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OSCILLATION THEOREMS FOR FOURTH-ORDER HYBRID NONLINEAR FUNCTIONAL DYNAMIC EQUATIONS WITH DAMPING

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Abstract

In this paper, we will establish some oscillation criteria for the fourth-order hybrid nonlinear functional dynamic equations with damping. The authors present new oscillation criteria to check whether all solutions of an equation, in this class, oscillate. This study aims to present some new sufficient conditions for the oscillatory of solutions to a class of fourth-order hybrid nonlinear functional dynamic equations by use of Riccati technique and other method. Illustrative examples are also provided.

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1 Introduction

The theory of oscillations is an important branch of the applied theory of dynamic equations related to the study of oscillatory phenomena in technology and natural and social sciences. In recent years, there has been much research activity concerning the oscillation of solutions of various dynamic equations on time scales.

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Motivated by the above articles, now, in this article, we are interested in the oscillation of solutions of the fourth-order hybrid nonlinear dynamic equation

$$\left(\frac{a(t)\left(u^{(2)}(t)\right)^{\beta}}{f(t,u(t))}\right)^{(2)}+\sum_{i=1}^{i=n} b_i(t)\, u^{\beta}(\tau_i(t)) = g(t,u(\eta(t))), \quad \text{for all } t\in J_{t_0}, \tag{1}$$

where $J_{t_0}=[t_0,\infty)$, n is an integer and β is a quotient of odd integer, such as $\beta>0$ and $n\geq 1$. Since we are interested in oscillation, we assume throughout this paper that the given interval of the form $J_{t_0}:=[t_0,\infty)$. The equation (1) will be studied under the following assumptions:

(C_1) The function $f: J_{t_0}\times\mathbb{R}\to\mathbb{R}$ such that $f\in C(J_{t_0}\times\mathbb{R},\mathbb{R})$, $uf(t,u)>0$, for all $(t,u)\in J_{t_0}\times\mathbb{R}-\{0\}$ and there is $\rho\in C(J_{t_0},[0,\infty))$ such that

$$f(t,u)\geq\rho(t), \quad \text{for all } (t,u)\in J_{t_0}\times\mathbb{R}-\{0\}. \tag{2}$$

(C_2) The function $g: J_{t_0}\times\mathbb{R}\to\mathbb{R}$ such that $g\in C(J_{t_0}\times\mathbb{R},\mathbb{R})$, $ug(t,u)>0$, for all $(t,u)\in J_{t_0}\times\mathbb{R}-\{0\}$ and there is $\sigma\in C(J_{t_0},[0,\infty))$ such that

$$u^{-\beta}g(t,u)\leq\sigma(t), \quad \text{for all } (t,u)\in J_{t_0}\times\mathbb{R}-\{0\}. \tag{3}$$

(C_3) $a,\{b_i\}_{i\in\{1,..,n\}}\in C(J_{t_0},[0,\infty))$, such as

$$\int_{t_0}^{\infty}\left(\frac{s\rho(s)}{a(s)}\right)^{\frac{1}{\beta}}ds=\infty. \tag{4}$$

and

$$B_n(t):=\sum_{i=1}^{i=n} b_i(t)-\sigma(t)\geq 0, \quad \text{for all } t\in J_{t_0}. \tag{5}$$

(C_4) $\{\tau_i\}_{i\in\{1,..,n\}},\eta\in C(J_{t_0},J_{t_0})$ such as τ,η are strictly increasing,

$$\lim_{t\to\infty}\tau_i(t)=\lim_{t\to\infty}\eta(t)=\infty.$$

and

$$\eta(t)\leq t\leq\tau_i(t), \quad \text{for all } t\in J_{t_0}, \quad \text{for all } i\in\{1,...,n\} \tag{6}$$

By a solution of (1) we mean a nontrivial real-valued function $u\in C^4(J_{T_u},\mathbb{R})$, $T_u\in J_{t_0}$ which satisfies (1) on J_{T_u}. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution u of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory. Recently, there has been an increasing interest in obtaining sufficient conditions for oscillation and nonoscillation of solutions of various dynamic equations, we refer the reader to the articles [3, 4, 6, 10, 11, 13, 21, 22, 28, 36, 40] and the references cited therein. In recent years, there has been much research activity concerning the hybrid differential equation, the reason is that hybrid differential equations generalize ideas from dynamic systems. For more information on the oscillation of the theory of hybrid differential equations, we refer [8, 9, 12, 17, 27, 32, 33, 38, 39] to the reader and the references cited therein. On the other hand, the types of equations considered in the relevant literature are generally as follows. Using a comparison technique, J. Džurina et al. [10] studied the oscillation of solutions to fourth-order trinomial delay differential equations

$$y^{(4)}(t) + p(t)\, y^{'}(t) + q(t)\, y(\tau(t)) = 0, \quad \text{for all } t \in J_{t_0},$$

A. B. Trajkovict al. [36] studied the oscillatory behavior of intermediate solutions of Fourth-order nonlinear differential equations

$$\left(p(t)\left|x^{(2)}(t)\right|^{\alpha-1} x^{(2)}(t)\right)^{(2)} + q(t)\,|x(t)|^{\beta-1}\, x(t) = 0, \quad \text{for all } t \in J_{t_0},$$

under the assumption

$$\int_{t_0}^{\infty} \frac{t^{1+\frac{1}{\alpha}}}{p^{\frac{1}{\alpha}}(t)} dt < \infty.$$

S. R. Grace et al. [15] have considered the oscillation of fourth-order delay differential equations

$$\left(r_3\left(r_2\left(r_1 y^{'}\right)^{'}\right)^{'}\right)^{'} + q(t)\, y(\tau(t)) = 0,$$

under the assumption

$$\int_{t_0}^{\infty} \frac{1}{r_{i(t)}} dt < \infty, \qquad i \in \{1, 2, 3\}.$$

Motivated by the papers mentioned above and other papers, here we wish to establish some new oscillation criteria for equation (1) which is considered a form that generalizes several differential equations and is similar to papers in a special case, for example, if $f(t,u) = 1$ and $g(t,u) = 0$, then equation (1) is reduced to the half-linear differential equations of fourth order with unbounded neutral coefficients

$$\left(a(t)\left(u^{(2)}(t)\right)^{\beta}\right)^{(2)} + \sum_{i=1}^{i=n} b_i(t)\, u^{\beta}(\tau_i(t)) = 0, \quad \text{for all } t \in J_{t_0}. \tag{7}$$

If $a(t) = 1$ and $\beta = 1$, then equation (7) is reduced to the linear differential equations of fourth order with unbounded neutral coefficients

$$u^{(4)}(t) + \sum_{i=1}^{i=n} b_i(t)\, u^{\beta}(\tau_i(t)) = 0, \quad \text{for all } t \in J_{t_0}, \tag{8}$$

which include several equations, the equation that has been studied by many authors [24, 18].

