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INITIAL VALUE PROBLEMS FOR CAPUTO-FABRIZIO IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS IN b-METRIC SPACES

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Abstract

This article deals with some existence results for some classes of Caputo–Fabrizio implicit fractional differential equations in b-metric spaces with initial conditions. The results are based on some fixed point theorems. We illustrate our results by some examples in the last section.

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1 Introduction

Fractional calculus has sparked the interest from researchers ever since its beginning. Fractional differential equations arise from a variety of applications, in various areas such as, applied sciences, physics, chemistry, biology, etc. [1, 5, 6, 18, 21, 24, 25].

In 2015, Caputo an Fabrizio published a new paper [12] proposing a new fractional derivative with a non-singular kernel. Next, another one by Losada and Nieto [19] discussing some properties of the so-called Caputo-Fabrizio fractional derivative. Fractional differential equations involving this new derivative have been developed and studied by many authors; see [2, 3, 9, 10, 11, 17, 22], and the references therein.

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The notion of b-metric was proposed by Czerwik [14, 15]. Following these initial papers, the existence fixed point for the various classes of operators in the setting of b-metric spaces have been investigated extensively; see [13, 16, 20, 23], and related references therein.

Implicit fractional differential equations were studied in several papers. We mention the monograph [1] and the papers [4, 22]. In this paper, we investigate the existence and uniqueness of solutions for the following class of initial value problems of Caputo–Fabrizio fractional differential equations

$$\begin{cases} ({}^{CF}D_0^r u)(t) = f(t, u(t), ({}^{CF}D_0^r u)(t)); \ t \in I := [0, T], \\ u(0) = u_0, \end{cases} \tag{1}$$

where $T > 0$, $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0, 1)$, and $u_0 \in \mathbb{R}$.

In the last section, we present some examples illustrating the presented results.

2 Preliminaries

Let $C(I)$ be the Banach space of all real continuous functions on I with the norm

$$\|u\|_\infty = \sup_{t \in I} |u(t)|.$$

By $L^1(I)$ we denote the Banach space of measurable functions $u : I \to \mathbb{R}$ with are Lebesgue integrable, equipped with the norm

$$\|u\|_{L^1} = \int_0^T |u(t)| dt.$$

Definition 1. *[12, 19] The Caputo-Fabrizio fractional integral of order $0 < r < 1$ for a function $h \in L^1(I)$ is defined by*

$${}^{CF}I^r h(\tau) = \frac{2(1-r)}{M(r)(2-r)} h(\tau) + \frac{2r}{M(r)(2-r)} \int_0^\tau h(x) dx, \quad \tau \geq 0,$$

where $M(r)$ is normalization constant depending on r.

Definition 2. *[12, 19] The Caputo-Fabrizio fractional derivative for a function $h \in C^1(I)$ of order $0 < r < 1$, is defined by*

$${}^{CF}D^r h(\tau) = \frac{(2-r)M(r)}{2(1-r)} \int_0^\tau \exp\left(-\frac{r}{1-r}(\tau - x)\right) h'(x) dx; \ \tau \in I.$$

Note that $({}^{CF}D^r)(h) = 0$ if and only if h is a constant function.

Lemma 1. *Let $h \in L^1(I,\mathbb{R})$. A function $u \in C(I)$ is a solution of problem*

$$\begin{cases} ({}^{CF}D_0^r u)(t) = h(t); \quad t \in I := [0,T] \\ u(0) = u_0, \end{cases} \tag{2}$$

if and only if u satisfies the following integral equation

$$u(t) = C + a_r h(t) + b_r \int_0^t h(s)ds. \tag{3}$$

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)},$$
$$C = u_0 - a_r h(0).$$

proof. Suppose that u satisfies (2). From Proposition 1 in [19]; the equation

$$({}^{CF}D_0^r u)(t) = h(t),$$

implies that

$$u(t) - u(0) = a_r(h(t) - h(0)) + b_r \int_0^t h(s)ds.$$

Thus from the initial condition $u(0) = u_0$, we get

$$u(t) = u(0) + a_r h(t) - a_r h(0) + b_r \int_0^t h(s)ds.$$

So; we get (3).
Conversely, if u satisfies (3), then $({}^{CF}D_0^r u)(t) = h(t)$; for $t \in I$,
and $u(0) = u_0$.

