R_2$:

$$-\frac{(C_0-\mu)}{2}C_3 \leq \int_{B(R)} (\operatorname{div} S_{f,\omega})(X)dv_g - \int_{B(R)} X(f)_\omega dv_g \leq \frac{(C_0-\mu)}{2}C_3 \quad (26)$$

from (23), (24) and (26), we obtain

$$\begin{aligned} R\int_{\partial B(R)} f_\omega \varphi ds_g &\geq \int_{B(R)} < S_{f,\omega}, \frac{1}{2}L_X g > dv_g + \int_{B(R)} (\operatorname{div} S_{f,\omega})(X)dv_g \\ &\geq \int_{B(R)} \Big[< S_{f,\omega}, \frac{1}{2}L_X g > + X(f)_\omega \Big] dv_g - \frac{(C_0-\mu)}{2}C_3 \\ &\geq (C_0-\mu)\int_{B(R)} f_\omega dv_g - \frac{(C_0-\mu)}{2}C_3 \\ &\geq \frac{(C_0-\mu)}{2}C_3. \end{aligned},$$

that is:

$$\int_{\partial B(R)} f_\omega \varphi ds_g \geq \frac{(C_0-\mu)}{2R}C_3. \quad (27)$$

From (27) and $|\nabla r| = \varphi^{-1}$, we have:

$$\begin{aligned} \lim_{R\to\infty} \int_{B(R)} \frac{f_\omega}{\psi(r(x))}dv_g &= \int_0^\infty \frac{dR}{\psi(R)} \int_{\partial B(R)} f_\omega \varphi ds_g \\ &\geq \int_{R_2}^\infty \frac{dR}{\psi(R)} \int_{\partial B(R)} f_\omega \varphi ds_g \\ &\geq \int_{R_2}^\infty \frac{(C_0-\mu)}{2R}C_3 \frac{dR}{\psi(R)} \\ &\geq (C_0-\mu)C_3 \int_{R_2}^\infty \frac{dR}{2R\psi(r)} = \infty \end{aligned}$$

which contradicts (21), therefore $\omega \equiv 0$ □

Theorem 4. *Suppose that $\omega \in A^p(\xi)$ $(p \geq 1)$, satisfies the generalized integral f-conservation law, i.e.*

$$\int_M (\operatorname{div} S_{f,\omega})(X)dv_g = \int_M X(f)_\omega dv_g$$

and $\xi : E \longrightarrow (M,g)$ is a smooth Riemannian vector bundle over $(M, \varphi^2 g_0)$. If $C_0 - \mu > 0$, φ satisfies (φ_1) and (φ_2), with $\int_M f_\omega dv_g < \infty$ and $s\frac{\partial f}{\partial s} \leq f$, then $\omega \equiv 0$.

Proof. We take $X = \phi(r)r\frac{\partial}{\partial r} = \frac{1}{2}\phi(r)\nabla^0 r^2$, where ∇^0 denote the covariant derivative determined by g_0 and $\phi(r)$ is a nonnegative function determined later. By a direct computation, we have:

$$\Big\langle S_{f,\omega}, \frac{1}{2}L_X g\Big\rangle = \phi(r)r\frac{\partial \log\varphi}{\partial r}\Big\langle S_{f,\omega}, g\Big\rangle + \frac{1}{2}\varphi^2\Big\langle S_{f,\omega}, L_{\phi(r)r\frac{\partial}{\partial r}} g_0\Big\rangle. \quad (28)$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal basis with respect to g_0 and $e_m = \frac{\partial}{\partial r}$. We may assume that $Hess_{g_0}(r^2)$ becomes a diagonal matrix with respect to $\{e_i\}_{i=1}^m$. Then $\{\tilde{e}_i = \varphi^{-1} e_i\}$ is an orthonormal basis with respect to g. We compute:

