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THE MECHANICS OF THE VIBRATING TRIANGLE SYSTEM

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Abstract

Vibrating and oscillation systems play an important task in physics and mathematics. In this paper, we investigated the dynamical behavior of an important system called the vibrating triangle (in some literature it is called the planar system). First of all, the Lagrangian equation of the system has been constructed. Secondly, we derived the Euler- Lagrange equations (ELEs). Thirdly, we solve the obtained ELEs numerically using MATLAB for some selected parameters, and for specified initial conditions.

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Key words: vibrating triangle, Lagrange equation, Euler-Lagrange equations, numerical solution.

1 Introduction

Spring mass problems are one of the familiar differential equations that have significant importance in mechanical engineering, physics and mathematics. Multi-atom molecules, oscillating circuits and elastic solids can be represented by two or more masses with two and more springs [13].

The masses attached to springs will have potential energy and as they are moved from their equilibrium position, they will start oscillations. Obtaining the

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frequency, amplitude and describing the motion of such oscillated systems is of great importance for scientists and material science engineers as these oscillations may be undesired in some situations as vibrations in planes and automobiles which can lead to structural defects due to fatigue [14], and desired in other situations as conducting heat in solids.

Describing and obtaining the equations of motion of such systems may be accomplished using Newtonian mechanics by investigating and analyzing the forces on each mass of the system and this needs a lot of labor and care as forces are vector quantities. Another easier approach to obtain these equations is through building the Lagrangian of the system which is the difference between kinetic and potential energies which are scalar quantities and easy to deal with and then by applying Lagrangian mechanics one can obtain the equations of motion. Interested readers can refer to some classical books to take a whole picture of how one can apply Lagrangain mechanics to solve vibrating and oscillating physical systems in addition to many other interesting physical systems [12, 17, 9]. It is important to point that upon applying Lagrangain equation and deriving the Euler- Lagrange equations (equations of motion), which are differential equations, one has to seek for their analytical solution if they are simple for this purpose, we refer to some texts on differential equations to learn how to solve them [6, 2, 19].

In many cases, the derived Euler- Lagrange equations are not so easily solved analytically and, in this case, we focus our attention On getting a numerical or a simulation result for the derived equations.

Many numerical methods have been introduced in literature to solve differential equations. These numerical methods are based on MATLAB, MATHEMATICA, MATCAD software and other computerized programs. For interested readers we advise them to refer to the following works [7, 16, 11, 8, 5] and reference therein. In many cases MATLAB has to used to obtain the simulation results. We point here to some recent works carried on, see for example [15, 1, 3, 4, 20] and references therein.

The structure of this work is systematized on the following picture: A physical description of the system is presented, where the Euler- Lagrange equations were derived from the Lagrangian equation in Sec. 2. In Sec. 3 a numerical and simulation technique used is discussed and presented, and in Sec. 4 results and discussions are reported on the dynamical behavior of the system. Finally, we close the paper with a conclusion in Sec. 5

2 The vibrating triangle

The arrangement shown in Fig. 1 is composed of three masses (m_1, m_2 and m_3) located at the corners of an equilateral triangle and connected by three spiral springs with stiffness (k_{12}, k_{13},k_{23}). In some literature this arrangement is known as the vibrating triangle or planar system [19], we suppose that masses are allowed to vibrate in the plane of the page only. This system is considered to be a good example on group and group representation theory and for more details one can refer reference [18] chapter 16.

Firstly, we begin our description by constructing the Lagrangian, which is the difference between the kinetic and potential energies, respectively.