We can conclude the following lemma:

Lemma 2. *A function u is a solution of problem (1), if and only if u satisfies the following integral equation*

$$u(t) = c + a_r g(t) + b_r \int_0^t g(s)ds,$$

where $g \in X$, with $g(t) = f(t, u(t), g(t))$ and

$$c = u_0 - a_r g(0).$$

Definition 3. *[7, 8] Let $c \geq 1$ and M be a set. A distance function $d : M \times M \to \mathbb{R}_+^*$ is called b-metric if for all $\mu, \nu, \xi \in M$, the following are fulfilled:*

- *(bM1) $d(\mu,\nu) = 0$ if and only if $\mu = \nu$;*
- *(bM2) $d(\mu,\nu) = d(\nu,\mu)$;*
- *(bM3) $d(\mu,\xi) \leq c[d(\mu,\nu) + d(\nu,\xi)]$.*

The tripled (M, d, c) is called a b-metric space.

Example 1. *[7, 8] Let $d : C(I) \times C(I) \to \mathbb{R}_+^*$ be defined by*

$$d(u, v) = \|(u - v)^2\|_\infty := \sup_{t\in I} |u(t) - v(t)|^2;\ for\ all\ u, v \in C(I).$$

It is clear that d is a b-metric with $c = 2$.

Example 2. *[7, 8] Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}_+^*$ be defined by*

$$d(x, y) = |x^2 - y^2|;\ for\ all\ x, y \in X.$$

It is clear that d is not a metric, but it is easy to see that d is a b-metric space with $r \geq 2$.

Let Φ be the set of all increasing and continuous function $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ satisfying the property: $\phi(c\mu) \leq c\phi(\mu) \leq c\mu$, for $c > 1$ and $\phi(0) = 0$. We denote by $\mathcal{F}$ the family of all nondecreasing functions $\lambda : \mathbb{R}_+^* \to [0, \frac{1}{c^2})$ for some $c \geq 1$.

Definition 4. *[7, 8] For a b-metric space (M, d, c), an operator $T : M \to M$ is called a generalized $\alpha - \phi-$Geraghty contraction type mapping whenever there exists $\alpha : M \times M \to \mathbb{R}_+^*$, and some $L \geq 0$ such that for*

$$D(x, y) = \max\left\{d(x, y), d(x, T(x)), d(y, T(y)), \frac{d(x, T(y)) + d(y, T(x))}{2s}\right\},$$

and

$$N(x, y) = \min\{d(x, y), d(x, T(x)), d(y, T(y))\},$$

we have

$$\alpha(\mu, \nu)\phi(c^3 d(T(\mu), T(\nu)) \leq \lambda(\phi(D(\mu, \nu))\phi(D(\mu, \nu)) + L\psi(N(\mu, \nu); \tag{4}$$

for all $\mu, \nu \in M$, where $\lambda \in \mathcal{F}$, $\phi\,\psi \in \Phi$.

Remark 1. *In the case when $L = 0$ in Definition 4, and the fact that*

$$d(x, y) \leq D(x, y);$$

for all $x, y \in M$, the inequality (4) becomes

$$\alpha(\mu, \nu)\phi(c^3 d(T(\mu), T(\nu)) \leq \lambda(\phi(d(\mu, \nu))\phi(d(\mu, \nu)). \tag{5}$$

Definition 5. *[7, 8] Let M be a non empty set, $T : M \to M$, and $\alpha : M \times M \to \mathbb{R}_+^*$ be a given mappings. We say that T is $\alpha-$admissible if for all $\mu, \nu \in M$, we have*

$$\alpha(\mu, \nu) \geq 1 \Rightarrow \alpha(T(\mu), T(\nu)) \geq 1.$$

Definition 6. *[7, 8] Let (M, d) be a b-metric space and let $\alpha : M \times M \to \mathbb{R}_+^*$ be a function. M is said to be $\alpha-$regular if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_n$ with $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

The following fixed point theorem plays a key role in the proof of our main results.

