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A REGULAR LIE GROUP ACTION YIELD SMOOTH SECTIONS OF THE TANGENT BUNDLE AND RELATEDNESS OF VECTOR FIELDS, DIFFEOMORPHISMS

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Abstract

In this paper, we have concentrated on a group action on the tangent bundle of some smooth/differentiable manifolds which has been built from a regular Lie group action on such smooth/differentiable manifolds. Interestingly, elements of orbit space yield smooth sections of the tangent bundle having beautiful algebraic properties. Moreover, each of those smooth sections behaves nicely as a left-invariant vector field with respect to Lie group action by G. We have explained here a simple isomorphism between the set of such smooth sections and each tangent space of that smooth/differentiable manifold. Also we have discussed more about F-relatedness and have introduced vector field relatedness by notations $rel_{\mathfrak{X}(M)}(F), rel_{Diff(M)}(X)$, etc. which are sets based on both vector field related diffeomorphisms and diffeomorphism related vector fields. We have presented consequences based on the algebraic structure on $rel_{\mathfrak{X}(M)}(F), rel_{Diff(M)}(X)$, etc. sets and built some related group actions. We have placed some interrelationship between the both kinds of rel operations.

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1 Introduction

Vector fields on smooth manifolds represents a major study in differential geometry. The entire theory of vector fields has majority applications in physics especially in the field of gravity, electricity, and magnetism [1]. More so when

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Erlanger program was initiated by F. Klein in 1872's which characterized the geometries with the help of projective geometry and group theory thereby helping modern Physics mainly in symmetry. On the other hand, Poincare extensively used group theory for topological invariants that has shown remarkable developments in differential topology as well as in algebraic topology.

It is well-known a smooth (c^r-differentiable) section of the tangent bundle of a smooth (c^r-differentiable) manifold is a vector field, and indeed assigns a unique vector in the tangent space at each element of the smooth (c^r-differentiable) manifold. In general, magnitude and direction to each vector assigned by a vector field cannot be specified unless the ambient space is a Riemannian manifold. One can easily visualize the vector fields in three-dimensional Euclidean space and its subspaces from the ideas of curl, divergence etc. But concerning vector fields invariance, existence, the existence of nowhere vanishing vector fields, complete vector fields, killing vector fields, transitive of a class of vector fields, etc. on a given smooth manifold needs a lot of enquiry to visualize as well as to absorb their full utilization. For instance, the Hairy ball theorem on vector fields is one such result that constitutes more usage, and its special case i.e. existence of a vector field on compact manifolds characterized by a topological property-Euler characteristics [4].

Theory of bundles, the tangent bundle and the cotangent bundle in both Lie group and differential manifolds have a prominent role in solutions of ordinary and partial differential equations, classifications of forms, tensors, and metrics on Riemannian manifolds. Fundamental vector fields are instruments that explain the infinitesimal behaviour of a Lie group action on a smooth manifold. Such vector fields find predominant applications in the study of Lie groups, representation theory, symplectic geometry and the study of Hamiltonian group actions.

[7] Sophus Lie (1842-1899) introduced the theory of Lie groups. It was introduced basically to model the continuous symmetries of differential equations as a tool to simplify or solve the partial differential equations, ordinary differential equations found in Physics. The dynamics and bifurcation theory of group-invariant vector fields is a much-sought branch of Mathematics. A part of this i.e. left-invariant vector fields is also a classical study on a Lie group which is a geometrical invariance under the translation group action of Lie group on itself. Some developments have been taken in differential forms, Riemannian metrics, etc. for the Lie group. Later, studies have been produced on the existence of such an invariant vector field subject to some conditions. For instance, one can see a result in [18]-let G be an odd-dimensional Lie group with a left-invariant metric then there is a left-invariant minimal unit vector field on G, and J. C. Gonzalez-Davila and et al in [6] showed that every uni-modular Lie group admits a left-invariant harmonic unit vector field. Recently in 2014, an isomorphic notion between two left-invariant vector fields and fundamental applications including some remarkable consequences are presented in [5].

Top space is a smooth manifold with a binary operation satisfying certain axioms (see [3, 14]) like a Lie group. A similar work of left-invariant vector fields has been built on the Top space with its binary operation. Moreover, a similar theory of Lie bracket, Lie algebras has been developed on a Top space [3, 14, 15, 16, 17].

The set of invariant vector fields of a Lie group (of dimension n) is a subspace (equal to the dimension n) of the set of vector fields on it, but it is different in the case of left-invariant metrics such collection is of $\frac{n(n+1)}{2}$ dimension [9]. Concerning the vector fields, the main issue in this area is to construct left-invariant vector fields, left-invariant unit vector fields, left-invariant harmonic unit vector fields etc. for a Lie group as well for an arbitrary manifold under a group action by a Lie group. Handling the collection of such smooth vector fields pertaining to certain conditions is another issue, one can see that in the study of [2, 10]. The positive semi orbit of a family of left-invariant vector fields through a point of a manifold leads to the definition of transitive to the family of vector fields of a Lie group. Using this, in [2] B. Bonnard and et al. characterized the family of the left-invariant vector fields of a Lie group to be transitive. In the context of the mentioned issues, in this paper we have developed and discussed a method by a Lie group action on some smooth manifolds that yield vector fields and which induces Lie algebra on the tangent space of each element of its manifold. Also, we have discussed F-related vector fields and have introduced X-related diffeomorphisms. Further, by defining new notions $rel(F), rel(X)$, etc. we have deduced algebraic structure on the respective notions. For the family of left-invariant vector fields, we have mentioned an interrelationship with the transformation group of the Lie group. We shall further extend it to the manifold that undergoes regular Lie group actions, and the same for left-invariant unit vector fields in a forthcoming paper.