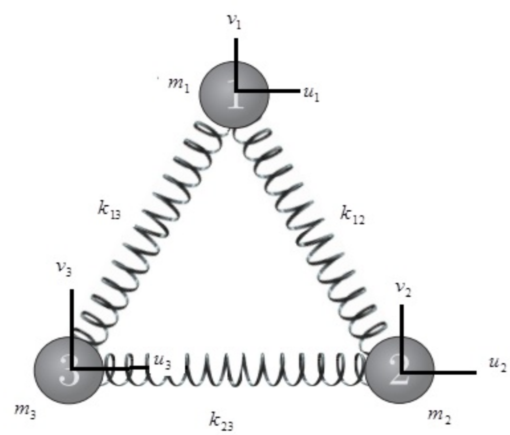


Figure 1: The Vibrating Triangle

Let us describe the displacements from the equilibrium positions by u_1, v_1, u_2, v_2, u_3, v_3. The kinetic energy T of the system is then obtained from:

$$T = \frac{1}{2}m_1(\dot{u_1}^2 + \dot{v_1}^2) + \frac{1}{2}m_2(\dot{u_2}^2 + \dot{v_2}^2) + \frac{1}{2}m_3(\dot{u_3}^2 + \dot{v_3}^2) \tag{1}$$

The potential energy stored in the springs attached between masses (m_1, m_2) , (m_1, m_3) and (m_2, m_3) respectively is given as:

$$U = U_{12} + U_{13} + U_{23}$$

$$= \frac{1}{2}k[(u_1 - u_2)^2 + (v_1 - v_2)^2 + (u_2 - u_3)^2 + (v_2 - v_3)^2 + (u_3 - u_1)^2 + (v_3 - v_1)^2] \tag{2}$$

According to the definition of the Lagrangian (L) and Eqs. (1), (2), we can write:

$$L = T - U$$

$$= \tfrac{1}{2}m[(\dot{u_1}^2 + \dot{v_1}^2) + (\dot{u_2}^2 + \dot{v_2}^2) + (\dot{u_3}^2 + \dot{v_3}^2)] -$$
$$\tfrac{1}{2}k[(u_1 - u_2)^2 + (v_1 - v_2)^2 + (u_2 - u_3)^2 + (v_2 - v_3)^2 + (u_3 - u_1)^2 + (v_3 - v_1)^2] \tag{3}$$

Applying

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\partial L}{\partial \dot{q}} = 0$$

to Eq. (3) with $q = u_1, u_2, u_3, v_1, v_2, v_3$ respectively, the equations of motion for our system read:

$$\ddot{u}_1 = \frac{k_{13}}{m_1}(u_3 - u_1) - \frac{k_{12}}{m_1}(u_1 - u_2) \tag{4}$$

$$\ddot{u}_2 = \frac{k_{12}}{m_2}(u_1 - u_2) - \frac{k_{23}}{m_2}(u_2 - u_3) \tag{5}$$

$$\ddot{u}_3 = \frac{k_{23}}{m_3}(u_2 - u_3) - \frac{k_{13}}{m_3}(u_3 - u_1) \tag{6}$$

$$\ddot{v}_1 = \frac{k_{13}}{m_1}(v_3 - v_1) - \frac{k_{12}}{m_1}(v_1 - v_2) \tag{7}$$

$$\ddot{v}_2 = \frac{k_{12}}{m_2}(v_1 - v_2) - \frac{k_{23}}{m_2}(v_2 - v_3) \tag{8}$$

$$\ddot{v}_3 = \frac{k_{23}}{m_3}(v_2 - v_3) - \frac{k_{13}}{m_3}(v_3 - v_1) \tag{9}$$

We will focus our attention on the case where all masses are identical (i.e., $m_1 = m_2 = m_3 = m$), in addition three cases of interests will be studied as follows:

2.1 Case 1: all springs are identical ($k_{12} = k_{13} = k_{23} = k$)

For this case equations (4-9) read

$$\ddot{u}_1 = w^2 u_3 + w^2 u_2 - 2w^2 u_1 \tag{10}$$

$$\ddot{u}_2 = w^2 u_1 + w^2 u_3 - 2w^2 u_2 \tag{11}$$

$$\ddot{u}_3 = w^2 u_2 + w^2 u_1 - 2w^2 u_3 \tag{12}$$

$$\ddot{v}_1 = w^2 v_3 + w^2 v_2 - 2w^2 v_1 \tag{13}$$

$$\ddot{v}_2 = w^2 v_1 + w^2 v_3 - 2w^2 v_2 \tag{14}$$

$$\ddot{v}_3 = w^2 v_2 + w^2 v_1 - 2w^2 v_3 \tag{15}$$

2.2 Case 2: The two springs on the sides are only identical, and the third spring is stiffer than the side springs ($k_{12} = k_{13} = k, k_{23} = 10k$)

