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QUASI INVO-CLEAN RINGS

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Abstract

An element v of an arbitrary ring R is called an *involution* if $v^2 = 1$ and a *quasi-involution* if either v or $1 - v$ is an involution. We thereby define R to be *quasi invo-clean* as the one whose elements are written in the form of a sum of an idempotent and a quasi-involution. This considerably extends the class of invo-clean rings introduced by the present author in Commun. Korean Math. Soc. (2017) and the class of weakly tripotent rings introduced by Breaz and Cîmpean in Bull. Korean Math. Soc. (2018). We, moreover, prove the curious fact that the newly defined class of quasi invo-clean rings actually coincides with the class of weakly invo-clean rings defined by Danchev in Afr. Mat. (2017).

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1 Introduction and Preliminaries

Throughout the text of the current article, all rings R are assumed to be associative possessing the identity element 1 which differs from the zero element 0. Our standard terminology and notation are mainly in agreement with [7]. For instance, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R, $J(R)$ the Jacobson radical of R, and $C(R)$ the center of R. The specific notions and notations will be given explicitly in what follows.

Recall that an element y of a ring R is called *tripotent* if the equality $y^3 = y$ holds. If each element of R is equipped with this property, the ring R is said to be *tripotent* as well. The complete description of such rings is given in [6]. Specifically, they are a subdirect product (= a subring of a direct product) of a family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

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It is worth noticing that special tripotent elements are the involutions, i.e., the units v of order at most 2 (writing $v^2 = 1$) as well as all idempotents (writing $e^2 = e$).

In a similar vein, in [1] were recently defined and explored the so-called *weakly tripotent* rings that are rings in which, for every element x, at least one of the elements x or $1 + x$ is a tripotent. It is immediate that weakly tripotent rings of characteristic 3 are always tripotent, because $(1+x)^3 = 1+x^3 = 1+x$ yields that $x^3 = x$. Also, referring to [1, Proposition 2], the given conditions on x and $1 + x$ to be tripotents are obviously equivalent to either x or $1 - x$ to be tripotent by replacing x with $-x$.

On the other hand, recently in [2] *invo-clean* rings were introduced as those rings R such that, for each element $r \in R$, there exists an involution $v \in R$ and an idempotent $e \in R$ depending on r such that $r = v + e$. In particular, if $ve = ev$, invo-clean rings are termed *strongly invo-clean*. We hereafter shall respectively call such elements r invo-clean and strongly invo-clean as well. These rings were intensively studied in [4], [8] and [9], respectively. Some further generalization to such rings was established in [3] by defining the class of *weakly invo-clean* rings R for which at least one of the equalities $r = v + e$ or $r = v - e$ holds – henceforth we shall call such elements r weakly invo-clean too. We will somewhat comment in detail these rings below.

In the light of the above element-wise elementary observations about weakly tripotents, one can note the simple but useful fact that the element r is (strongly) invo-clean exactly when the element $1-r$ is (strongly) invo-clean. In fact, $r = v+e$ implies that $1 - r = (-v) + (1 - e)$, where $-v$ is an involution and $1 - e$ is an idempotent. Reciprocally, the writing $1 - r = w + f$, for some involution w and an idempotent f, forces that $r = (-w) + (1 - f)$, as required.

In that way, what can be said for any weakly invo-clean element r, is the following:

- r is weakly invo-clean $\iff$ r or $1 + r$ is invo-clean.

Indeed, if $r = v - e$, then $r + 1 = v + (1 - e)$. Conversely, $r + 1 = w + f$ for some involution w and some idempotent f gives that $r = w - (1 - f)$, as needed.

- r is weakly invo-clean $\iff$ $r = v + e$, where v or $1 + v$ is an involution.

Indeed, $r = v - e$ with involution v can be equivalently written as $r = (v - 1) + (1 - e)$ with involution $1 + (v - 1) = v$. Reversibly, $r = v + e$ with involution $1 + v$ is equivalent to $r = (1 + v) - (1 - e)$, as expected.