In this article, we are dealing with the oscillation of the solutions of the fourth-order hybrid nonlinear functional dynamic equations with damping (1) by using the generalized Riccati transformations and an integral averaging method, the contribution is orginnal, as no results on the oscillation of fourth-order hybrid nonlinear functional dynamic equations having been reported in the literature. This paper is organized as follows. In Section 2, four lemmas are given to prove the main results. In Section 3, we establish new oscillation results for Equation (1) while in final section. In Section 4, we present some examples to illustrate the effectivenss of our main results. Some conclusions are discussed in Section 5.

2 Auxiliary result

The following auxiliary results may play a major role throughout the proofs of our main results. For simplification, we note

$$D := \{(t,s) \in \mathbb{R} : t \geq s \geq t_0\} \quad \text{and} \quad D_0 := \{(t,s) \in \mathbb{R} : t > s \geq t_0\}.$$

Definition 1. *[29]The function $H \in C(D, \mathbb{R})$ is said to belong to the function class P if*

i) $H(t,t) = 0$, for $t \geq t_0$ and $H(,s) > 0$ on D_0,

ii) H has a nonpositive continuous and partial derivative $\frac{\partial H}{\partial s}(t,s)$ on D_0 and there exists function $\eta \in C(J_{t_0}, \mathbb{R})$, such that

$$\frac{\partial H(t,s)}{\partial s} + H(t,s)\frac{\eta^{'}(s)}{\eta(s)} = \frac{h(t,s)}{\eta(s)}H(t,s). \tag{9}$$

Lemma 1 (Kiguarde's Lemma)**.** *[15, Theorem 2.2]. Let $n \in \mathbb{N}$ and $f \in C^n(J_{t_0}, \mathbb{R})$. Suppose that f is either positive or negative and $f^{(n)}$is not identically zero and is either nonnegative or nonpositive on J_{t_0}. Then there exist $t_1 \in J_{t_0}$, $m \in \{0, ..., n-1\}$ such that $(-1)^{n-m} f(t) f^{(n)}(t) \geq 0$ holds for all $t \in J_{t_1}$ with*

(1) $f(t) f^{(j)}(t) \geq 0$ holds for all $t \in J_{t_1}$ and $j \in \{0, ..., m-1\}$,

(2) $(-1)^{m+j} f(t) f^{(j)}(t) \geq 0$ holds for all $t \in J_{t_1}$ and $j \in \{m, ..., n-1\}$.

Lemma 2. *[15, Lemma 2.3] Let $f \in C^n(\mathbb{T}, \mathbb{R})$, with $n \geq 2$. Moreover, suppose that Kiguarde's Lemma 1 holds with $m \in \{1, ..., n-1\}$ and $f^{(n)} \leq 0$ on J_{t_0}. Then there exists a sufficiently large $t_1 \in J_{t_0}$ such that*

$$f^{(1)}(t) \geq \frac{(t-t_1)^{m-1}}{(m-1)!} f^{(m)}(t), \qquad \text{for all } t \in J_{t_1}.$$

Corollary 1. *[15, Corollary 2.4]Assume that the conditions of Lemma 2 hold. Then*

$$f(t) \geq \frac{(t-t_1)^m}{m!} f^{(m)}(t), \qquad \text{for all } t \in J_{t_1}.$$

Lemma 3. *[19]If $n \in \mathbb{N}$ and $f \in C^n(J_{t_0}, \mathbb{R})$ then the following statements are true.*

(1) $\liminf_{t\to\infty} f^{(n)}(t) > 0$ *implies* $\lim_{t\to\infty} f^{(k)}(t) = \infty$, *for all* $k \in \{1, ..., n-1\}$.

(2) $\limsup_{t\to\infty} f^{(n)}(t) < 0$ *implies* $\lim_{t\to\infty} f^{(k)}(t) = -\infty$, *for all* $k \in \{1, ..., n-1\}$.

Next, we need the following lemma see [16].

Lemma 4. *[16] If A and B are nonnegative and $\gamma > 0$, then*

$$\lambda A B^{\lambda-1} - A^{\lambda} \leq (\lambda - 1) B^{\lambda}, \tag{10}$$

where equality holds if and if $A = B$.

3 Oscillation Results

In this section, we establish some sufficient conditions which guarantee that every solution u of (1) oscillates on J_{t_0}. In this paper, we consider the operator $P_{f,\beta}$ is defined by :

$$P_{f,\beta} u(t) = \frac{a(t)\left(u^{(2)}(t)\right)^{\beta}}{f(t, u(t))}, \quad \text{for } t \in J_{t_0}.$$

For simplification, we note

$$\delta_+(t) = \max\{\delta(t), 0\}, \quad \text{for } t \in J_{t_0}.$$

Theorem 1. *Suppose that the assumptions $(C_1) - (C_3)$ hold. Assume that there exists a positive function $\tau \in C^1(J_{t_0}, \mathbb{R})$ such that for all sufficiently large $t_1 \in J_{t_0}$, for some $t_2 \in J_{t_1}$ ant $t_3 \in J_{t_2}$, such that*

$$\limsup_{t\to\infty} \int_{t_3}^{t} \left(\tau(s) B_n(s) - \frac{1}{(\beta+1)^{\beta+1}} \frac{\left(\tau'_+(s)\right)^{\beta+1}}{\left(\varphi(s, t_1, t_2)\tau(s)\right)^{\beta}} \right) ds = \infty, \tag{11}$$

where

$$\varphi(t, t_1, t_2) := \int_{t_2}^{t} \left((s - t_1) \frac{\rho(s)}{a(s)} \right) ds, \quad \textit{for } t \in J_{t_2}.$$

If there exists a positive function $\theta \in C^1(J_{t_0}, \mathbb{R})$, such that

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\theta(s)\psi^{\frac{1}{\beta}}(s) - \frac{\left[\theta'(s)\right]^2}{4\theta(s)} \right) ds = \infty, \tag{12}$$

where

$$\psi(t) := \frac{\rho(t)}{a(t)} \int_t^{\infty} \int_s^{\infty} B_n(\lambda)\, d\lambda ds, \quad \textit{for } t \in J_{t_1}.$$