In this case, the equations of motion become:

$$\ddot{u_1} = w^2 u_3 + w^2 u_2 - 2w^2 u_1 \tag{16}$$

$$\ddot{u_2} = w^2 u_1 + 10w^2 u_3 - 11w^2 u_2 \tag{17}$$

$$\ddot{u_3} = w^2 u_1 + 10w^2 u_2 - 11w^2 u_3 \tag{18}$$

$$\ddot{v_1} = w^2 v_3 + w^2 v_2 - 2w^2 v_1 \tag{19}$$

$$\ddot{v_2} = w^2 v_1 + 10w^2 v_3 - 11w^2 v_2 \tag{20}$$

$$\ddot{v_3} = w^2 v_1 + 10w^2 v_2 - 11w^2 v_3 \tag{21}$$

2.3 Case 3: The two springs on the sides are only identical, while the third spring is less stiff than the side springs ($k_{12} = k_{13} = k, k_{23} = 0.01k$)

For this case, the equations (4-9) read:

$$\ddot{u_1} = w^2 u_3 + w^2 u_2 - 2w^2 u_1 \tag{22}$$

$$\ddot{u_2} = w^2 u_1 + 0.01w^2 u_3 - 1.01w^2 u_2 \tag{23}$$

$$\ddot{u_3} = 0.01w^2 u_2 + w^2 u_2 - 1.01w^2 u_3 \tag{24}$$

$$\ddot{v_1} = w^2 v_3 + w^2 v_2 - 2w^2 v_1 \tag{25}$$

$$\ddot{v_2} = w^2 v_1 + 0.01w^2 v_3 - 1.01w^2 v_2 \tag{26}$$

$$\ddot{v_3} = 0.01w^2 v_2 + w^2 v_1 - 1.01w^2 v_3 \tag{27}$$

where in the all the cases above $w = \sqrt{\frac{k}{m}}$ is the angular frequency. In the following, we aim to obtain a numerical solution for the equations of motion for the cases presented above for some selected initial conditions.

3 Numerical and simulation technique

MATLAB is good software for solving symbolic and differential equations using pre-defined and user defined functions. Several MATLAB packages are used for doing plots in 2-dimensional and 3-dimensional graphs. MATLAB is now used for mathematical calculations and data simulation in companies and government labs ranging from aerospace, civil engineering, car design, biology, signal analysis and hydrodynamics to instrument control and financial analysis [22].

Symbolic equations are important blocks in different mathematical problems. They are used in different implementations and systems as physical and chemical applications, control and electrical engineering, and economy [21]. MATLAB preprocesses the symbolic operations using various functions and commands such as $findsym(), collect(), expand(), factor(), simplify()$ and $pretty()$. In preprocessing phase, MATLAB restructures the symbolic equations as the general structure and style of the numerical operations before the solution. The pretty mathematical equations are solved using $solve()$ command if the equations are Algebra equations, for example the ordinary differential equations can be solved using the $dsolve()$ command [4, 22, 10].

The $dsolve()$ command returns a solution as a matrix of n-columns, the number of columns depends on the number of equation systems in the symbolic system of equations. The inputs of $dsolve()$ are equations system and their initial conditions but its output is the matrix of the solutions: the first column is the solution of the first variable, the second column is the solution of the second variable.

The output (the matrix of n-columns) of $dsolve()$ command lets the opportunity for researchers to plot charts and graphs easily. In MATLAB, symbolic differentiation for n-order equations can be implemented by utilizing the $diff()$ function [21, 23].

In the next section, simulation results for the dynamical behavior of the vibrating triangle system will be studied using $dsolve()$ and $plot()$. Several plots and graphs were created for many important states.

4 Results and discussion

In this section the dynamical behavior of the coordinates u_1, u_2 and u_3 will be depicted against time for the three cases explained in section 2 above, and for the specific three initial conditions listed below.