That is why, inspired by these two equivalencies and by analogy with the property of the already commented above weakly tripotent rings, it will be of interest to consider those records $v + e$ in a ring for which $e^2 = e$ and either $v^2 = 1$ or $(1 - v)^2 = 1$ – such an element v will be called in the sequel a *quasi-involution*.

So, taking into account the second bullet point listed above, we come to our key tool.

Definition 1.1. We shall say that a ring R is *quasi invo-clean* if, for any $r \in R$, there exists a quasi-involution $v \in R$ and an idempotent $e \in R$ both depending on r such that $r = v + e$.

It is evident that invo-clean rings are both weakly invo-clean and quasi invo-clean as this implication is extremely non-reversible by looking quickly at the field $\mathbb{Z}_5$. Even something more, in an invo-clean ring R each element $r \in R$ can be written simultaneously in the form $r = v + e = w + f$ such that $e, f \in Id(R)$ and both v and $1 - w$ are involutions. Indeed, the first record for r follows directly from the definition of an invo-clean clean. Now, since $r - 1$ also belongs to R, we may write that $r - 1 = u + f$ for some involution u and idempotent f of R. So, $r = w + f$, where $w := 1 + u$ and hence seeing at once that $1 - w = -u$ is an involution, too, as promised. We shall freely use this fact in Lemma 2.3 in the sequel.

On the other side, a direct inspection shows that the field $\mathbb{F}_4$ consisting of four elements is surely not a quasi invo-clean ring of characteristic 2.

Incidentally, the next relationship significantly strengthens [1, Corollary 4] and is our basic point of view.

Proposition 1.2. *Every weakly tripotent ring is strongly invo-clean.*

Proof. For such a ring R, we have $r^3 = r$ or $(1 - r)^3 = 1 - r$ whenever $r \in R$. By what we have shown above an element r is strongly invo-clean uniquely when the element $1 - r$ is strongly invo-clean, so it suffices to consider only the tripotent element r. Furthermore, one writes that $r = (1 - r^2) + (r^2 + r - 1)$. A direct manipulation shows that $1 - r^2 \in Id(R)$ as $r^2 \in Id(R)$ and that $(r^2 + r - 1)^2 = 1$, observing elementarily that these two elements commute, as desired. □

However, the next construction unambiguously illustrates that the reverse implication in this statement is impossible even in the commutative case. Specifically, the following assertion holds:

Example 1.3. There exists a commutative (strongly) invo-clean ring of characteristic 2 which is *not* weakly tripotent.

In fact, let us consider the group ring $R = BG$, where $B \ncong \mathbb{Z}_2$ is a Boolean ring and G is a group consisting only of elements of order at most 2. Clearly, the equality $r^4 = r^2$ is valid for each element $r \in R$ as $\mathrm{char}(R) = 2$. Consequently, one writes that $r = (1 + r^2) + (1 + r + r^2)$, where a routine direct check shows that $1 + r^2$ is an idempotent as so is r^2, and $1 + r + r^2$ is an involution, as required.

However, as B contains non-trivial idempotents, elementary calculations show that both inequalities $r^3 \neq r^2$ and $r^3 \neq r$ are fulfilled, which facts we leave to the interested reader for an easy inspection. This, in turn, assures us that R cannot be weakly tripotent, because the first given inequality $r^3 \neq r^2$ is amounting to the inequality $(1 - r)^3 \neq 1 - r$. This concludes the example and substantiates our claim.

It, however, manifestly seems that the non-commutative weakly tripotent rings are not comprehensively characterized in [1], so that some more material in this aspect is definitely needed. In fact, we already proved in Proposition 1.2 stated above that these rings are strongly invo-clean. Moreover, as showed in [2, Corollary 2.17], accomplishing this with the chief result from [5], if a ring R is strongly invo-clean, then R is decomposable as $R_1 \times R_2$, where $R_1 = \{0\}$ or $R_1/J(R_1)$ is Boolean with nil $J(R_1)$ of index of nilpotence at most 3, and $R_2 = \{0\}$ or R_2 is a subdirect product of a family of copies of the field $\mathbb{Z}_3$.