Theorem 1. *[7, 8] Let (M, d) be a complete b-metric space and $T : M \to M$ be a generalized $\alpha - \phi -$Geraghty contraction type mapping such that*

- *(i) T is $\alpha-$admissible;*
- *(ii) there exists $\mu_0 \in M$ such that $\alpha(\mu_0, T(\mu_0)) \geq 1$;*
- *(iii) either T is continuous or M is $\alpha-$regular.*

Then T has a fixed point. Moreover, if

- *(iv) for all fixed points μ, ν of T, either $\alpha(\mu, \nu) \geq 1$ or $\alpha(\nu, \mu) \geq 1$,*

then T has a unique fixed point.

3 Main Results

Let $(C(I), d, 2)$ be the complete b-metric space with $c = 2$, such that $d : C(I) \times C(I) \to \mathbb{R}_+^*$ is given by:

$$d(u, v) = \|(u - v)^2\|_\infty := \sup_{t \in I} |u(t) - v(t)|^2.$$

Then $(C(I), d, 2)$ is a b-metric space.

In this section, we are concerned with the existence results of problem (1).

Definition 7. *By a solution of problem (1) we mean a function $u \in C(I)$ that satisfies*

$$u(t) = c + a_r g(t) + b_r \int_0^t g(s) ds, \tag{6}$$

where $g \in C(I)$, with $g(t) = f(t, u(t), g(t))$ and

$$c = u_0 - a_r g(0).$$

The following hypotheses will be used in the sequel.

(H_1) There exist $p : C(I) \times C(I) \to (0, \infty)$ and $q : I \to (0, 1)$ such that for each $u, v, u_1, v_1 \in C(I)$ and $t \in I$

$$|f(t, u, v) - f(t, u_1, v_1)| \leq p(u, v)|u - u_1| + q(t)|v - v_1|,$$

with

$$\left\| 1 + 2a_r \frac{p(u, v)}{1 - q*} + b_r \int_0^t \frac{p(u, v)}{1 - q*} ds \right\|_\infty^2 \leq \phi(\|(u - v)^2\|_\infty).$$

(H_2) There exist $\phi \in \Phi$ and $\mu_0 \in C(I)$ and a function $\theta : C(I) \times C(I) \to \mathbb{R}$, such that

$$\theta\left(\mu_0(t), c + a_r g(t) + b_r \int_0^t g(s)ds\right) \geq 0,$$

where $g \in C(I)$, with $g(t) = f(t, \mu_0(t), g(t))$,

(H_3) For each $t \in I$, and $u, v \in C(I)$, we have:

$$\theta(u(t), v(t)) \geq 0$$

implies

$$\theta\left(c + a_r g(t) + b_r \int_0^t g(s)ds, c + a_r h(t) + b_r \int_0^t h(s)ds\right) \geq 0,$$

where $g, h \in C(I)$, with

$$g(t) = f(t, u(t), g(t)) \text{ and } h(t) = f(t, v(t), h(t)).$$

(H_4) If $u_{nn\in N} \subset C(I)$ with $u_n \to u$ and $\theta(u_n, u_{n+1}) \geq 1$, then

$$\theta(u_n, u) \geq 1,$$

(H_5) For all fixed solutions x, y of problem (1), either

$$\theta(x(t), y(t)) \geq 0,$$

or

$$\theta(y(t), x(t)) \geq 0.$$

Theorem 2. *Assume that the hypotheses* $(H_1) - (H_4)$ *hold. Then the problem (1) has at least one solution defined on* I*. Moreover, if* (H_5) *holds, then we get a unique solution.*

Proof. Consider the operator $N : C(I) \to C(I)$ such that,

$$(Nu)(t) = c + a_r g(t) + b_r \int_0^t g(s)ds,$$

where $g \in C(I)$, with $g(t) = f(t, u(t), g(t))$ and

$$c = u_0 - a_r g(0).$$

Using Lemma 2, it is clear that the fixed points of the operator N are solutions of our problem (1).