$i)u_1(0) = u_2(0) = u_3(0) = 0$, and $\dot{u_1}(0) = 0, \dot{u_2}(0) = 1, \dot{u_3}(0) = -1$

$ii)u_1(0) = u_2(0) = u_3(0) = 0$, and $\dot{u_1}(0) = 1, \dot{u_2}(0) = 0, \dot{u_3}(0) = 0$

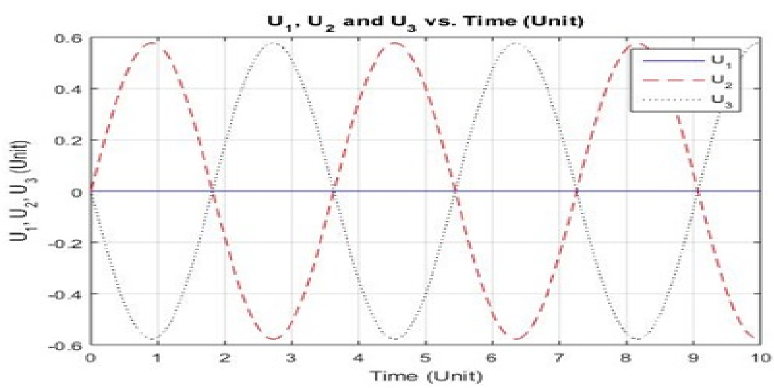
$iii)u_1(0) = u_2(0) = -u_3(0) = 1$, and $\dot{u_1}(0) = 1, \dot{u_2}(0) = 0, \dot{u_3}(0) = 0$

The following parameters will be considered in our numerical results $m = 1.00$, and $k = 1.00$, respectively with the following initial conditions:

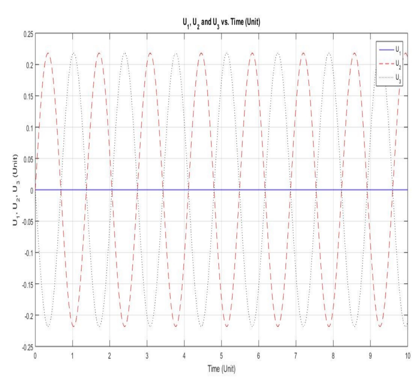
Figures(2 - 4) below show the dynamical behavior of the coordinates u_1, u_2 and u_3 against time, for the above cases and initial conditions specified and listed before. In Fig.2 the first initial condition was used, where Fig.2(a) shows the dynamical behavior of the coordinates u_1, u_2 and u_3 against time for case 1, while in Fig.2(b) it represents case 2, and finally Fig.2(c) belongs to case 3.

In Fig.3 the initial second initial condition was applied, where Fig.3(a) shows the dynamical behavior of the coordinates u_1, u_2 and u_3 against time for case 1, while in Fig.3(b) it represents case 2, and finally Fig.3(c) belongs to case 3.

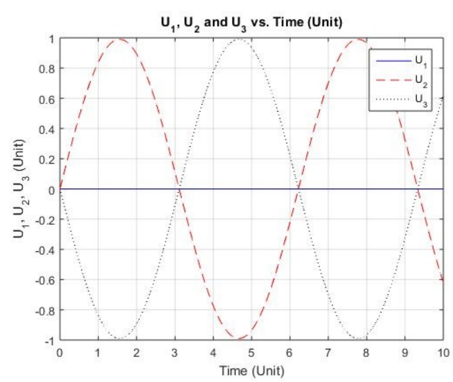
Finally, in Fig.4 the third initial condition was used, where Fig.4(a) shows the dynamical behavior of the coordinates u_1, u_2 and u_3 against time for case 1, while in Fig.4(b) it represents case 2, and finally Fig.4(c) belongs to case 3.