Resuming all details, our motivation in writing up the paper is to encompass the two classes of weakly tripotent rings mentioned above and invo-fine rings generalizing them to this new context of quasi invo-clean rings by obtaining a complete description of their crucial properties.

2 Main Results and Problems

We begin our work with the following technicality which is a common strengthening of [1, Lemma 1 (3)].

Proposition 2.1. *Each quasi invo-clean ring R decomposes as $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are either zero rings or non-zero quasi invo-clean rings for which $8 = 0$ in R_1, $3 = 0$ in R_2 and $5 = 0$ in R_3, respectively.*

Proof. Every element can be written as $v+e$, where $e^2 = e$ and $v^2 = 1$ or $v^2 = 2v$. Since any of the elements $0, 1, 2, 3$ clearly has such a trivial record, we approach the element $4 = v+e$. In the first case, we have that $(4-e)^2 = 1$ ensuring $15 = 7e$. Squaring this, one has that $225 = 49e = 105$ which means that $120 = 8.3.5 = 0$. In the second case, we deduce that $(4 - e)^2 = 2(4 - e)$ assuring $8 = 5e$. The squaring leads to $64 = 25e = 40$ which amounts to $24 = 8.3 = 0$. Finally, we derive in both cases that $2^3.3.5 = 0$, as promised.

But since $(8, 3, 5) = 1$ we, therefore, apply the Chinese Remainder Theorem to get the wanted direct decomposition into the indicated above rings R_1, R_2 and R_3. It now easily follows by element-wise arguments that the three direct factors R_1, R_2 and R_3 are quasi invo-clean, as claimed. □

Our next pivotal instruments are the following ones.

Lemma 2.2. *A ring R of characteristic 5 is isomorphic to $\mathbb{Z}_5$ if, and only if, every element of R satisfies at least one of the equations $x^3 = x$ or $x^3 = -x$.*

Proof. "**Necessity.**" It is pretty straightforward to verify that all elements of $\mathbb{Z}_5 = \{0, 1, 2, 3, 4 \mid 5 = 0\}$ satisfy at least one of the equations $x^3 = x$ or $x^3 = -x$.

"**Sufficiency.**" Letting P be the subring of R generated by 1, we see that $P \cong \mathbb{Z}_5$. We claim that $P = R$, so to show that we assume in a way of contradiction that there exists $b \in R \setminus P$. With no loss of generality, we shall also assume that $b^3 = b$ since $b^3 = -b$ obviously implies that $(2b)^3 = 2b$ as $5 = 0$ and $b \notin P \iff 2b \notin P$.

Let us now $(1+b)^3 = -(1+b)$. Hence $b = b^3$ along with $5 = 0$ enable us that $b^2 = 1$. This allows us to conclude that $(1+2b)^3 \neq \pm(1+2b)$, however. In fact, if $(1+2b)^3 = 1+2b$, then one deduces that $2b = 3 \in P$ (and so $b = -1 = 4 \in P$) which is, certainly, manifestly untrue. If now $(1+2b)^3 = -1-2b$, then one infers that $2b = 2 \in P$ which is, of course, false as well. That is why, $(1+b)^3 = 1+b$ must hold. This, in turn, guarantees that $b^2 = -b$. Moreover, $b^3 = b$ is equivalent to $(-b)^3 = -b$ and, by what we have proved so far applied to $-b \notin P$, it follows that $-b = b^2 = (-b)^2 = -(-b) = b$. Consequently, $2b = 0 = 6b = b \in P$ because $5 = 0$, which is the wanted contradiction. We thus conclude that $P = R$, as claimed. □

We, however, notice the interesting fact that if R is a ring of characteristic 5 whose elements satisfy the equation $x^5 = x$, then R is a subdirect product of a family of copies of the field $\mathbb{Z}_5$ (see, for instance, [7]). In our situation both $x^3 = x$ and $x^3 = -x$ imply $x^5 = x$, but accomplishing this with the previously mentioned fact is absolutely not enough to conclude our claim.