2 Preliminaries

In the entire paper, whatever theory we develop for smooth manifolds one can also do the same to the C^r-differentiable manifolds. Lie group is a smooth manifold equipped with a group structure in which group operations are smooth maps [12, 13]. Generally a group action on a set M by a group $(G, \star)$ is group homomorphism $\phi : G \to Sym(M)$ by $g \to \phi(g) = \phi_g$, or it is a map μ from product space of G and M to M satisfying $\mu(e, x) = x$ and $\mu(g, \mu(h, x)) = \mu(g \star h, x)$ for all $x \in M$ and $g \in G$ [8, 12, 13] and in this paper, often in the many proofs of propositions, we shall use the second definition for group action. Out of many types of group actions, a group action that is both transitive and free is called a regular group action. A Lie group action is a group action by a Lie group on a smooth manifold M such that map $\mu : G \times M \to M$ is smooth. Every Lie group action gives a transformations group which is a subgroup of $Diff(M)$.

Tangent bundle TM of a smooth manifold M, itself is a smooth manifold of dimension equal to twice of $dim(M)$. Every smooth map from a manifold to a manifold induces a linear map at each point x, called derivative map at x denoted by F_{*x}. We often use the standard projection π from tangent space to M.

Our intuition in this paper is to obtain the smooth sections by group actions.

Left-invariant vector fields are frequently studied in the theory of Lie groups and we have studied the same for some kind of general manifolds. For a differentiable manifold M, generally, $Diff(M)$ acts on the tangent bundle of manifold M, by $(f, V_x) \to f_{*x}(V_x)$. But while computing the orbit of each vector, one can see this contains more than two vectors from the same tangent space. See example, for the manifold real line $\mathbb{R}$, under the same group action by $Diff(\mathbb{R})$ on its tangent bundle as we stated, for any non-zero vector $V_1 \in T_1\mathbb{R}$, the orbit of such vector contains at least two different vectors $(f(x) = 10x - 9)_{*1}(V_1)$ and $(f(x) = x)_{*1}(V_1)$ based at 1. Pertaining to this point, to place a unique vector to each point of the manifold will not be possible from the orbit set of an element by this group action. Not only the stated group action, but many group actions finally have such an issue. But there is a group action on the tangent bundle of some manifold from a special group, which has been built from a special kind of group action on such manifold. Moreover here, one can see each element of the orbit space yields a smooth section of the tangent bundle.

3 Regular group action yield smooth sections of the tangent bundle on some smooth manifolds

Let $(G, \star)$ be a Lie group, M be a smooth manifold and $\mu_1 : G \times M \to M$ be any regular (free and transitive) left Lie group action (one can take right group action also). This induces a group homomorphism $\phi : G \to Sym(M)$ by $g \to \phi(g) = \phi_g$, where $\phi_g : M \to M$ is given by $\phi_g(x) = \mu_1(g, x)$, it is obvious ϕ_g is a smooth diffeomorphism for each $g \in G$. Since Kernel of ϕ is trivial, hence ϕ is a monomorphism. Choosing $\phi(G)$ =Image of ϕ on G as a codomain, ϕ becomes an isomorphism. For further discussion, we denote $\phi(G) = Diff_{\mu_1}(M) = \{l_g = \phi_g : g \in G\}$ (l_g is for the left group action) which is a subgroup of $Sym(M)$ under the composition of the functions. The group $Diff_{\mu_1}(M)$ naturally builds a group action on the tangent bundle, which we shall discuss in this section. The orbit of each element of the manifold under this group action gives smooth sections of the tangent bundle.

In this section, we confine (from Lemma 3.1 to Proposition 3.17) M as a smooth manifold that undergoes a regular smooth group action μ_1 by a Lie group G.

Lemma 3.1. *If $x, y \in M$ then there exists a unique $l_g \in Diff_{\mu_1}(M)$ such that $l_g(x) = y$.*

Proof. Let $x \in M$ and for all $y \in M$ the transitive of μ_1 on M gives there exists $g \in G$ such that $\mu_1(g, x) = l_g(x) = y$. The uniqueness of g is obvious, suppose there exist two such $g, h \in G$ such that $\mu_1(g, x) = \mu_1(h, x)$ since the group action μ_1 is free so $g = h$. □

Tangent bundle TM of the manifold stated in the introductory part of section 3 naturally undergoes a group action by the group $Diff_{\mu_1}(M)$, which we present below.

Proposition 3.2. *For the manifold M, define a map $\mu_2 : Diff_{\mu_1}(M) \times TM \to TM$ by $\mu_2(l_g, V_x) = (l_g)_{*x}(V_x)$ then it is a non-transitive group action (where $(l_g)_*$ is derivative map of l_g at x).*

Proof. Since each l_g is a diffeomorphism, whose derivative map is a linear isomorphism from T_xM to $T_{\mu_1(g,x)}M$. Hence $(l_g)_{*x}(V_x) \in T_{\mu_1(g,x)}M \subset TM, \forall V_x \in TM$ and $\forall l_g \in Diff_{\mu_1}(M)$. Also $\mu_2(l_e, V_x) = (l_e = Id_M)_{*x}(V_x) = V_x$ and $\mu_2(l_g, \mu_2(l_h, V_x)) = \mu_2(l_g, (l_h)_{*x}(V_x)) = l_{g*\mu_1(h,x)}((l_h)_{*x}(V_x)) = (l_g o l_h)_{*x}(V_x) = \mu_2(l_g \circ l_h, V_x)$. Hence μ_2 is group action on TM.
We have orbit i.e. $Orb(V_x) = \{\mu_2(l_g, V_x) = (l_g)_{*x}(V_x) : \forall l_g \in Diff_{\mu_1}(M)\}$ one can see that each $(l_g)_{*x}$ assign single vector in each tangent space T_xM. Therefore it does not cover the entire TM. Therefore μ_2 is a non-transitive group action. □

Throughout the paper, we use the fact that $Orb(V_x)$ can be replaced by $[V_x]$ which is an equivalence class containing V_x under the equivalence relation implied by the group action μ_2. The quotient space or orbit space produced by group action μ_2 is denoted by $TM/Diff_{\mu_1}(M)$.