(a) Case a



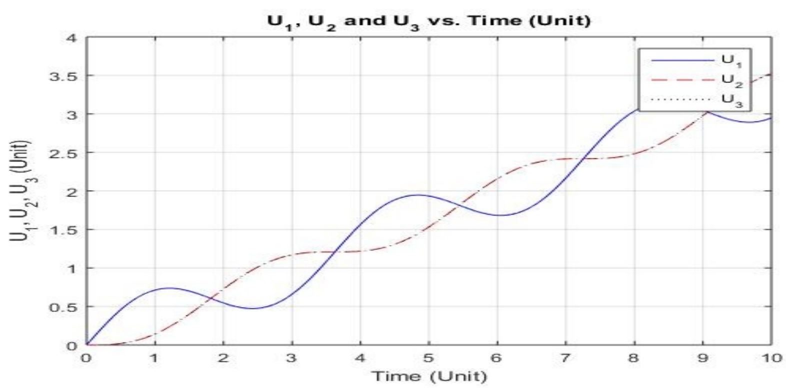
(b) Case b



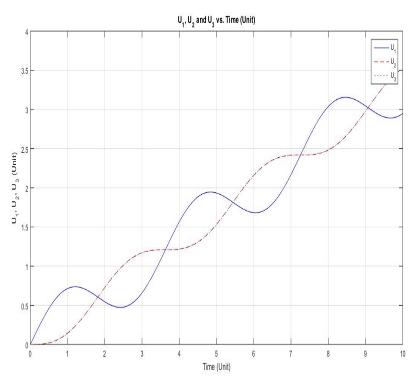
(c) Case c

Figure 2: The dynamical behavior of the coordinates u_1, u_2 and u_3 against time for the first initial condition $u_1(0) = u_2(0) = u_3(0) = 0$, and $\dot{u_1}(0) = 0, \dot{u_2}(0) = 1, \dot{u_3}(0) = -1$.

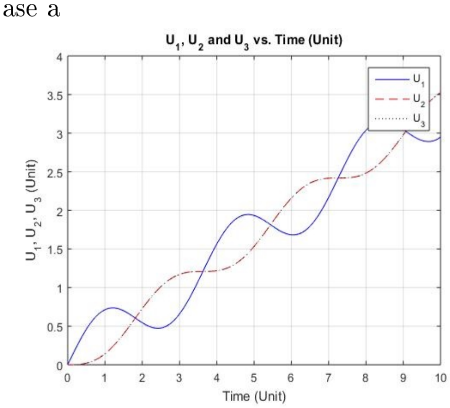
It is clear from Fig.2(a, b, c) that masses 2 and 3 undergo simple harmonic motion with the same amplitude and frequency for each case but they are out phase, while mass 1 remains at rest. The important three features for a simple harmonic motion are the amplitude, period, and frequency, and its clear that for this initial condition for case 1 the amplitude (0.6 units), period ($T = 3.6363$) units, and the frequency ($f = 0.275$) units, while for case 2 the amplitude is nearly (0.22 units), the period ($T = 1.3793$) units, and the frequency ($f = 0.725$) units. Finally, for case 3 the amplitude is nearly (1.00 units), the period ($T = 6.25$) units, and the frequency ($f = 0.16$) units. Changing spring constant k between the two masses 2 and 3 in the base of the triangle directly affects the vibration frequency of these masses which clearly increases by increasing k_{23}.