Then any solution of (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution u on J_{t_0}. We may assume without loss of generality that there exists $t_1 \in J_{t_0}$, such that

$$u(t) > 0, \quad u^{\beta}(\tau_i(t)) > 0, \quad u(\eta(t)) > 0 \quad \text{for } t \in J_{t_1},\ i \in \{1, ..., n\}.$$

Since similar arguments can be made, for the case $u(t) < 0$, eventually. Then u' is of constant sign eventually, that is to say, we have two cases. The first case if $u'(t) \geq 0$, for $t \in J_{t_1}$. From (5) and (6), we have

$$\sum_{i=1}^{i=n} b_i(t)\, u^{\beta}(\tau_i(t)) - \sigma(t)\, u^{\beta}(\eta(t)) \geq B_n(t)\, u^{\beta}(t), \quad \text{for } t \in J_{t_1}. \tag{13}$$

The second case if $u'(t) \leq 0$, for $t \in J_{t_1}$, by (5) and (6), we obtain

$$\sum_{i=1}^{i=n} b_i(t)\, u^{\beta}(\tau_i(t)) - \sigma(t)\, u^{\beta}(\eta(t)) \geq B_n(t)\, u^{\beta}(\eta(t)), \quad \text{for } t \in J_{t_1}.$$

Now, from (1), (13) and the above inequality, we obtain

$$\left(P_{f,\beta} u(t)\right)'' \leq \sigma(t)\, u^{\beta}(\eta(t)) - \sum_{i=1}^{i=n} b_i(t)\, u^{\beta}(\tau_i(t)) < 0, \quad \text{for } t \in J_{t_1}.$$

Thus, $t \to \left(P_{f,\beta} u(t)\right)'$ is decreasing on J_{t_1}. We claim that $t \to P_{f,\beta} u(t) > 0$, for $t \in J_{t_1}$. If not, then there exists a $t_2 \in J_{t_1}$ and $m > 0$, such that

$$\left(P_{f,\beta} u(t)\right)' \leq -m < 0, \quad \text{for } t \in J_{t_2}.$$

Integrating the above inequality from t_2 to t, we obtain

$$P_{f,\beta} u(t) \leq -m(t - t_2) + c, \quad \text{for } t \in J_{t_2},$$

where $c := P_{f,\beta} u(t_2)$, we can choose $t_3 \in J_{t_2}$, such as

$$u''(t) \leq -\left(\frac{m}{2} \frac{t}{a(t)} f(t, u(t))\right)^{\frac{1}{\beta}} \leq -\left(\frac{mk}{2}\right)^{\frac{1}{\beta}} \left(\frac{t\rho(t)}{a(t)}\right)^{\frac{1}{\beta}}, \quad \text{for } t \in J_{t_3}.$$

Integrating the above inequality from t_3 to t, we obtain

$$u'(t) \leq -\frac{mk}{2} \int_{t_2}^{t} \frac{s\rho(s)}{a(s)} ds + u(t_2), \quad \text{for } t \in J_{t_3}.$$

which implies that $\lim_{t\to\infty} u'(t) = -\infty$. By lemma 3, we obtain $\lim_{t\to\infty} u(t) = -\infty$, which is a contradiction.

Then, there is $t_2 \geq t_1$, such that only one of the following two cases happens.

Case 1. Let $u''(t) > 0$, for all $t \in J_{t_1}$, then $u'(t) > 0$ for all $t \in J_{t_1}$, due to $\left(P_{f,\beta} u(t)\right)' > 0$. Define the function ω by:

$$\omega_1(t) := \frac{\tau(t)}{u^{\beta}(t)} \left(P_{f,\beta} u(t)\right)' > 0, \quad \text{for all } t \in J_{t_1}.$$

Computing the derivative of ω_1 and from (1), we get

$$\omega_1'(t) = \frac{\tau'(t)}{\tau(t)}\omega_1(t) - \frac{\tau(t)}{u^\beta(t)}\left(\sum_{i=1}^{i=n} b_i(t)\,u^\beta(\tau_i(t)) - g(t,u(\eta(t)))\right) - \beta\frac{u'(t)}{u(t)}\omega_1(t). \tag{14}$$

It follows from $u'(t) > 0$ for all $t \geq t_1$ and (6), (3) that

$$\sum_{i=1}^{i=n} b_i(t)\,u^\beta(\tau_i(t)) - g(t,u(\eta(t))) \geq B_n(t)\,u^\beta(t). \tag{15}$$

Therefore, $t \to (P_{f,\beta}u(t))'$ is a nonincreasing function on J_{t_1}. Then, we obtain

$$\begin{aligned} P_{f,\beta}u(t) &= \int_{t_1}^{t}(P_{f,\beta}u(t))'\,ds + P_{f,\beta}u(t_2) \\ &\geq (t-t_1)(P_{f,\beta}u(t))', \text{ for all } t \in J_{t_1}. \end{aligned} \tag{16}$$

Hence,

$$\left(\frac{P_{f,\beta}u(t)}{t-t_1}\right)' = \frac{(P_{f,\beta}u(t))'}{t-t_1} - \frac{P_{f,\beta}u(t)}{(t-t_1)^2} \leq 0.$$

Thus, $t \to \dfrac{P_{f,\beta}u(t)}{t-t_1}$ is a nonincreasing function on J_{t_2}. Then, we obtain

$$\begin{aligned} u'(t) &\geq \int_{t_2}^{t}\frac{P_{f,\beta}u(t)}{s-t_1}\frac{f(s,u(s))(s-t_1)}{a(s)}ds \\ &\geq \frac{1}{f(t,u(t))}\left(\frac{a(t)}{t-t_1}\int_{t_2}^{t}\frac{\rho(s)(s-t_1)}{a(s)}ds\right)u''(t) \\ &\geq (P_{f,\beta}u(t))'\int_{t_2}^{t}\left((s-t_1)\frac{\rho(s)}{a(s)}\right)ds \\ &= \varphi(t,t_1,t_2)(P_{f,\beta}u(t))'. \end{aligned} \tag{17}$$

Substituting (17) and (15) in (14), we have

$$\begin{aligned} \omega_1'(t) &\leq \frac{\tau'(t)}{\tau(t)}\omega_1(t) - \tau(t)B_n(t) - \beta\frac{\omega_1(t)\varphi(t,t_1,t_2)}{u(t)}(P_{f,\beta}u(t))' \\ &\leq -\tau(t)B_n(t) + \frac{\tau_+'(t)}{\tau(t)}\omega_1(t) - \beta\frac{\varphi(t,t_1,t_2)}{\tau^{\frac{1}{\beta}}(t)}(\omega_1(t))^{1+\frac{1}{\beta}}. \end{aligned} \tag{18}$$