Lemma 2.3. *The direct product $P \times L$ of two rings P and L is a quasi invo-clean ring if, and only if, both P and L are quasi invo-clean rings and at least one of P or L is an invo-clean ring.*

Proof. Given the ring $P \times L$ is quasi invo-clean, it readily follows by simple element-wise arguments that both P and L are also quasi invo-clean. Further, we assume in a way of contradiction that neither P nor L is invo-clean. As already commented above, there will be elements $a \in P$ and $b \in L$ such that a is not presentable as $a = v + e$ with $v^2 = 1$ and $e^2 = e$ for some $v, e \in P$ and b is not presentable as $b = w + f$ with $(1-w)^2 = 1$ and $f^2 = f$ for some $w, f \in L$. What we claim now is that the pair $(a, b) \in P \times L$ is definitely not a quasi invo-clean element of the direct product $P \times L$, which fact has an easy direct verification and thus leaving it to the interested reader for a confirmation, and which fact is exactly the desired contradiction.

Conversely, given for concreteness that P is quasi invo-clean and L is invo-fine, it easily follows from routine technical element-wise arguments by considering the pair $(a, b) \in P \times L$, which details we leave to the interested reader for an inspection, that the direct product $P \times L$ is also quasi invo-clean, as claimed. If, however, both P and L are invo-clean rings, then it follows directly from [2] that the direct product $P \times L$ is also an invo-clean ring, and thus it is quasi invo-clean, as required. □

Let us remember the definition of a nil-clean ring which states as follows (see, e.g., [5]): A ring is said to be *nil-clean* if every its element can be written as a sum of a nilpotent and an idempotent.

We now arrive at our central result.

Theorem 2.4. *A ring R is quasi invo-clean if, and only if, $R \cong R_1 \times R_2 \times R_3$, where $R_1 = \{0\}$ or R_1 is an invo-clean ring of characteristic not exceeding* 8 *which* *is nil-clean, $R_2 = \{0\}$ or R_2 is a subdirect product of a family of copies of $\mathbb{Z}_3$, and $R_3 = \{0\}$ or $R_3 \cong \mathbb{Z}_5$.*

Proof. "$\Rightarrow$". In virtue of Proposition 2.1, one may write that $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are quasi invo-clean rings such that $\mathrm{char}(R_1) \leq 8$ is a power of 2, $\mathrm{char}(R_2) = 3$ and $\mathrm{char}(R_3) = 5$.

We will now characterize any of the three direct factors R_1, R_2, R_3 separately as follows:

Characterizing R_1: Here $8 = 0$ and so 2 is a nilpotent. Our tactic here is to show that R_1 is an invo-clean ring for which $z^2 = 2z$ whenever $z \in Nil(R_1)$ and so it will follow from [2, Proposition 2.10] that R_1 is nil-clean as well. In doing that, we shall consider the three possibilities about the characteristic of R_1 separately, namely:

• $\mathrm{char}(R_1) = 2$. For any $r \in R_1$, one writes that $r = v + e$, where $v, e \in R_1$ with $v^2 = 1$ or $v^2 = 2v = 0$, and $e^2 = e$. In the case when $v^2 = 1$, we write that $r = (v + 1) + (1 + e)$, where $(v + 1)^2 = 0$ and $(1 + e)^2 = 1 + e$, so we are set.