$$\begin{aligned}
\varphi^2 \Big\langle S_{f,\omega}, \frac{1}{2} L_{\phi(r) r \frac{\partial}{\partial r}} g_0 \Big\rangle &= \varphi^2 \sum_{i,j=1}^m S_{f,\omega}(\tilde{e}_i, \tilde{e}_j)(L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(\tilde{e}_i, \tilde{e}_j) \\
&= \varphi^2 \Big[\sum_{i,j=1}^m f_\omega g(\tilde{e}_i, \tilde{e}_j)(L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(\tilde{e}_i, \tilde{e}_j) \\
&\quad - f_\omega^{'}(\omega \odot \omega)(\tilde{e}_i, \tilde{e}_j)(L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(\tilde{e}_i, \tilde{e}_j) \Big] \\
&= \sum_{i=1}^m f_\omega (L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(e_i, e_i) \\
&\quad - \sum_{i,j=1}^m f_\omega^{'}(\omega \odot \omega)(\tilde{e}_i, \tilde{e}_j)(L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(e_i, e_j) \\
&= \phi(r) \sum_{i=1}^m f_\omega Hess_{g_0}(r^2)(e_i, e_i) + 2 f_\omega r \phi^{'}(r) \\
&\quad - \phi(r) \sum_{i,j=1}^m f_\omega^{'}(\omega \odot \omega)(\tilde{e}_i, \tilde{e}_j) Hess_{g_0}(r^2)(e_i, e_j) \\
&\quad - 2 f_\omega^{'} r \phi^{'}(r)(\omega \odot \omega)(\tilde{e}_m, \tilde{e}_m) \\
&\geq \phi(r) f_\omega [2 + (m-1)\lambda_{\min}] \\
&\quad - \phi(r) f_\omega^{'} \max(2, \lambda_{\max}) \sum_{i=1}^{m-1} (\omega \odot \omega)(\tilde{e}_i, \tilde{e}_i) \\
&\quad + 2 f_\omega r \phi^{'}(r) - 2 f_\omega^{'} r \phi^{'}(r)(\omega \odot \omega)(\tilde{e}_m, \tilde{e}_m) \\
&\geq \phi(r) f_\omega [2 + (m-1)\lambda_{\min}] \\
&\quad - \phi(r) f_\omega^{'} \max(2, \lambda_{\max}) p |\omega|^2 2 f_\omega r \phi^{'}(r) \\
&\quad - 2 f_\omega^{'} r \phi^{'}(r)(\omega \odot \omega)(\tilde{e}_m, \tilde{e}_m).
\end{aligned}$$

As $s \frac{\partial f}{\partial s} \leq f$ we find that $\frac{|\omega|^2}{2} f_\omega^{'} \leq f_\omega$, we obtain:

$$\begin{aligned}
\varphi^2 \Big\langle S_{f,\omega}, \frac{1}{2} L_{\phi(r) r \frac{\partial}{\partial r}} g_0 \Big\rangle \geq\ & \phi(r) f_\omega [2 + (m-1)\lambda_{\min} \\
& - 2p \max(2, \lambda_{\max})] + 2 f_\omega r \phi^{'}(r) \\
& - 2 f_\omega^{'} r \phi^{'}(r)(\omega \odot \omega)(\tilde{e}_m, \tilde{e}_m).
\end{aligned} \tag{29}$$

From (14), (28) , (29) and (φ_1), (φ_2):

$$\begin{aligned}
\Big\langle S_{f,\omega}, \frac{1}{2} L_X g \Big\rangle \geq\ & \phi(r) r \frac{\partial \log \varphi}{\partial r} [m f_\omega - 2p f_\omega] + \phi(r) f_\omega [1 + \frac{(m-1)}{2} \lambda_{\min} - p \max(2, \lambda_{\max})] \\
& + f_\omega r \phi^{'}(r) - f_\omega^{'} r \phi^{'}(r)(\omega \odot \omega)(\tilde{e}_m, \tilde{e}_m)
\end{aligned}$$

$$\begin{aligned}
&\geq \phi(r) r \frac{\partial \log \varphi}{\partial r} f_\omega [m-2p] + \phi(r) f_\omega [1 + \frac{(m-1)}{2}\lambda_{\min} - p\max(2,\lambda_{\max})] \\
&\quad + f_\omega r \phi^{'}(r) - f_\omega^{'} r \phi^{'}(r)(\omega \odot \omega)(\tilde{e}_m, \tilde{e}_m) \\
&\geq \phi(r) f_\omega [(m-2p) r \frac{\partial \log \varphi}{\partial r} + 1 + \frac{(m-1)}{2}\lambda_{\min} - p\max(2,\lambda_{\max})] \\
&\quad + f_\omega r \phi^{'}(r) - f_\omega^{'} r \phi^{'}(r)(\omega \odot \omega)(\tilde{e}_m, \tilde{e}_m) \\
&\geq C_0 \phi(r) f_\omega + f_\omega r \phi^{'}(r) - f_\omega^{'} r \phi^{'}(r)(\omega \odot \omega)(\tilde{e}_m, \tilde{e}_m).
\end{aligned} \tag{30}$$