If we apply Lemma 4, we see that

$$\frac{\tau_+'(t)}{\tau(t)}\omega_1(t) - \beta\frac{\varphi(t,t_1,t_2)}{\tau^{\frac{1}{\beta}}(t)}(\omega_1(t))^{1+\frac{1}{\beta}} \leq \frac{1}{(\beta+1)^{\beta+1}}\frac{(\tau_+'(t))^{\beta+1}}{(\varphi(t,t_1,t_2)\tau(t))^\beta}. \tag{19}$$

Using (19) in (18), we obtain

$$\omega_1'(t) \leq -\tau(t)B_n(t) + \frac{1}{(\beta+1)^{\beta+1}}\frac{(\tau_+'(t))^{\beta+1}}{(\varphi(t,t_1,t_2)\tau(t))^\beta}.$$

Integrating the above inequality over $[t_3, t)$ yields

$$\int_{t_3}^{t}\left(\tau(s)B_n(t)-\frac{1}{(\beta+1)^{\beta+1}}\frac{\left(\tau'_+(s)\right)^{\beta+1}}{(\varphi(s,t_1,t_2)\tau(s))^{\beta}}\right)ds\leq\omega_1(t_3),$$

which contradicts (11).

Case 2. Let $u''(t)<0$, for all $t\in J_{t_1}$, then $u'(t)>0$ for all $t\in J_{t_1}$, due to $u(t)>0$. Integrating (1) over $[t,s)$, we get

$$\begin{aligned}\int_t^s\left(P_{f,\beta}u(\tau)\right)^{(2)}d\tau &= \left(P_{f,\beta}u(s)\right)'-\left(P_{f,\beta}u(t)\right)'\\ &\leq -\int_t^s\left(\sum_{i=1}^{i=n}b_i(\lambda)u^{\beta}(\tau_i(\lambda))-g\left(t,u^{\beta}(\eta(\lambda))\right)\right)d\lambda\\ &\leq -\int_t^s\left(\sum_{i=1}^{i=n}b_i(\lambda)u^{\beta}(\tau_i(\lambda))-\sigma(\lambda)u^{\beta}(\eta(\lambda))\right)d\lambda.\end{aligned}$$

When s tends to ∞ in the above inequality, we obtain

$$\left(P_{f,\beta}u(t)\right)'\geq\int_t^{\infty}\left(\sum_{i=1}^{i=n}b_i(\lambda)u^{\beta}(\tau_i(\lambda))-\sigma(t)u^{\beta}(\eta(\lambda))\right)d\lambda.$$

It follows from $u'(t)>0$, for all $t\in J_{t_1}$, and (15), we have

$$\left(P_{f,\beta}u(t)\right)'\geq\int_t^{\infty}B_n(s)u^{\beta}(s)ds\geq u^{\beta}(t)\int_t^{\infty}B_n(s)ds. \tag{20}$$

Integrating above inequality over $[t,s)$, we get

$$P_{f,\beta}u(s)-P_{f,\beta}u(t)\geq\int_t^s\left(u^{\beta}(\rho)\int_{\rho}^{\infty}B_n(\lambda)d\lambda\right)d\rho$$

When s tends to ∞ in the above inequality, we obtain

$$-P_{f,\beta}u(t)\geq u^{\beta}(t)\int_t^{\infty}\int_s^{\infty}B_n(\lambda)d\lambda ds.$$

This means

$$-\left(\frac{u''(t)}{u(t)}\right)^{\beta}\geq\frac{\rho(t)}{a(t)}\int_t^{\infty}\int_s^{\infty}B_n(\lambda)d\lambda ds=\psi(t).$$

Then, we define the function ω_2 by:

$$\omega_2(t):=\theta(t)\frac{u'(t)}{u(t)}>0,\quad\text{for all } t\in J_{t_1}.$$

Computing the derivative of ω_2, we have

$$\begin{aligned} \omega_2'(t) &= \frac{\theta'(t)}{\theta(t)}\omega_2(t) + \theta'(t)\frac{u''(t)}{u(t)} - \theta'(t)\left|\frac{u'(t)}{u(t)}\right|^2 \\ &\leq \frac{\theta'(t)}{\theta(t)}\omega_2(t) - \theta(t)\psi^{\frac{1}{\beta}}(t) - \frac{\omega_2^2(t)}{\theta(t)} \\ &\leq -\theta(t)\psi^{\frac{1}{\beta}}(t) + \frac{1}{4}\frac{\left(\theta'(t)\right)^2}{\theta(t)}. \end{aligned}$$

Integrating the above inequality from t_1 to t, we obtain

$$\int_{t_1}^{t}\left(\theta(s)\psi^{\frac{1}{\beta}}(s) - \frac{\left[\theta'(s)\right]^2}{\theta(s)}\right)ds \leq \omega_2(t_1),$$

which contradicts (12). This completes the proof. □

Corollary 2. *Suppose that the assumptions* $(C_1) - (C_3)$ *hold, such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t} B_n(s)\,ds = \infty, \tag{21}$$

where B_n *is defined as in Theorem 1.*
Then any solution of (1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 1, we put $\tau(t) = \theta(t)$ in Equations (11) and (12), we find Equations (21) and (34). □

Theorem 2. *Suppose that the assumptions* $(C_1) - (C_3)$ *hold. Assume that there exists a positive function* $\tau \in C^1(J_{t_0}, \mathbb{R})$ *such that for all sufficiently large* $t_1 \in J_{t_0}$*, for some* $t_2 \in J_{t_1}$*, and* $t_3 \in J_{t_2}$*, such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_3}^{t} H(t,s)\left(\tau(s)B_n(s) - \frac{h^{\beta+1}(t,s)}{(\beta+1)^{\beta+1}\varphi^{\beta}(s,t_1,t_2)\tau^{\beta}(s)}\right)ds = \infty, \tag{22}$$

where $\varphi(.,t_1,t_2)$ *and* B_n *are defined as in Theorem 1.*
If there exists a positive functions $\theta \in C^1(J_{t_0}, \mathbb{R})$ *such that* (12) *holds.*
Then any solution of (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution u on J_{t_0}. We may assume without loss of generality that there exists $t_1 \in J_{t_0}$, such that

$$u(t) > 0, \quad u^{\beta}(\tau_i(t)) > 0, \quad u(\eta(t)) > 0 \quad \text{for } t \in J_{t_1},\ i \in \{1,...,n\}.$$

Since similar arguments can be made, for the case $u(t) < 0$, eventually. Then there are only the following two possible cases.