• $\mathrm{char}(R_1) = 4$. Writing for any $r \in R_1$ that $r = v + e$ for $v \in R_1$ with $v^2 = 1$ or $v^2 = 2v$ and for $e \in R_1$ with $e^2 = e$, we are ready in the case when $v^2 = 2v$, because one observes that $v^3 = 2v^2 = 0$. However, in the case when $v^2 = 1$, one modifies the record for r as $r = (v - 1 + 2e) + (1 - e)$, where $v-1 \in Nil(R_1)$ since $(v-1)^2 = 2(1-v)$ and $2 \in Nil(R_1)$ as $4 = 0$. Consequently, it is not too hard to check that $v - 1 + 2e \in Nil(R_1)$ because $(v - 1 + 2e)^2 = (v-1)^2+2(v-1)e+2e(v-1) = 2[(1-v)+(v-1)e+e(v-1)]$ and so $(v-1+2e)^4 = 0$.

• $\mathrm{char}(R_1) = 8$. Since for any $v \in R_1$ with $v^2 = 2v$ it must be that $v^4 = 4v^2 = 0$, the same manipulation as in the previous case allows us to conclude that R_1 is too nil-clean in this situation.

We now will prove that each nilpotent element z lying in R_1 possesses the identity $z^2 = 2z$. In fact, if we write $z = v + e$ for some $v \in R_1$ with $v^2 = 1$ and $e \in Id(R_1)$, the application of [2, Corollary 2.6] leads to $e = 1$, and thus $z = v+1$ which, by squaring, gives that $z^2 = 2z$, as pursued. But if now $z = w+f$ for some $w \in R_1$ with $(1 - w)^2 = 1$ (that is, $w^2 = 2w$) and $f \in Id(R_1)$, one modifies this record as $-z + 2 = (1 - w) + (1 - f)$. Since $-z + 2$ is still a nilpotent in R_1, as so are both z and 2, again by applying [2, Corollary 2.6] we infer that $1 - f = 1$, i.e., $f = 0$. Therefore, $z = w$ and immediately $z^2 = 2z$, as asked for.

Characterizing R_2: Here $3 = 0$, that is, $\mathrm{char}(R_2) = 3$. We claim that for any $x \in R_2$ the equality $x^3 = x$ holds and thus we may employ the major result from [6] to get the desired characterization. To that goal, we foremost assert that the ring R_2 is reduced and hence abelian (that is, all its idempotents are central). In fact, given $u \in U(R_2)$ with $u^3 = 1$, we will derive that $u^2 = 1$. So, write $u = v + e$, where $e \in Id(R_2)$ and $v \in R_2$ with $v^2 = 1$ or $(1 - v)^2 = 1$ as the latter guarantees that $v^2 = 2v = -v$. In the first case when $v^2 = 1$, one finds that $1 = u^3 = v + vev + ve + ev + eve$. Multiplying this by the right with e leads to

$ve + e = veve - eve$. Also, the multiplication of the last equality with v by the left allows us to write that $e + ve = eve - veve$. Thus, comparing both equations, we extract that $-ve - e = e + ve$, i.e., that $2e = -2ve$, i.e., $e = -ve$. Hence $ue = ve + e = 0$ which means that $e = 0$ and $u = v$ with $u^2 = 1$. In the second case when $v^2 = -v$, one gets by the same token that $1 = u^3 = v + vev + eve + e$ and $(1 - e) - v = vev + eve$, so by multiplying the last with e from the left enables us that $-ev = evev + eve$. Therefore, its multiplication from the right with v is a guarantor that $ev = -evev + evev = 0$. Thus $1 - e = v$ and squaring this implies that $1 - e = e - 1$, that is, $2e = 2$, that is, $-e = -1$, that is, $e = 1$. Finally, $v = 0$ and $u = 1$ forcing $u^2 = 1$. That is why, in both cases, we infer that u is a unit of order at most 2, as expected.

Furthermore, given $q \in Nil(R_2)$ with $q^2 = 0$, it follows that $q^3 = 0$ and thus $(1 + q)^3 = 1$. By what we have shown above, $(1 + q)^2 = 1$ which insures that $2q = -q = 0$, i.e., $q = 0$, as asserted. This substantiates our initial claim that R_2 is a reduced whence abelian ring. Consequently, for every its element $x = w + f$ with $f \in Id(R_2)$ and $w \in R_2$ such that $w^2 = 1$ or $w^2 = -w$ (so, in both cases, $w^3 = w$ as well as $f^3 = f$), one concludes that $x^3 = (w + f)^3 = w^3 + f^3 = w + f = x$, as claimed.