Note 3.3. *The group action μ_2 is free and necessarily faithful but not regular since it is non-transitive. Also we have stabilizers $Stab_{Diff_{\mu_1}(M)}(V_x)=\{l_g \in Diff_{\mu_1}(M) : \mu_2(l_g, V_x)=(l_g)_{*x}(V_x) = V_x\} = \{l_e\}$ and $Stab_{Diff_{\mu_1}(M)}(TM) = \bigcap_{V_x \in TM} Stab_{Diff_{\mu_1}(M)}(V_x) = \{l_e\}$ and Fixed point set of TM is $TM^{Diff_{\mu_1}(M)} = \{V_x \in TM : \mu_2(l_g, V_x) = (l_g)_{*x}(V_x) = V_x, \forall l_g \in Diff_{\mu_1}(M)\}$ since $(l_g)_{*x}$ are isomorphisms hence $TM^{Diff_{\mu_1}(M)} = \{O_x : x \in M\}$. This set contains all zero vectors of each tangent space of manifold M, moreover one can visualize which is like the zero vector field on M.*

Proposition 3.4. *Let $x \in M$ and V_x, W_x be two distinct elements from T_xM then $[V_x] \neq [W_x]$.*

Proof. We will prove this by the contradiction method, suppose $[V_x] = [W_x]$ implies V_x is related to W_x under the relation built by μ_2, this implies there exists $l_g \in Diff_{\mu_1}(M)$ such that $(l_g)_{*x}(V_x) = W_x$. Since $(l_g)_{*x} : T_xM \to T_{(\mu_1(g,x))}M$ so $(l_g)_{*x}$ send vector into $T_{(\mu_1(g,x))}M$ therefore $(l_g)_{*x}(V_x) = W_x \in T_{(\mu_1(g,x))}M$. This implies the vector W_x based at x that equals $\mu_1(g, x)$. That is $\mu_1(g, x) = x$, since the group action μ_1 is free implies $g = e$. Since $l_e = Id_M$, therefore we can see $(l_e)_{*x}(V_x) = V_x = W_x$ this is a contradiction to the hypothesis distinct of V_x, W_x. Hence $[V_x] \neq [W_x]$. □

Corollary 3.5. *Distinct linearly dependent vectors in TM have different equivalence classes. That is for a non-zero vector $V_x \in T_xM$ and $1 \neq \alpha \in \mathbb{R}$ then $[V_x] \neq [\alpha V_x]$.*

Proof. It follows from Proposition 3.4. since $V_x \neq \alpha V_x$. □

Proposition 3.6. *Let $[V_x] \in TM/Diff_{\mu_1}(M)$ and for given y in M then there exists a unique vector $V_y \in [V_x]$.*

Proof. Consider the base point x in M, and any y in M the transitive of group action μ_1 on M implies there exists a unique $g \in G$ such that $\mu_1(g,x) = y$. We can choose a unique element $l_g \in Diff_{\mu_1}(M)$ and also $l_g(x) = \mu_1(g,x) = y$. For this l_g, the vector $(l_g)_{*x}(V_x)$ is in the orbit of V_x. Hence $(l_g)_{*x}(V_x) \in [V_x]$. Indeed, $(l_g)_{*x} : T_xM \to T_{(\mu_1(g,x)=y)}M$ so $(l_g)_{*x}(V_x) \in T_{(\mu_1(g,x)=y)}M$. Say $(l_g)_{*x}(V_x) = V_y \in [V_x]$ and this vector $(l_g)_{*x}(V_x)$ is unique because $l_g \in Diff_{\mu_1}(M)$ is unique such that $l_g(x) = y$. □

Proposition 3.6 can be exploited to get special kinds of smooth sections of tangent bundle TM. This is explicitly expressed as follows.

Corollary 3.7. *Let $[V_x] \in TM/Diff_{\mu_1}(M)$ then there exists a smooth section which is obtained by $[V_x]$, defined by $X_{[V_x]} : M \to TM$, by $X_{[V_x]}(y) = (l_g)_{*x}(V_x) = V_y$ (Which is explicitly defined in the Proposition 3.6).*

Proof. The map defined by $X_{[V_x]} : M \to TM$, by $X_{[V_x]}(y) = (l_g)_{*x}(V_x) = V_y$ is well-defined, because for every element $y \in M$, the $X_{[V_x]}$ assigns a unique vector $(l_g)_{*x}(V_x)$ based at y. Hence by composition with standard projection $\pi : TM \to M$ gives identity i.e. $\pi \circ X_{[V_x]} = Id_M$ trivially. Thus $X_{[V_x]}$ is a section of TM.
[12, 13] For the smoothness, it suffices to show that for any smooth function f on M, the function $X_{[V_x]}f$ is also C^∞. Choose a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = x$, and $\gamma'(0) = X_{[V_x]}(x) = V_x$. For every $y \in M$ there exists unique $g \in G$, we can have $l_g(x) = y$. For this $g \in G$, then $\mu_1(g,\gamma(t))$ is a curve at $\mu_1(g,\gamma(0)) = \mu_1(g,x) = l_g(x)$ with initial vector $X_{[V_x]}(y) = (l_g)_{*x}(V_x)$ and,

$$(\mu_1(g,\gamma(t)))'(0) = (l_g)_{*x}(\gamma'(0)) = (l_g)_{*x}(X_{[V_x]}(x)) = (l_g)_{*x}(V_x) = X_{[V_x]}(y).$$