Case 1. If $u^{''}(t) > 0$ and $u^{'}(t) > 0$, for all $t \in J_{t_1}$. Multiplying both sides of (18) by $H(t,s)$, integrating it with respect to s from t_2 to t and using the property (9), we get

$$\begin{aligned}\int_{t_3}^{t} H(t,s)\tau(s)B_n(s)ds &\leq -\int_{t_3}^{t} H(t,s)\omega_1^{'}(s)ds + \int_{t_3}^{t} H(t,s)\frac{\tau_+^{\prime}(s)}{\tau(s)}\omega_1(s)ds \\ &\quad -\beta\int_{t_3}^{t} H(t,s)\frac{\varphi(s,t_1,t_2)}{\tau^{\frac{1}{\beta}}(s)}(\omega_1(s))^{1+\frac{1}{\beta}}ds \\ &\leq H(t,t_3)\omega_1(t_2) + \int_{t_3}^{t}\left(\frac{h(t,s)}{\tau(s)}H(t,s)\right)\omega_1(s)ds \\ &\quad -\beta\int_{t_3}^{t} H(t,s)\frac{\varphi(s,t_1,t_2)}{\tau^{\frac{1}{\beta}}(s)}(\omega_1(s))^{1+\frac{1}{\beta}}ds.\end{aligned}$$

If we apply Lemma 4, we see that

$$\int_{t_3}^{t} H(t,s)\tau(s)B_n(s)ds \leq H(t,t_2)\omega_1(t_2) + \int_{t_3}^{t}\frac{H(t,s)}{(\beta+1)^{\beta+1}}\frac{h^{\beta+1}(t,s)}{\varphi^{\beta}(s,t_1,t_2)\tau^{\beta}(s)}ds.$$

which implies that

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t} H(t,s)\left(\tau(s)B_n(s) - \frac{1}{(\beta+1)^{\beta+1}}\frac{h^{\beta+1}(t,s)}{\varphi^{\beta}(s,t_1,t_2)\tau^{\beta}(s)}\right)ds \leq \omega_1(t_3),$$

which contradicts (22).
The proof of **case (2)** is the same as that of **case (2)** in Theorem 1, and so is omitted.
This completes the proof. □

As a Theorem of the previous result, we deduce the following Corollars.

Corollary 3. *Suppose that the assumptions $(C_1) - (C_3)$ hold. Assume that there exists $m \in \mathbb{N}$ such that, for all sufficiently large $t_1 \in J_{t_0}$, for some $t_2 \in J_{t_1}$, and $t_3 \in J_{t_2}$, such that*

$$\limsup_{t\to\infty} t^{-m}\int_{t_3}^{t}(t-s)^m B_n(s) - \left(\frac{n}{\beta+1}\right)^{\beta+1}\frac{(t-s)^{-(\beta+1)}}{\varphi^{\beta}(s,t_1,t_2)}ds = \infty, \qquad (23)$$

where $\varphi(.,t_1,t_2)$ and B_n are defined as in Theorem 1.
If there exists a positive function $\theta \in C^1(J_{t_0},\mathbb{R})$ such that (12) holds.
Then any solution of (1) is oscillatory.

Proof. The proof is similar to that of Theorem 1, we put $\tau(t) = 1$ and $H(t,s) = (t-s)^m$, for $t > s > t_0$ in Equation (22), we find Equation (23). □

Theorem 3. *Assume that conditions* (C_1)-(C_3) *hold. Assume that there exists a positive function* $\varphi \in C^1(J_{t_0}, \mathbb{R})$, *such that for all sufficiently large* $t_1 \in J_{t_0}$, *for some* $t_2 \in J_{t_1}$, *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(B_n(s) \int_{t_1}^{s} \Lambda(\rho)\, d\rho \right) ds = \infty, \tag{24}$$

and

$$\varphi(t) - \varphi'(t)(t - t_1) \leq 0, \quad \text{for all } t \in J_{t_2}. \tag{25}$$

where

$$\Lambda(t) := \left((t - t_1) \int_{t}^{\infty} B_n(s)\, ds \right)^{\frac{1}{\beta}} \int_{t_1}^{t} \left(\frac{\varphi(s)\rho(s)}{a(s)} \right)^{\frac{1}{\beta}} ds.$$

If there exists a positive function $\theta \in C^1(J_{t_0}, \mathbb{R})$ *such that* (12) *holds. Then any solution of* (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution u on J_{t_0}. We may assume without loss of generality that there exists $t_1 \in J_{t_0}$, such that

$$u(t) > 0, \quad u^{\beta}(\tau_i(t)) > 0, \quad u(\eta(t)) > 0 \quad \text{for } t \in J_{t_1},\ i \in \{1, ..., n\}.$$

Since similar arguments can be made, for the case $u(t) < 0$, eventually. Then there are only the following two possible cases.

Case 1. If $u''(t) > 0$ and $u'(t) > 0$, for all $t \in J_{t_1}$. Using (1), it follows from (16) that

$$\left(\frac{P_{f,\beta} u(t)}{\varphi(t)} \right)' \leq \frac{P_{f,\beta} u(t)}{\varphi^2(t)(t - t_1)} \left(\varphi(t) - \varphi'(t)(t - t_1) \right) \leq 0.$$

Thus, $t \to \dfrac{P_{f,\beta} u(t)}{\varphi(t)}$ is a nonincreasing function on J_{t_1}. Then,

$$\begin{aligned} u'(t) &\geq \left(\frac{P_{f,\beta} u(t)}{\varphi(t)} \right)^{\frac{1}{\beta}} \int_{t_1}^{t} \left(\frac{\varphi(s) f(t, u(s))}{a(s)} \right)^{\frac{1}{\beta}} ds \\ &\geq \left(\frac{P_{f,\beta} u(t)}{\varphi(t)} \right)^{\frac{1}{\beta}} \int_{t_1}^{t} \left(\frac{\varphi(s)\rho(s)}{a(s)} \right)^{\frac{1}{\beta}} ds. \end{aligned} \tag{26}$$

It follows from (16) and (20) that

$$\begin{aligned} u'(t) &\geq u(t) \left((t - t_1) \int_{t}^{\infty} B_n(s)\, ds \right)^{\frac{1}{\beta}} \int_{t_1}^{t} \left(\frac{\varphi(s)\rho(s)}{a(s)} \right)^{\frac{1}{\beta}} ds \\ &= \Lambda(t) u(t), \text{ for all } t \in J_{t_1}. \end{aligned}$$