Characterizing R_3: Here $5 = 0$, that is, $\text{char}(R_3) = 5$. Firstly, we claim that R_3 is reduced. To show this, given $q \in Nil(R_3)$ with $q^2 = 0$, we write $q = v + e$ for some $v \in R_3$ with $v^2 = 1$ or $v^2 = 2v$ and some $e \in Id(R_3)$. In the case when $v^2 = 2v$, one obtains by squaring that $2v + ve + ev + e = 0$. Thus, multiplying subsequently by e from the right and from the left, one gets that $3ve + eve + e = 0$ and that $4eve + e = 0$. Hence $eve = e$ and substituting this above leads to $2ve = 2e$. Similarly, using the same tricks, one has that $2ev = 2e$ and so $2ve = 2ev$. Furthermore, with this at hand, since $4v + 2ve + 2ev + 2e = 0$ is tantamount to $4v + 6e = 4v + e = 0$, one deduces that $e = -4v = v$. Therefore, $e = e^2 = 2e$ gives that $e = 0 = v$ and thus $q = 0$, as expected. Next, in the case when $v^2 = 1$, the squaring leads to $1 + ve + ev + e = 0$. The multiplication of this equality from the right by e implies that $2e + ve + eve = 0$ and the multiplication of the last equation by $2e$ from the left yields that $eve = -e$. The substitution of this relation above enables us that $ve = -e$. Analogously, utilizing the same arguments, one has that $ev = -e$ and, after all, $ev = ve$. Consequently, $1 - e - e + e = 0$, i.e., $e = 1$ whence $v = -1$. Finally, $q = 0$, as claimed.

Secondly, what we now assert is that $Id(R_3) = \{0, 1\}$. To prove that, we assume in a way of contradiction that there exists $e \in Id(R_3) \setminus \{0, 1\}$. By what we have already shown, e lies in $C(R_3)$, hence we may decompose $R_3 = R_3e \oplus R_3(1 - e) \cong P \times L$, where $P \cong R_3e$ and $L \cong R_3(1 - e)$ are both quasi invo-clean rings of characteristic 5. However, as neither of them is not invo-clean, this manifestly contradicts Lemma 2.3 and thereby guarantees our initial claim that R_3 is indecomposable.

Thirdly, we are sure that all elements of R_3 satisfy at least one of the equations $x^3 = x$ or $x^3 = -x$ and thus Lemma 2.2 now applies to detect the desired isomorphism between R_3 and $\mathbb{Z}_5$. In fact, one has for any $r \in R_3$ that $r = v$ or

$r = v + 1$ for some $v \in R_3$ such that $v^2 = 1$ or $v^2 = 2v$.

- $r = v$ with $v^2 = 1$. Hence $r^2 = 1$ implying that $r^3 = r$.
- $r = v$ with $v^2 = 2v$. Hence $r^2 = 2r$ and $r^3 = 2r^2 = 4r = -r$.
- $r = v + 1$ with $v^2 = 1$. Hence $r^2 = 2r$ and $r^3 = 2r^2 = 4r = -r$.
- $r = v + 1$ with $v^2 = 2v$. Hence $v^4 = -v^2$ and $(v^4)^2 = v^4$. Then either $v^4 = 0$ or $v^4 = 1$, i.e., $v^2 = 0$ or $v^2 = -1$, i.e., $2v = 0$ or $2v = -1$. Since $r^2 = 1 - v$, we have that $r^2 = r$ when $2v = 0$ whence $r^3 = r$. If now $2v = -1$, then $-v = 4v = -2$, so that $v = 2$ and $r = 3 = -2$ guaranteeing that $r^3 = -r$.

Having in mind these four bullet points, we are done.