(a) Case a



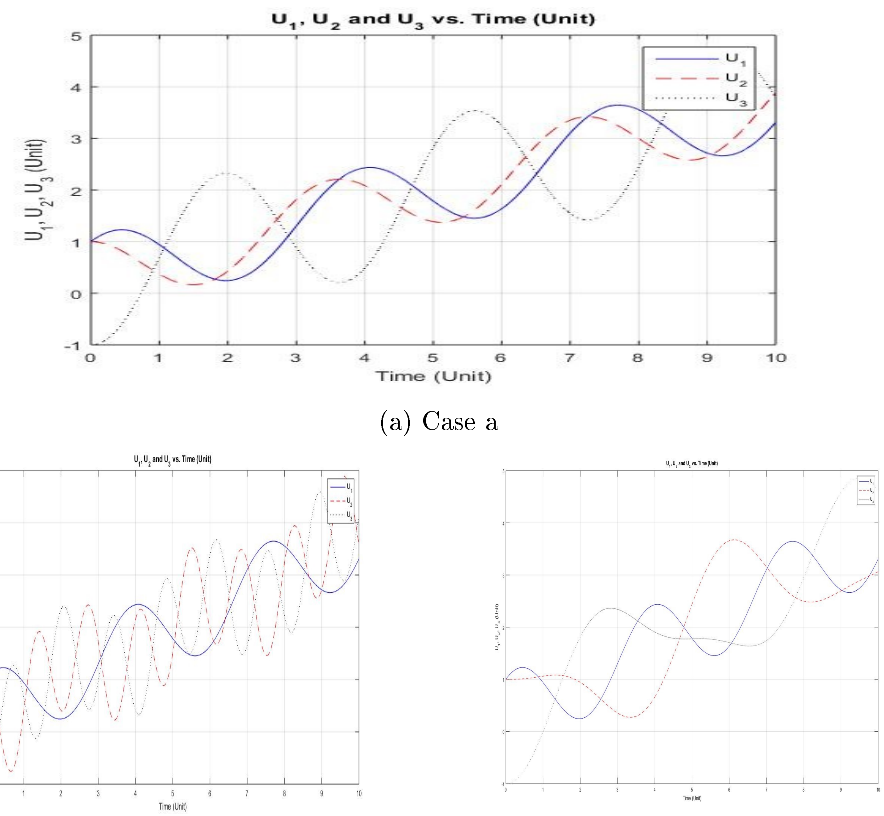
(b) Case b



(c) Case c

Figure 3: The dynamical behavior of the coordinates u_1, u_2 and u_3 against time for the second initial condition $u_1(0) = u_2(0) = u_3(0) = 0$, and $\dot{u_1}(0) = 1, \dot{u_2}(0) = 0, \dot{u_3}(0) = 0$.

In Fig.3 above, no displacement is given to the three masses and they start from equilibrium position but a push is given to the first mass. One can easily notice that the two masses 2, and 3 are undergoing a harmonic motion in phase while mass 1 is undergoing a harmonic motion out of phase with masses 2, and 3. It is important here to mention that the harmonic motion of the three masses is not completely the known simple harmonic motion, whereas here after the system starts the vibrating under the second initial condition each of the three masses start the harmonic motion every time in a new equilibrium position. In other words the masses starts from the original equilibrium position reaching a maximum displacement and instead of returning to the same original equilibrium position they returned to a new equilibrium position and underwent again a harmonic motion and so on every time the equilibrium position is shifted to a new one which means that we have a mixed motion composed of harmonic plus translational one.



(a) Case a

(b) Case b

(c) Case c

Figure 4: The dynamical behavior of the coordinates u_1, u_2 and u_3 against time for the second initial condition $u_1(0) = u_2(0) = -u_3(0) = 1$, and $\dot{u_1}(0) = 1, \dot{u_2}(0) = 0, \dot{u_3}(0) = 0$.

In Fig.4 and according to the third initial condition all masses don't start from their equilibrium positions and are displaced by the same amount but m_1 and m_2 are displaced in the same direction while m_3 is displaced in the opposite direction and again the equilibrium position is shifted to a new one each new oscillation as in Fig.3. In case 1 we can notice that the vibration of m_3 is out of phase with m_1 while m_2 was not in step nor out of step with m_1, it has a small phase shift. In case 2 and 3 both m_2 and m_3 are nearly out of phase in motion. The oscillation frequency of m_1 is the same in the three cases while the frequency of m_2 and m_3 are affected and directly proportional to the spring constant k_{23}.

5 Conclusion

In this paper several user-defined and predefined functions in addition to different structures have been applied successfully to find a truthful numerical solution for the vibrating triangle system. The position of each mass is depicted for the time interval [0, 10]. We examine the motion for three different cases and in each case three different initial conditions were taken into consideration.

We noticed from the figures that mainly two main featured motions are obtained depending on the chosen initial conditions. In Fig.2 a totally simple harmonic motion is obtained, while in Fig.3 and Fig.4 the simple harmonic motion is shifted, and for each case the oscillations of the system start at a new equilibrium position.

Furthermore, we believe that this method is effective for predicting the numerical solutions in many branches of science and engineering problems.

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