Clearly $u'(t) > 0$, for $t \in J_{t_1}$, then there exists $\ell > 0$, such that

$$u(t) \geq \ell \int_{t_1}^{t} \Lambda(s)\, ds, \text{ for all } t \in J_{t_1}.$$

Using (1), (5), and the above inequality, we obtain

$$(P_{f,\beta}u(t))^{(2)} \leq -\ell B_n(t)\int_{t_1}^{t}\Lambda(s)\,ds.$$

Integrating the above inequality over $[t_1, t)$, we obtain

$$(P_{f,\beta}u(t))' \leq (P_{f,\beta}u(t_1))' - \ell\int_{t_1}^{t}\left(B_n(s)\int_{t_1}^{s}\Lambda(\rho)\,d\rho\right)ds.$$

By (12), this gives

$$\liminf_{t\to\infty}(P_{f,\beta}u(t))' = -\infty.$$

Lemma 3, give us $\lim_{t\to\infty} P_{f,\beta}u(t) = -\infty$, which is a contradiction.
The proof of **case (2)** is the same as that of **case (2)** in Theorem 1, and so is omitted.
This completes the proof. □

Theorem 4. *Assume that conditions* (C_1)-(C_3) *hold. Assume that there exists a positive functions* $\varphi, \xi \in C^1(J_{t_0}, \mathbb{R})$, *such that for all sufficiently large* $t_1 \in J_{t_0}$, *for some* $t_2 \in J_{t_1}$, *such that*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\frac{\beta\varphi(s)\xi(s)}{\pi(s)(s-t_1)^{\beta+1}} - \frac{\xi(s)}{(s-t_1)^{\beta+1}\pi^{\beta}(s)} - \frac{\xi'(s)\varphi(s)}{\pi(s)(s-t_1)^{\beta}}\right)ds = \infty, \tag{27}$$

where φ *is defined as in Theorem 3, and*

$$\xi(t) + (t-t_1)\pi^{\beta-1}(t)\xi'(t)\varphi(t) \leq \beta\pi^{\beta-1}(t)\varphi(t)\xi(t), \text{ for all } t \in J_{t_2}.$$

$$\pi(t) := \int_{t_1}^{t}\left(\frac{\varphi(s)\rho(s)}{a(s)}\right)^{\frac{1}{\beta}}ds, \text{ for all } t \in J_{t_2}.$$

If there exists a positive function $\theta \in C^1(J_{t_0}, \mathbb{R})$ *such that* (12) *holds.*
Then any solution of (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution u on J_{t_0}. We may assume without loss of generality that there exists $t_1 \in J_{t_0}$, such that

$$u(t) > 0, \quad u^{\beta}(\tau_i(t)) > 0, \quad u(\eta(t)) > 0 \quad \text{for } t \in J_{t_1},$$

Since similar arguments can be made, for the case $u(t) < 0$, eventually. Then there are only the following two possible cases.
Case 1. If $u''(t) > 0$ and $u'(t) > 0$, for all $t \in J_{t_1}$. We define the function ω_3 by:

$$\omega_3(t) := \frac{\xi(t)}{u^{\beta}(t)}P_{f,\beta}u(t) > 0, \quad \text{for all } t \in J_{t_1}.$$

Using (26) and (16), we arrive at

$$\frac{u'(t)}{u(t)} \geq \left(\frac{\pi(t)}{\varphi(t)\xi(t)}\right)^{\frac{1}{\beta}}\omega_3^{\frac{1}{\beta}}(t), \quad \text{for all } t \in J_{t_1}. \tag{28}$$

$$\left(P_{f,\beta}u(t)\right)' \leq \frac{\left(u'(t)\right)^{\beta}}{(t-t_1)\,\pi^{\beta}(t)}, \quad \text{for all } t \in J_{t_1}. \tag{29}$$

It follows from Corollary 1, we have

$$u(t) \geq u'(t)(t-t_1), \quad \text{for all } t \in J_{t_1}.$$

This implies that

$$\omega_3(t) \leq \frac{\varphi(t)\,\xi(t)}{\pi(t)(t-t_1)^{\beta}}, \quad \text{for all } t \in J_{t_2}. \tag{30}$$

By (29) and as the above inequality, we get

$$\frac{\left(P_{f,\beta}u(t)\right)'}{u^{\beta}(t)} \leq \frac{1}{(t-t_1)^{\beta+1}\,\pi^{\beta}(t)}, \quad \text{for all } t \in J_{t_2}. \tag{31}$$

Computing the derivative of ω_3, we have

$$\omega_3'(t) = \frac{\xi'(t)}{\xi(t)}\omega_3(t) + \frac{\xi(t)}{u^{\beta}(t)}\left(P_{f,\beta}u(t)\right)' - \beta\omega_3(t)\frac{u'(t)}{u(t)}.$$

Substituting (31), (30) and (28) in the above equality, we obtain

$$\begin{aligned}\omega_3'(t) &\leq \frac{\xi(t)}{(t-t_1)^{\beta+1}\,\pi^{\beta}(t)} + \frac{\xi'(t)}{\xi(t)}\omega_3(t) - \beta\left(\frac{\pi(t)}{\varphi(t)\,\xi(t)}\right)^{\frac{1}{\beta}}\omega_3^{1+\frac{1}{\beta}}(t)\\ &\leq \frac{\xi(t)}{(t-t_1)^{\beta+1}\,\pi^{\beta}(t)} + \frac{\xi'(t)\varphi(t)}{\pi(t)(t-t_1)^{\beta}} - \frac{\beta\varphi(t)\,\xi(t)}{\pi(t)(t-t_1)^{\beta+1}} \leq 0.\end{aligned}$$

Integrating the above inequality over $[t_2, t)$, we obtain

$$\int_{t_2}^{t}\left(\frac{\beta\varphi(s)\,\xi(s)}{\pi(s)(s-t_1)^{\beta+1}} - \frac{\xi(s)}{(s-t_1)^{\beta+1}\,\pi^{\beta}(s)} - \frac{\xi'(s)\varphi(s)}{\pi(s)(s-t_1)^{\beta}}\right)ds \leq \omega_3(t_2),$$

which contradicts (27).
The proof of **case (2)** is the same as that of **case (2)** in Theorem 1, and so is omitted.
This completes the proof. □

Let $\xi(t) = t - t_1$, for $t \in J_{t_2}$. Then Theorem 4 yields the following result.