It is well-known, $X_{[V_x]}(f)(y) = X_{[V_x]}(y)(f) = \frac{d}{dt}_{t=0} f(\mu_1(g,\gamma(t)))$. But the function $f(\mu_1(g,\gamma(t)))$ is smooth, since it is the composition of smooth functions as given below.

$$Id \times \gamma : G \times \mathbb{R} \to G \times M, \mu_1 : G \times M \to M, f : M \to \mathbb{R}$$

$$(g,t) \to (g,\gamma(t)) \to \mu_1((g,\gamma(t))) \to f(\mu_1((g,\gamma(t))))$$

Hence $X_{[V_x]}(f)$ is smooth at each $y \in M$. □

We denote $\mathfrak{X}_{\mu_2}(M)$ for the set of all smooth sections/vector fields generated by all of the equivalence classes $[V_x]$ in fact by the group action μ_2, i.e. $\mathfrak{X}_{\mu_2}(M) = \{X_{[V_x]} : \text{all } [V_x] \in TM/Diff_{\mu_1}(M)\}$ and from now onwards we use the same.

Proposition 3.8. *For all $x \in M$, a map $\Omega_x : T_xM \to \mathfrak{X}_{\mu_2}(M)$ defined by $\Omega_x(V_x) = X_{[V_x]}$ is a bijection.*

Proof. The map $\Omega_x : T_xM \to \mathfrak{X}_{\mu_2}(M)$ defined by $\Omega_x(V_x) = X_{[V_x]}$ is well-defined from Proposition 3.7.
For injection: for any $V_x \neq W_x \in T_xM$, by Proposition 3.4, $[V_x] \neq [W_x]$. It is

obvious that $\Omega_x(V_x) = X_{[V_x]} \neq X_{[W_x]} = \Omega_x(W_x)$, because for the point $x \in M$, $X_{[V_x]}(x) = V_x \neq W_x = X_{[W_x]}(x)$.
For surjection: for every $X_{[V_s]} \in \mathfrak{X}_{\mu_2}(M)$, and for $[V_s]$ there exists a unique $V_x \in [V_s]$ by Proposition 3.6. we can see $[V_s] = [V_x]$. Therefore, there exists this vector $V_x \in T_xM$, such that $\Omega_x(V_x) = X_{[V_x]} = X_{[V_s]}$. Hence Ω_x is a bijection. □

Note 3.9. *i) There is inverse for Ω_x i.e. $\Omega_x^{-1} : \mathfrak{X}_{\mu_2}(M) \to T_xM$ is given by $\Omega_x^{-1}(X_{[V_s]}) = [V_x']$, where $[V_s] = [V_x']$.*
ii) The vector field $X_{[O_x]}$ is a zero vector field. one can see Ω_x maps zero vector to the zero vector field.

Proposition 3.10. *Every non-zero vector field in $\mathfrak{X}_{\mu_2}(M)$ has support equal to whole M.*

Proof. For every non-zero vector field $X_{[V_x]} \in \mathfrak{X}_{\mu_2}(M)$. The support is the closure of $\{y \in M : X_{[V_x]}(y) \neq 0\}$, which is M, because each $X_{[V_x]}$ is a nowhere vanishing vector field. □

Proposition 3.11. *Every vector field $X_{[V_x]} \in \mathfrak{X}_{\mu_2}(M)$ is a left-invariant vector field with respect to group action μ_1 (or G-invariant vector field on M or $Diff_{\mu_1}(M)$-related vector field, see definition 4.12).*

Proof. Every vector field $X_{[V_x]} \in \mathfrak{X}_{\mu_2}(M)$ and every element $l_g \in Diff_{\mu_1}(M), \forall y \in M$ consider the $(l_g)_{*y} \circ X_{[V_x]}(y) = (l_g)_{*y}((l_h)_{*x}(V_x))$ for some $l_h \in Diff_{\mu_1}(M)$, and $l_h(x) = \mu_1(h,x) = y$. Hence $(l_g)_{*y} \circ X_{[V_x]}(y) = (l_g \circ l_h)_{*x}(V_x) = (l_{g\star h})_{*x}(V_x)$. On the other hand, consider the $X_{[V_x]} \circ l_g(y) = X_{[V_x]}(\mu_1(g,y)) = (l_k)_{*x}(V_x)$, for some $l_k \in Diff_{\mu_1}(M)$, and $l_k(x) = \mu_1(k,x) = \mu_1(g,y)$. By substituting the value of y we can see $\mu_1(k,x) = \mu_1(g,\mu_1(h,x)) = \mu_1(g \star h, x)$ the free group action implies $k = g \star h$. Hence $X_{[V_x]} \circ l_g(y) = (l_{g\star h})_{*x}(V_x)$. Comparing the discussion, we can see $(l_g)_* \circ X_{[V_x]} = X_{[V_x]} \circ l_g$. Therefore $X_{[V_x]}$ is a left-invariant vector field with respect to group action μ_1. □

It is well-known that the set $\mathfrak{X}(M)$ of all smooth vector fields on M is an infinite-dimensional linear space over $\mathbb{R}$ under point wise addition (+) and scalar multiplication $(\cdot)$. We can get a linear structure on $\mathfrak{X}_{\mu_2}(M)$ under these operations. Moreover, we can also define addition in another way but that coincides with the usual operations trivially.