Let $\alpha : C(I) \times C(I) \to]0, \infty)$ be the function defined by:

$$\begin{cases} \alpha(u,v) = 1; & if \ \theta(u(t), v(t)) \geq 0, \ t \in I, \\ \alpha(u,v) = 0; & eles. \end{cases}$$

First, we prove that N is a generalized α-ϕ-Geraghty operator:
For any $u, v \in C(I)$ and each $t \in I$, we have

$$|(Nu)(t) - (Nv)(t)| \leq |c_g - c_h| + a_r|g(t) - h(t)| + b_r \int_0^t |g(s) - h(s)|ds$$

where $g, h \in C(I)$, with $g(t) = f(t, u(t), g(t))$ and $h(t) = f(t, v(t), h(t))$.
From (H_1) we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, v(t), h(t))| \\ &\leq p(u,v)|u(t) - v(t)| + q(t)|g(t) - h(t)| \\ &\leq p(u,v)(|u(t) - v(t)|^2)^{\frac{1}{2}} + q(t)|g(t) - h(t)|. \end{aligned}$$

Thus,

$$\|g - h\|_\infty \leq \frac{p(u,v)}{1 - q*} \|(u - v)^2\|_\infty^{\frac{1}{2}},$$

where $q* = \sup_{t \in I} |q(t)|$.
Next, we have

$$\begin{aligned} |(Nu)(t) - (Nv)(t)| &\leq \|(u - v)^2\|_\infty^{\frac{1}{2}} + 2a_r \tfrac{p(u,v)}{1-q*} \|(u - v)^2\|_\infty^{\frac{1}{2}} \\ &+ b_r \int_0^t \tfrac{p(u,v)}{1-q*} \|(u - v)^2\|_\infty^{\frac{1}{2}} ds. \end{aligned}$$

Thus

$$\begin{aligned} \alpha(u,v)|(Nu)(t) - (Nv)(t)|^2 &\leq \|(u - v)^2\|_\infty \alpha(u,v) \\ &\quad \left\| 1 + 2a_r \tfrac{p(u,v)}{1-q*} + b_r \int_0^t \tfrac{p(u,v)}{1-q*} ds \right\|_\infty^2 . \\ &\leq \|(u - v)^2\|_\infty \phi(\|(u - v)^2\|_\infty). \end{aligned}$$

Hence

$$\alpha(u,v)\phi(2^3 d(N(u), N(v)) \leq \lambda(\phi(d(u,v))\phi(d(u,v)),$$

where $\lambda \in \digamma$, $\phi \in \Phi$, with $\lambda(t) = \frac{1}{8}t$, and $\phi(t) = t$.
So, N is generalized α-ϕ-Geraghty operator.

Let $u, v \in C(I)$ such that

$$\alpha(u,v) \geq 1.$$

Thus, for each $t \in I$, we have

$$\theta(u(t), v(t)) \geq 0.$$

This implies from (H_3) that

$$\theta(Nu(t), Nv(t)) \geq 0,$$

which gives

$$\alpha(N(u), N(v)) \geq 1.$$

Hence, N is a α-admissible.

Now, from (H_2), there exists $\mu_0 \in C(I)$ such that

$$\alpha(\mu_0, N(\mu_0)) \geq 1.$$

Finally, From (H_4), If $\mu_{n n\in N} \subset M$ with $\mu_n \to \mu$ and $\alpha(\mu_n, \mu_{n+1}) \geq 1$, then

$$\alpha(\mu_n, \mu) \geq 1.$$

From an application of Theorem 1, we deduce that N has a fixed point u which is a solution of problem (1).

Moreover, (H_5), implies that if x and y are fixed points of N, then either $\theta(x, y) \geq 0$ or $\theta(y, x) \geq 0$. This implies that either $\alpha(x, y) \geq 0$ or $\alpha(y, x) \geq 0$. Hence; problem (1) has the uniqueness.