Corollary 4. *Assume that conditions (C_1)-(C_3) hold. Assume that there exists a positive function $\varphi \in C^1(J_{t_0}, \mathbb{R})$, such that for all sufficiently large $t_1 \in J_{t_0}$, for some $t_2 \in J_{t_1}$, such that*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\frac{\beta\varphi(s)\,\pi^{\beta-1}(s) - 1}{\pi^{\beta}(s)(s-t_1)^{\beta}}ds = \infty,$$

where φ and π are defined as in Theorem 4, and

$$\pi^{\beta-1}(t)\,\varphi(t) \geq \frac{1}{\beta}, \quad \textit{for all } t \in J_{t_2}.$$

If there exists a positive function $\theta \in C^1(J_{t_0}, \mathbb{R})$ such that (12) holds.
Then any solution of (1) is oscillatory.

4 Examples and Discussions

In this section, we give an example to illustrate our main result.

Example 1. *Consider the neutral differential equation*

$$\left(\frac{u^{(2)}(t)}{e^t}\right)^{(2)} + \sum_{i=1}^{i=n} e^{-t} u(t-i) = 0, \quad \text{for all } t \geq n+1. \tag{32}$$

Here, $\beta = 1$*,* $r(t) = 1$*,* $n \in \mathbb{N}$*,* $a(t) = 1$*,* $\tau_i(t) = t - i$*,* $b_i(t) = e^{-t}$*, for all* $i \in \{1, 2, ..., n\}$*,* $f(t, u) = e^t$ *and* $g(t, u) = 0$*. Then* $\rho(t) = e^t$*,* $\sigma(t) = 0$ *and the hypotheses* $(C_1) - (C_4)$ *hold. On the other hand, we see that*

$$B_n(t) = \sum_{i=1}^{i=n} b_i(t) - \sigma(t) = ne^{-t}, \quad \text{for all } t \geq n+1.$$

$$\varphi(t, t_1, t_2) \geq \frac{t}{2} e^t, \quad \text{for } t \text{ large enough},$$

$$\psi(t) = n, \quad \text{for } t \text{ large enough}.$$

Let $\tau(t) = e^t$ *and* $\theta(t) = 1$*, for all* $t \geq n+1$*. Thus,* (11) *and* (12) *hold. By Theorem 1, equation* (32) *is oscillatory.*

Example 2. *Consider the hybrid differential equation*

$$\left(e^{-u^2 - t} \sqrt[3]{u^{(2)}(t)}\right)^{(2)} + e^{-t} \sqrt[3]{u(t)} = 0, \quad \text{for all } t \geq 0, \tag{33}$$

Here, $\beta = \frac{1}{3}$*,* $a(t) = 1$*,* $n = 1$*,* $b_1(t) = e^{-t}$*,* $\tau_1(t) = t$*,* $f(t, x) = e^{u^2 + t}$*, and* $g(t, u) = 0$*, Then* $\rho(t) = e^t$*,* $\sigma(t) = 0$ *and the hypotheses* (C_1)*-*(C_4) *hold. Let* $\varphi(t) = e^{3t}$*, for* $t \geq 0$*, then* (25) *holds,*

$$\Lambda(t) \geq d \sqrt[3]{t} e^t, \quad \text{for } t \text{ large enough},$$

where $d > 0$*, then* (24) *holds. Therefore, we have*

$$\psi(t) = 1, \quad \text{for } t \text{ large enough},$$

Let $\theta(t) = 1$*, for all* $t \geq 0$*. Thus,* (12) *holds. By Theorem 3, equation* (33) *is oscillatory.*

Remark 1. *These results show that the coefficient functions* $\{b_i\}_{i \in \{1,..,n\}}$ *play an important role in oscillation of fourth-order hybrid nonlinear dynamic equation; see the details in Example 1 and differences between Corollary 2 and Theorem 1, Theorem 2, Theorem 3.*

5 Conclusion

In this paper, the main aim to provide a study of oscillation of the fourth-order hybrid nonlinear functional dynamic equations with damping by using the following methods:

(1) The generalized Riccati transformation technique.

(2) The Integral averaging technique.

The results presented complement a number of results reported in the literature. Furthermore, the findings of this paper can be extended to study a class of systems of higher order hybrid advanced differential equations, for example

$$\left(\frac{a(t)\left(u^{(k-2)}(t)\right)^{\beta}}{f(t,u(t))}\right)^{(2)}+\sum_{i=1}^{i=n} b_i(t)\,u^{\beta}(\tau_i(t))=g(t,u(\eta(t))), \quad \text{for all } t\in J_{t_0},$$

Remark 2. *If we consider a fourth-order hybrid nonlinear functional dynamic equation with damping on time scale*

$$\left(\frac{a(t)\left(u^{\Delta^2}(t)\right)^{\beta}}{f(t,u(t))}\right)^{\Delta^2}+\sum_{i=1}^{i=n} b_i(t)\,u^{\beta}(\tau_i(t))=g(t,u(\eta(t))), \tag{34}$$

on an arbitrary time scale $\mathbb{T}$ *with* $\sup\mathbb{T}=\infty$*. Thus, equation* (1) *becomes a special case of equation* (34) *in a case* $\mathbb{T}=\mathbb{R}$*. From the method given in this paper, one can obtain some oscillation criteria for* (34)*. It means obtaining generalizations of Theorems 1, 2, 3 and 4. The details are left to the reader.*

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References

[1] Agwa, H.A., Khodier, A. M.M. and Arafa, H.M., *Oscillation of second-order nonlinear neutral dynamic equations with mixed arguments on time scales*, Journal of Basic and Applied Research International, **17** (2016), 49–66.

[2] Agwa, H.A., Ahmed, Khodier, M. and Arafa, H,M., New *Oscillation Results of Second-Order Mixed Nonlinear Neutral Dynamic Equations with Damping on Time Scales*, J. Ana. Num. Theor, **5**,(2017), 137-145 , DOI:10.18576/jant/050208.

[3] Benaissa Cherif, A., Ladrani, F.Z. and Hammoudi, A., *Oscillation theorems for higher order neutral nonlinear dynamic equations on time scales*, Malaya. J. Mat, **4** (2016), 599–605.

[4] Benaissa Cherif, A. and Ladrani, F.Z., *Asymptotic behavior of solution for a fractional Riemann-Liouville differential equations on time scales*, Malaya. J. Mat, **5** (2017), 561–568.

[5] Beniani, A., Benaissa Cherif, A., Zennir, Kh. and Ladrani, F.Z., *Oscillation theorems for higher order nonlinear functional dynamic equations with unbounded neutral coefficients on time scales*, Novi Sad J. Math. **76** (2012), no. 4.

[6] Bohner, M., Grace, S.R. and Jadlovská, I., *Oscillation criteria for second-order neutral delay differential equations*, Elec. J. Qualitative Theory. Diff. Equ, **60** (2017), 1-12.