Proposition 3.12. *i) On the set $\mathfrak{X}_{\mu_2}(M)$, a map $\oplus : \mathfrak{X}_{\mu_2}(M) \times \mathfrak{X}_{\mu_2}(M) \to \mathfrak{X}_{\mu_2}(M)$ defined by $\oplus(X_{[V_s]}, X_{[V_u]}) = X_{[V_s + V_s']}$, (where $[V_s'] = [V_u]$) is well-defined. Further, this addition coincides with the usual point wise addition operation.*
ii) A map $\odot : \mathbb{R} \times \mathfrak{X}_{\mu_2}(M) \to \mathfrak{X}_{\mu_2}(M)$ defined by $\odot(\alpha, X_{[V_s]}) = X_{[\alpha \cdot V_s]}$ is well-defined. Further, this scalar multiplication coincides with the usual point wise scalar multiplication operation.
iii) Under the operations $\oplus$ and $\odot$ the set $\mathfrak{X}_{\mu_2}(M)$ becomes a linear space.

Note 3.13. *The set $\mathfrak{X}_{\mu_2}(M)$ can acquire a linear structure of tangent spaces of the manifold M, indeed which is identically equal to the operations in Proposition 3.12.*

Proposition 3.14. *For any point $x \in M$ then T_xM is linearly isomorphic to $\mathfrak{X}_{\mu_2}(M)$ under the map Ω_x.*

Proof. The map $\Omega_x : T_xM \to \mathfrak{X}_{\mu_2}(M)$ given by $\Omega_x(V_x) = X_{[V_x]}$ is a bijection from Proposition 3.8 and we have for linearity. We have $\Omega_x(V_x + W_x) = X_{[V_x+W_x]} = X_{[V_x]} \oplus X_{[W_x]} = \Omega_x([V_x]) \oplus \Omega_x([W_x])$, because $X_{[V_x+W_x]}(y) = (l_g)_{*x}(V_x + W_x) = (l_g)_{*x}(V_x) + (l_g)_{*x}(W_x) = X_{[V_x]}(y) + X_{[W_x]}(y) = X_{[V_x]} \oplus X_{[W_x]}(y)$ for all $y \in M$, where $l_g(x) = y$. Similarly, for any real α, we have $\Omega_x(\alpha[V_x]) = X_{[\alpha V_x]} = \alpha \odot X_{[V_x]} = \alpha \odot \Omega_x([V_x])$, because $X_{[\alpha V_x]}(y) = (l_g)_{*x}(\alpha V_x) = \alpha(l_g)_{*x}(V_x) = \alpha X_{[V_x]}(y) = (\alpha \odot X_{[V_x]})(y)$ for all $y \in M$, where $l_g(x) = y$. Hence the result. □

Note 3.15. *The linear space $\mathfrak{X}_{\mu_2}(M)$ is a finite-dimensional (equal to dimensional of manifold M). Hence it is a Lie group*

Proposition 3.16. *For the manifold M if $\mathfrak{B} = \{v^1{}_x, v^2{}_x, ...v^n{}_x\}$ is basis for T_xM for a point x then $\mathfrak{B}' = \{X_{[v^1{}_x]}, X_{[v^2{}_x]}, ..., X_{[v^n{}_x]}\}$ forms a smooth frame.*

Proof. Since the map Ω_x is a linear isomorphism, it will send basis to basis, and each vector field in the basis is smoothly varying, therefore $\mathfrak{B}'$ forms a smooth frame on M. □

Restriction of the Lie bracket of $\mathfrak{X}(M)$ on $\mathfrak{X}_{\mu_2}(M)$ forms a Lie subalgebra hence all tangent space can be viewed as a Lie algebra induced by $\mathfrak{X}_{\mu_2}(M)$ as follows.

Proposition 3.17. *i) With the usual definition of $[X_{[V_u]}, X_{[V_s]}] = X_{[V_u]}X_{[V_s]} - X_{[V_s]}X_{[V_u]}$ for all $X_{[V_u]}, X_{[V_s]} \in \mathfrak{X}_{\mu_2}(M)$ then $\mathfrak{X}_{\mu_2}(M)$ becomes a Lie subalgebra of $\mathfrak{X}(M)$.*
ii) Let $V_x, W_x \in T_xM$ and define $[|V_x, W_x|] = {\Omega_x}^{-1}([X_{[V_x]}, X_{[W_x]}])$ then $(T_xM, [|, |])$ forms Lie algebra, and all tangent spaces are Lie algebras isomorphic to one another.

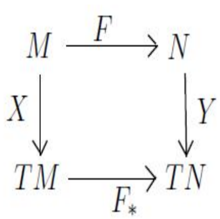
Proof. Both results are quite simple i) is obtained from the usual Lie bracket of $\mathfrak{X}(M)$ restricting on $\mathfrak{X}_{\mu_2}(M)$.
ii) Here, Lie bracket comes from Lie bracket of $\mathfrak{X}_{\mu_2}(M)$ under the isomorphism Ω_x. □

In the theory of Lie groups usually one can see the fact that T_eG is isomorphic to the set of left-invariant vector fields $L(G)$. We immediately conclude which is a case of the general discussion for smooth manifolds undergoing a regular Lie group action made above. Since for the Lie group $(G, \star)$ the G-action translation $\mu_1(g, h) = g \star h$ is a regular group action. This group action yields group $Diff_{\mu_1}(G)$, which further acts on TG, by $\mu_2(l_g, V_h) = (l_g)_{*h}(V_h)$. In the group action μ_2, the orbit of each element yields a vector field of G. Moreover it is a left-invariant vector field, and commonly, the collection of all left-invariant vector fields is denoted by $L(G)$ or $L_{\mathfrak{X}(G)}(G)$. Under the isomorphism $(l_g)_{*e}$ all tangent spaces are isomorphic to T_eG and under $X \to X(e)$ which become an isomorphism between $L_{\mathfrak{X}(G)}(G)$ and T_eG. Hence $L_{\mathfrak{X}(G)}(G)$ is finite-dimensional linear space, with induced Lie bracket operation on it, and thus it becomes a Lie subalgebra. The tangent spaces T_eG become Lie algebra denoted by $\mathfrak{g}$, which absorbs it from the Lie algebra structure of $L_{\mathfrak{X}(G)}(G)$. Therefore that translation group action on G itself is a case in general regular group action defined for any smooth manifold M by a Lie group.