"$\Leftarrow$". Since $R_1 \times R_2$ is invo-clean, the proof follows immediately by application of Lemma 2.3. □

Interestingly, as an immediate consequence, we derive the following surprising assertion.

Corollary 2.5. *The classes of quasi invo-clean rings and weakly invo-clean rings coincide.*

Proof. For convenience, we restate the main characterization theorem [3, Theorem 4.18] concerning weakly invo-clean rings in the following slightly modified form: *A ring R is weakly invo-clean $\iff$ $R \cong R_1 \times R_2 \times R_3$, where $R_1 = \{0\}$ or R_1 is an invo-clean ring of characteristic 2^k with $k \in \{1, 2, 3\}$, $R_2 = \{0\}$ or R_2 is embedding into the direct product of a family of copies of $\mathbb{Z}_3$, and either $R_3 = \{0\}$ or $R_3 \cong \mathbb{Z}_5$.* Comparing this isomorphism characterization with Theorem 2.4, which sounds in the same manner, and bearing in mind that every invo-clean ring is simultaneously weakly invo-clean and quasi invo-clean, the statement follows after all. □

The next commentaries are somewhat elaborating on that.

Remark 2.6. As an alternative direct argument to show that the classes of quasi invo-clean and weakly invo-clean rings do coincide, we also propose the following one: As it was already mentioned above, an element $r \in R$ is weakly invo-clean if, and only if, r or $r + 1$ is invo-clean. Now, let us assume that $r \in R$ is quasi invo-clean. Then either r is invo-clean or $r = v + e$ with $e^2 = e$ and $(1 - v)^2 = 1$. In the second case we have $r - 1 = (v - 1) + e$ and $(v - 1)^2 = (1 - v)^2 = 1$. Hence an element $r \in R$ is quasi invo-clean if, and only if, r or $r - 1$ is invo-clean. That is why, it is clear that quasi-invo-clean rings are exactly weakly invo-clean rings, and vice versa.

Another more direct approach could be the following one: Firstly, assume that R is quasi invo-clean and let $r \in R$. Then $1 - r$ admits the decomposition $1 - r = v + e$ with e an idempotent and either v or $1 - v$ an involution. In the first case, $r = 1 - (1 - r) = 1 - (v + e) = -v + (1 - e)$ with $-v$ an involution and $1 - e$ an idempotent. In the second case, $r = 1 - (1 - r) = 1 - (v + e) = (1 - v) - e$ with

$1-v$ an involution and e an idempotent. So, in both cases, r is a weakly invo-clean element, as needed. Secondly, assume that R is weakly invo-clean and let $r \in R$. Then $1-r$ admits either the decomposition $r = v+e$ or the decomposition $r = v-e$ with e an idempotent and v an involution. In the first case, and as above demonstrated, $r = 1-(1-r) = 1-(v+e) = -v+(1-e)$ with v an involution and $1-e$ an idempotent. In the second case, $r = 1-(1-r) = 1-(v-e) = (1-v)+e$ with $v = 1-(1-v)$ an involution and e an idempotent. Thus, in both cases, r is a quasi invo-clean element, as required.

Likewise, in the light of the corresponding results from [1] concerning weakly tripotent rings discussed above and in accordance with Corollary 2.5, it is worthwhile emphasizing the fact that we actually have proved the incidental equivalence that a ring R is quasi invo-clean if, and only if, each element $r \in R$ has a special representation as $v+e$, where $v, e \in R$ such that e is an idempotent and v or $1-v$ is an involution, if and only if, each element $r \in R$ has a special representation as $w+f$, where $w, f \in R$ such that f is an idempotent and w or $1+w$ is an involution. This, however, is perhaps not true for any single element of R.

We close our work with the following intriguing and non-trivial question. Precisely, regarding [3, Problem 4] and the main result from [4], one may ask the following:

Problem 2.7. If the ring R is quasi invo-clean, does it follow that the corner subring eRe is also quasi invo-clean for any $e \in Id(R)$?

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