From $\mu = \sup_{M\times\mathbb{R}_+^*} \frac{r}{f}|\frac{\partial f}{\partial r}|$, we have:

$$\mu \geq -r\frac{1}{f(x,s)}(\frac{\partial f}{\partial r})_{(x,s)}, \forall (x,s) \in M \times \mathbb{R},$$

so that:

$$\mu \geq -r\frac{1}{f_\omega}(\frac{\partial f}{\partial r})_\omega \frac{\phi(r)}{\phi(r)},$$

that is:

$$X(f)_\omega \geq -\mu f_\omega \phi(r).$$

We get:

$$\left\langle S_{f,\omega}, \frac{1}{2}L_X g \right\rangle + X(f)_\omega \geq (C_0 - \mu)\phi(r) f_\omega + f_\omega r \phi^{'}(r) - f_\omega^{'} r \phi^{'}(r)(\omega \odot \omega)(\tilde{e}_m, \tilde{e}_m). \tag{31}$$

For any fixed $R > 0$, we take a smooth function $\phi(r)$ which takes value 1 on $B_{\frac{R}{2}}(x_0)$, 0 outside $B_R(x_0)$ and $0 \leq \phi(r) \leq 1$ on $T_R(x_0) = B_R(x_0) - B_{\frac{R}{2}}(x_0)$. And $\phi(r)$ also satisfies the condition: $|\phi^{'}(r)| \leq \frac{C_2}{r}$ on M, where C_2 is a positive constant.
From (12) and (31), with the condition $s\frac{\partial f}{\partial s} \leq f$, we have:

$$\begin{aligned}
0 &\geq \int_M (C_0-\mu)\phi(r) f_\omega dv_g + \int_M f_\omega r\phi^{'}(r) dv_g - \int_M f_\omega^{'} r \phi^{'}(r)(\omega\odot\omega)(\tilde{e}_m,\tilde{e}_m) dv_g \\
&\geq \int_{B_{\frac{R}{2}}(x_0)} (C_0-\mu) f_\omega dv_g + \int_{T_R(x_0)} f_\omega r \phi^{'}(r) dv_g - \int_{T_R(x_0)} f_\omega^{'} r\phi^{'}(r)(\omega\odot\omega)(\tilde{e}_m,\tilde{e}_m) dv_g \\
&\geq \int_{B_{\frac{R}{2}}(x_0)} (C_0-\mu) f_\omega dv_g - C_2 \int_{T_R(x_0)} f_\omega dv_g - C_2 p \int_{T_R(x_0)} f_\omega^{'} |\omega|^2 dv_g \\
&\geq (C_0-\mu)\int_{B_{\frac{R}{2}}(x_0)} f_\omega dv_g - C_2\int_{T_R(x_0)} f_\omega dv_g - 2C_2 p \int_{T_R(x_0)} f_\omega dv_g \\
&\geq (C_0-\mu)\int_{B_{\frac{R}{2}}(x_0)} f_\omega dv_g - (1+2p)C_2 \int_{T_R(x_0)} f_\omega dv_g.
\end{aligned} \tag{32}$$

From $\int_M f_\omega dv_g < \infty$, we obtain $\lim_{R\longrightarrow\infty} \int_{T_R(x_0)} f_\omega dv_g = 0$, so by equation (32), we have:

$$0 \geq (C_0 - \mu) \lim_{R\longrightarrow\infty} \int_{B_{\frac{R}{2}}(x_0)} f_\omega dv_g,$$

that is:

$$0 \geq (C_0 - \mu) \int_M f_\omega dv_g.$$

So that $\omega = 0$. □

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