4 Vector field-related diffeomorphisms, diffeomorphism-related Vector fields and their algebra

The left-invariant vector field is a particular case of the concept F-related vector field which is a more general notion. Pullback and pushforward are the motivations to develop such concepts [12, 11, 13], in the theory of vector fields, we commonly see such notions. Here, we enhance that idea to diffeomorphisms called vector field relatedness, and we develop some interrelationships between both rel operations in this section.

Definition 4.1. *[12, 8, 13] Let M, N be smooth manifolds and $F : M \to N$ be a smooth map then a vector field $X \in \mathfrak{X}(M)$ is said to be $F-$related to a vector field $Y \in \mathfrak{X}(N)$ if $F_* \circ X = Y \circ F$, or the following diagram commute,*



At the same time, we call F is pair (X, Y)-related smooth map. Here we can define a relation $Q : C^\infty(M, N) \to \mathfrak{X}(M) \times \mathfrak{X}(N)$, by $Q(F) = (X, Y)$, if $F_* \circ X = Y \circ F$. With reference to the above definitions, we can define the following notions,

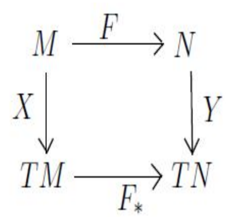
i) For a given $F \in C^\infty(M, N)$ then the set of all pairs of vector fields from $\mathfrak{X}(M) \times \mathfrak{X}(N)$ for which $F_* \circ X = Y \circ F$, is called $F-$related pairs of vector fields, is denoted by $rel_{\mathfrak{X}(M)\times\mathfrak{X}(N)}(F)$. Moreover, we can see $Q(F) = rel_{\mathfrak{X}(M)\times\mathfrak{X}(N)}(F)$.

ii) For a given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ then the set of all smooth maps F from M to N for which $F_* \circ X = Y \circ F$, is called pair $(X, Y)-$related smooth maps, is denoted by $rel_{C^\infty(M,N)}(X, Y)$. Moreover, we can see $Q^{-1}(\{(X, Y)\}) = rel_{C^\infty(M,N)}(X, Y)$.

For a given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ the set $rel_{C^\infty(M,N)}(X, Y)$ may sometimes be empty, since the relation is non-universal. However, $rel_{\mathfrak{X}(M)\times\mathfrak{X}(N)}(F)$ is always non-empty. More interestingly, for a given smooth map F, the $rel_{\mathfrak{X}(M)\times\mathfrak{X}(N)}(F)$ is a linear subspace of $\mathfrak{X}(M) \times \mathfrak{X}(N)$ under coordinate wise addition and scalar multiplication.

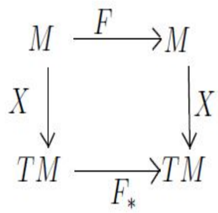
Definition 4.1 can be used for a diffeomorphism, but we are defining separately for a diffeomorphism as a new definition to avoid confusion in notation and also for the frequent usage of relatedness in terms of diffeomorphisms.

Definition 4.2. *[12, 8, 13] Let M, N be smooth manifolds and $F : M \to N$ be a diffeomorphism, a vector field $X \in \mathfrak{X}(M)$ is said to be $F-$related to a vector field $Y \in \mathfrak{X}(N)$ if $F_* \circ X = Y \circ F$, or the following diagram commute,*



At the same time, we call F is pair $(X, Y)-$related diffeomorphism. Here we can define a relation $R : Diff(M, N) \to \mathfrak{X}(M) \times \mathfrak{X}(N)$, by $R(F) = (X, Y)$, if $F_* \circ X = Y \circ F$. With reference to the above definitions, we can define the following notions,
i) For a given $F \in Diff(M, N)$ then the set of all pairs of vector fields from $\mathfrak{X}(M) \times \mathfrak{X}(N)$ for which $F_* \circ X = Y \circ F$, is called $F-$related pairs of vector fields, is denoted by $rel_{\mathfrak{X}(M)\times\mathfrak{X}(N)}(F)$. Moreover, we can see $R(F) = rel_{\mathfrak{X}(M)\times\mathfrak{X}(N)}(F)$.
ii) For a given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ then the set of all diffeomorphisms F from M to N for which $F_* \circ X = Y \circ F$, is called pair $(X, Y)-$related diffeomorphisms, is denoted by $rel_{Diff(M,N)}(X, Y)$. Moreover, we can see $R^{-1}(\{(X, Y)\}) = rel_{Diff(M,N)}(X, Y)$.
For a given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ the set $rel_{Diff(M,N)}(X, Y)$ may sometimes be empty since the relation is non-universal. However, $rel_{\mathfrak{X}(M)\times\mathfrak{X}(N)}(F)$ is always non-empty. More interestingly, for a given diffeomorphism F, the $rel_{\mathfrak{X}(M)\times\mathfrak{X}(N)}$ is a linear subspace of $\mathfrak{X}(M) \times \mathfrak{X}(N)$, under coordinate wise addition and scalar multiplication.