[7] Candan, T. and Dahiya, R.S., *On the oscillation of certain mixed neutral equations*, Appl. Math. Lett., **21** (2008), 222–226.

[8] Chatzarakis, G.E., Deepa, M., Nagajothi, N., Sadhasivam, V., *Some new oscillation criteria for second-order hybrid differential equations*, Hacet. J. Math. Stat, **49** (2020), 1334–1345.

[9] Dhage, B.C. and Lakshmikantham, V., *Basic results on hybrid differential equations*, Nonlinear Anal. Hybrid Syst. **4** (2010), 414-424.

[10] Džurina, J., Baculíková, B. and Jadlovská, I., *Oscillation of solutions to fourth-order trinomial delay differential equations*, Ele. J. Diff. Equ, **70** (2015), 1–10.

[11] Džurina, J., Jadlovska, I. and Stavroulakis, I.P., *Oscillatory results for second-order non-canonical delay differential equations*, Opuscula. Math, **39** (2019) , 483-495.

[12] Ge, H. and Xin, J., *On the existence of a mild solution for impulsive hybrid fractional differential equations*, Adv. Difference Equ. **211** (2013).

[13] Grace, S. R., Džurina, J., Jadlovská, I. and Li, T., *On the oscillation of fourth-order delay differential equations*, Advances. Diff. Equ , **118** (2019), DOI:10.1186/s13662-019-2060-1.

[14] Grace, S. R., *On the oscillations of mixed neutral equations*, J. Math. Anal. Appl., **194** (1995), 377–388.

[15] Grace, S. R., *On the Oscillation of nth Order Dynamic Equations on Time Scales*, Mediterr. J. Math, **10** (2013), 147–156.

[16] Hardy, G. H., Littlewood, I. E. and Polya, G., *Inequalities. University Press*, Cambridge, 1959.

[17] Herzallah, M.A.E. and Baleanu, D., *On fractional order hybrid differential equations*, Abstr. Appl. Anal. (2014), article ID 389386.

[18] Howard, H., *Oscillation criteria for fourth-order linear differential equations*, Transactions of the American Mathematical Society, **96** (1960), 296–311.

[19] Karpuz, B., *Sufficient conditions for the oscillation and asymptotic beaviour of higher-order dynamic equations of neutral type*, Applied Mathematics and Computation, **221** (2013), 453 - 462.

[20] Kumar, S., Pandey, P.,Das, S. and Craciun, E. M., *Numerical solution of two dimensional reaction-diffusion equation using operational matrix method based on Genocchi polynomial. Part I: Genocchi polynomial and operational matrix*, Proc. Rom. Acad., Ser. A: Math. Phys. Tech. Sci. Inf. Sci. **20** (2020), 393-399.

[21] Kusano, T. and Manojlovic, J. V., *Precise asymptotic behavior of regularly varying solutions of second order half-linear differential equations*, Elec. J. Qual. Theory. Diff. Equ, **62** (2016), 1-24.

[22] Ladrani, F. Z., Hammoudi, A. and Benaissa Cherif, A., *Oscillation theorems for fourth-order nonlinear dynamic equations on time scales*, Elec. J. Math. Anal. Appl, **3** (2005), 46-58.

[23] Ladrani, F. Z. and Benaissa Cherif, A., *Oscillation tests for conformable fractional differential equations with damping*, Punjab Univ. J. Math, **52** (2019), 45–55.

[24] Leighton, W., *Quadratic functionals of second order*, Transactions of the American Mathematical Society, **151** (1970), 309–322.

[25] Li, T. , *Comparison theorems for second-order neutral differential equations of mixed type*, Electron. J. Differ. Eq., **167** (2010), 1–7.

[26] Marin, M., *Generalized solutions in elasticity of micropolar bodies with voids*, Rev. Acad. Canar. Cienc., **8** (1996), 101-106.

[27] Micheau, P. and Coirault, P., *A harmonic controller of engine speed oscillations for hybrid vehicles*, Elsevier IFAC publications, (2005), 19-24.

[28] Özdemir, O., *Oscillation results for second order half-linear functional dynamic equations with unbounded neutral coefficients on time scales*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69** (2020), 668-683.

[29] Philos, Ch. G., *Oscillation theorems for linear differential equations of second order*, Arch. Math, **53** (1989), 482-492.

[30] Qi, Y., and Yu, J., *Oscillation of second order nonlinear mixed neutral differential equations with distributed deviating arguments*, Bull. Malays. Math. Sci. Soc., **38** (2015) , 543–560.

[31] Ramesh, R., Julio, G. D, Harikrishnan, R. and Prakash P., *Oscillation criteria for solution to partial dynamic equations on time scales*, Hacet. J. Math. Stat, **49** (2020), 1788–1797.

[32] Sadhasivam, V. and Deepa, M., *Oscillation criteria for fractional impulsive hybrid partial differential equations*, Probl. Anal. Issues Anal, **26** (2019), 73-91.

[33] Sitho, S., Ntouyas, S.K. and Tariboon, J., *Existence results for hybrid fractional integro differential equations*, Bound. Value Probl, **113** (2015), 1-13.

[34] Sui, Y. and Han,Z., *Oscillation of second order neutral dynamic equations with deviating arguments on time scales*, Advances. Diff. Equ, (2018), article 337.

[35] Thandapani, E., Padmavathi, S. and Pinelas, S., *Oscillation criteria for even-order nonlinear neutral differential equations of mixed type*, Bull. Math. Anal. Appl., **6** (2014), 9–22.

[36] Trajkovic, A. B. and Manojlovic, J. V., *Asymptotic behavior of intermediate solutions of Fourth-order nonlinear differential equations with regularly varying coefficients*, Ele. J. Diff. Equ, (2016), 1-32.

[37] Yan, J., *Oscillations of second order neutral functional diFFerential equations*, Appl. Math. Comput., **83** (1997), 27–41.

[38] Yuan, F. and Diclemente, D., *Hybrid voltage-controlled oscillator with low phase noise andlarge frequency tunig range*, Analog Integr Circ Sig Process, **82** (2015), 471-478.

[39] Zhao,Y., Sun, S., Han, Z. and Li, Q., *Theory of fractional hybrid differential equations*, Comput. Math. Appl. **62** (2011), 1312-1324.

[40] Zhang, M., Chen, W., Sheikh, M., Sallam,R.A., Hassan, A. and Li, T., *Oscillation criteria for second-order nonlinear delay dynamic equations of neutral type*, Advances. Diff. Equ, **2018** (2018), article 26.