Definition 4.3. *[12, 8, 13] Let M be smooth manifold and $F : M \to M$ be a diffeomorphism, a vector field $X \in \mathfrak{X}(M)$ is said to be $F-$related or $F-$invariant vector field if $F_* \circ X = X \circ F$, or the following diagram commute,*



At the same time, we are calling F is $X-$related diffeomorphism. Define a relation $S : Diff(M) \to \mathfrak{X}(M)$, by $S(F) = X$, if $F_* \circ X = X \circ F$. With reference to the above definitions, we can define the following notions,
i) For a given $F \in Diff(M)$ then the set of all vector fields on M for which $F_* \circ X = X \circ F$, is called $F-$related vector fields or $F-$invariant vector fields, is denoted by $rel_{\mathfrak{X}(M)}(F) = \{X \in \mathfrak{X}(M) : F_* \circ X = X \circ F\}$. Moreover, we can see $S(\{F\}) = rel_{\mathfrak{X}(M)}(F)$.
ii) For a given $X \in \mathfrak{X}(M)$ then the set of all diffeomorphisms F on M for which $F_* \circ X = X \circ F$, is called $X-$related diffeomorphisms, is denoted by $rel_{Diff(M)}(X)$. Explicitly $rel_{Diff(M)}(X) = \{F \in Diff(M) : F_* \circ X = X \circ F\}$, also which equal to $S^{-1}(\{X\})$.
For a given $X \in \mathfrak{X}(M)$ the set $rel_{Diff(M)}(X)$ is always non-empty, since $Id_* \circ X = X \circ Id$ for every vector field X, therefore $Id \in rel_{Diff(M)}(X)$. And also always set $rel_{\mathfrak{X}(M)}(F)$ is non-empty since $F_* \circ O = O \circ F$.

Proposition 4.4. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ then the set $rel_{Diff(M)}(X)$ is a subgroup of $Diff(M)$.*

Proof. Obviously $rel_{Diff(M)}(X)$ is a non-empty subset of the set of all diffeomorphisms on M. Take any two $F, G \in rel_{Diff(M)}(X)$, consider $(FoG)_* \circ X = F_* \circ G_* \circ X = F_* \circ X \circ G = X \circ (F \circ G)$. Therefore $F \circ G \in rel_{Diff(M)}(X)$. Associativity is true which is coming from the general property of the composition. Since $Id \in rel_{Diff(M)}(X)$ and it is the identity in $rel_{Diff(M)}(X)$. Also for any $F \in rel_{Diff(M)}(X)$ one can see $F^{-1} \in rel_{Diff(M)}(X)$ because consider $F_*^{-1} \circ X = F_*^{-1} \circ X \circ F \circ F^{-1} = F_*^{-1} \circ F_* \circ X \circ F^{-1} = X \circ F^{-1}$. Hence $rel_{Diff(M)}(X)$ is a subgroup. □

Note 4.5. *Let M be a smooth manifold and O from $\mathfrak{X}(M)$ be the zero vector field then $rel_{Diff(M)}(O) = Diff(M)$.*

Definition 4.6. *Let M be a smooth manifold and $\mathfrak{B} \subset \mathfrak{X}(M)$ then set of all diffeomorphisms F on M for which $F_* \circ X = X \circ F$, for all $X \in \mathfrak{B}$ is called $\mathfrak{B}-$related diffeomorphisms, denoted by $rel_{Diff(M)}(\mathfrak{B})$. i.e. $rel_{Diff(M)}(\mathfrak{B}) = \{F \in Diff(M) : F_* \circ X = X \circ F$, for all $X \in \mathfrak{B}\}$.*

Proposition 4.7. *Let M be a smooth manifold and $\mathfrak{B} \subset \mathfrak{X}(M)$ be a subset of vector fields on M then the set $rel_{Diff(M)}(\mathfrak{B})$ is a subgroup of $Diff(M)$.*

Proof. Similar to Proposition 4.4. □

Note 4.8. *i)* $rel_{Diff(M)}(\mathfrak{X}(M)) = \{Id\}$.

Discussed results are groups, which can also be seen as an isotropy subgroup of a point in $\mathfrak{X}(M)$ or stabilizer of a vector field and stabilizer of a set under a crucial group action on $\mathfrak{X}(M)$ by $Diff(M)$.

Proposition 4.9. *Let M be a smooth manifold and $\mu : Diff(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ a map defined by $(F, X) = F_* \circ X \circ F^{-1}$, [12] then,*
i) μ is a non-transitive group action (One can also define right group action).
ii) Isotropy$(X) = stab_{Diff(M)}(X) = rel_{Diff(M)}(X)$ for all $X \in \mathfrak{X}(M)$.
iii)For a given subset of vector fields $\mathfrak{B} \subseteq \mathfrak{X}(M)$, then the $rel_{Diff(M)}(\mathcal{B}) = \bigcap_{X\in\mathcal{B}} stab_{Diff(M)}(X)$.
iv)The fixed point set $\mathfrak{X}(M)^{Diff(M)} = \{O\}$.

Proof. i) Let M be a smooth manifold, the group $Diff(M)$ act by $\mu(F, X) = F_* \circ X \circ F^{-1}$. Because, it is clear that $Id_* \circ X \circ Id^{-1} = X$ for all vector fields, also $\mu(F, \mu(G, X)) = \mu(F, G_* \circ X \circ G^{-1}) = (F \circ G)_* \circ X \circ (F \circ G)^{-1} = \mu(F \circ G, X)$. The orbit of zero vector fields under this group action becomes a singleton trivial vector field, that is $orb_\mu(O) = \{\mu(F, O) : \text{for all } F \in Diff(M)\} = \{O\}$, hence group action is non-transitive.
ii) For each $X \in \mathfrak{X}(M)$ let us compute $stab_{Diff(M)}(X) = Isotropy(X)$,

$$\begin{aligned} Stab_{Diff(M)}(X) &= \{F \in Diff(M) : \mu(F, X) = F_* \circ X \circ F^{-1} = X\} \\ &= \{F \in Diff(M) : F_* \circ X = X \circ F\} \\ &= rel_{Diff(M)}(X) \end{aligned}$$

It is well-known that isotropy(X) is a subgroup of $Diff(M)$ so the $rel_{Diff(M)}(X)$. Hence $rel_{Diff(M)}(X)$ is a subgroup.
iii) For a given subset of vector fields $\mathfrak{B} \subseteq \mathfrak{X}(M)$, we compute,

$$\begin{aligned} rel_{Diff(M)}(\mathcal{B}) &= \{F \in Diff(M) : \mu(F, X) = F_* \circ X \circ F^{-1} = X, \forall X \in \mathcal{B}\} \\ &= \{F \in Diff(M) : F_* \circ X = X \circ F, \forall X \in \mathcal{B}\} \\ &= \bigcap_{X\in\mathcal{B}} stab_{Diff(M)}(X) \end{aligned}$$

iv) For the fixed point set $\mathfrak{X}(M)^{Diff(M)} = \{X \in \mathfrak{X}(M) : \mu(F, X) = F_* \circ X \circ F^{-1} = X, \text{ for every } F \in Diff(M)\}$. Pushforward of a vector field under all homeomorphisms is only the zero vector field. Therefore fixed point set $\{O\}$. □

Proposition 4.10. *Let M be a smooth manifold and for each $F \in Diff(M)$ then set $rel_{\mathfrak{X}(M)}(F)$ is a subspace of $\mathfrak{X}(M)$.*

Proof. Obviously $rel_{\mathfrak{X}(M)}(F)$ is a non-empty subset of the set of all vector fields on M. Choose arbitrary $X, Y \in rel_{\mathfrak{X}(M)}(F)$, consider $F_* \circ (X + Y) = F_* \circ X + F_* \circ Y = X \circ F + Y \circ F = (X + Y) \circ F$. Therefore $X + Y \in rel_{\mathfrak{X}(M)}(F)$. Also for every $\alpha \in \mathbb{R}$ and $X \in rel_{\mathfrak{X}(M)}(F)$, let us consider $F_* \circ (\alpha \cdot X) = \alpha \cdot F_* \circ (X) = \alpha \cdot X \circ F$. Therefore $\alpha \cdot X \in rel_{\mathfrak{X}(M)}(F)$. Thus $rel_{\mathfrak{X}(M)}(F)$ is a subspace. □

Note 4.11. *Let M be a smooth manifold and for $Id \in Diff(M)$ then $rel_{\mathfrak{X}(M)}(Id) = \mathfrak{X}(M)$.*

Definition 4.12. *Let M be a smooth manifold and $\mathcal{H} \subseteq Diff(M)$ then the set of all vector fields $X \in \mathfrak{X}(M)$ such that X is $\mathcal{H}-$related i.e $F_* \circ X = X \circ F$, for all $F \in \mathcal{H}$, is called $\mathcal{H}-$related vector fields or $\mathcal{H}-$ invariant vector fields, and is denoted by $rel_{\mathfrak{X}(M)}(\mathcal{H})$. i.e. $rel_{\mathfrak{X}(M)}(\mathcal{H}) = \{X \in \mathfrak{X}(M) : F_* \circ X = X \circ F$, for all $F \in \mathcal{H}\}$.*
For a given non-empty subset $\mathcal{H}$ of diffeomorphism group on M the set $rel_{\mathfrak{X}(M)}(\mathcal{H})$ is always non-empty, since $F_* \circ O = O \circ F$, for all $F \in \mathcal{H}$, therefore $O \in rel_{\mathfrak{X}(M)}(\mathcal{H})$.

Proposition 4.13. *Let M be a smooth manifold and $\mathcal{H} \subseteq Diff(M)$ then set $rel_{\mathfrak{X}(M)}(\mathcal{H})$ is a linear subspace of $\mathfrak{X}(M)$.*

Proof. It is quite similar to Proposition 4.10. (It is the intersection of all vector spaces $rel_{\mathfrak{X}(M)}(F)$, for all $F \in \mathcal{H}$). □

Note 4.14. *i) Let M be a smooth manifold then $rel_{\mathfrak{X}(M)}Diff(M) = \{O\}$.*
ii) In the theory of Lie group, Left-invariant vector field set $L_{\mathfrak{X}(G)}(G)$ and the diffeomorphism subgroup $\{l_g : l_g : G \to G\}$ yields by translation group action, respectively have rich interrelations by referring relatedness as follows
a) $rel_{Diff(M)}(L_{\mathfrak{X}(G)}(G)) = \{l_g : l_g : G \to G\}$ and b) $rel_{\mathfrak{X}(M)}(\{l_g : l_g : G \to G\}) = L_{\mathfrak{X}(G)}(G)$.

Here we would like to pose the following questions,
i) For every non-trivial proper subspace B of $\mathfrak{X}(M)$ is the $rel_{Diff(M)}(B)$ always equal to a non-trivial subgroup of the diffeomorphisms group, that emerges by a group action?
ii) For every non-trivial proper subgroup H of $Diff(M)$ is the $rel_{\mathfrak{X}(M)}(H)$ always equal to a finite-dimensional linear space?

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