4 An Example

Consider the Caputo-Fabrizio fractional differential problem

$$\begin{cases} ({}^{CF}D_0^r u)(t) = f(t, u(t), ({}^{CF}D_0^r u)(t)); \ t \in [0,1] \\ u(0) = 0, \end{cases} \tag{7}$$

where

$$f(t, u, v) = \frac{1 + \sin(|u|)}{4(1 + |u|)} + \frac{1}{4(1 + |v|)}; \ t \in [0,1].$$

Let $(C([0,1]), d, 2)$ be the complete b-metric space, such that $d : C([0,1]) \times C([0,1]) \to \mathbb{R}_+^*$ is given by:

$$d(u, v) = \|(u - v)^2\|_\infty := \sup_{t\in[0,1]} |u(t) - v(t)|^2.$$

For each $u,v\in C([0,1])$, we have Let $t\in(0,1]$, and $u,v,\bar{u},\bar{v}\in C([0,1])$. If $|u(t)|\le|v(t)|$, then

$$\begin{aligned}
|f(t,u(t),\bar{u}(t))-f(t,v(t),\bar{v}(t))| &\le \left|\frac{1+\sin(|u(t)|)}{4(1+|u(t)|+|\bar{u}(t)|)}-\frac{1+\sin(|v(t)|)}{4(1+|v(t)|+|\bar{v}(t)|)}\right|\\
&+ \frac{|\bar{u}(t)-\bar{v}(t)|}{4}\\
&\le \frac{1}{4}||u(t)|-|v(t)||+\frac{1}{4}|\sin(|u(t)|)-\sin(|v(t)|)|\\
&+ ||u(t)|\sin(|v(t)|)-|v(t)|\sin(|u(t)|)|\\
&+ \frac{|\bar{u}(t)-\bar{v}(t)|}{4}\\
&\le |u(t)-v(t)|+\frac{1}{4}|\sin(|u(t)|)-\sin(|v(t)|)|\\
&+ \frac{|\bar{u}(t)-\bar{v}(t)|}{4}\\
&+ ||v(t)|\sin(|v(t)|)-|v(t)|\sin(|u(t)|)|\\
&= |u(t)-v(t)|\\
&+ (1+|v(t)|)\,|\sin(|u(t)|)-\sin(|v(t)|)|\\
&+ \frac{|\bar{u}(t)-\bar{v}(t)|}{4}\\
&\le |u(t)-v(t)|+\frac{1}{2}(1+|v(t)|)\\
&\times \left|\sin\left(\frac{||u(t)|-|v(t)||}{2}\right)\right|\left|\cos\left(\frac{|u(t)|+|v(t)|}{2}\right)\right|\\
&\le (2+\|v\|_\infty)\|u-v\|_\infty+\frac{\|\bar{u}-\bar{v}\|_\infty}{4}.
\end{aligned}$$

The case when $|v(t)|\le|u(t)|$, we get

$$|f(t,u(t),\bar{u}(t))-f(t,v(t),\bar{v}(t))|\le(2+\|u\|_\infty)\|u-v\|_\infty+\frac{\|\bar{u}-\bar{v}\|_\infty}{4}.$$

Hence

$$|f(t,u(t),\bar{u}(t))-f(t,v(t),\bar{v}(t))|\le\min\{2+\|u\|_\infty,2+\|v\|_\infty\}\|u-v\|_\infty+\frac{\|\bar{u}-\bar{v}\|_\infty}{4}.$$

Thus, hypothesis (H_2) is satisfied with

$$p(u,v)=\min\{2+\|u\|_\infty,2+\|v\|_\infty\},\ \text{and}\ q(t)=\frac{1}{4}.$$

Define the functions $\lambda(t)=\frac{1}{8}t$, $\phi(t)=t$, $\alpha:C([0,1])\times C([0,1])\to\mathbb{R}_+^*$ with

$$\begin{cases}\alpha(u,v)=1;\ if\ \delta(u(t),v(t))\ge 0,\ t\in I,\\ \alpha(u,v)=0;\ else,\end{cases}$$

and $\delta : C([0,1]) \times C([0,1]) \to \mathbb{R}$ with $\delta(u,v) = \|u - v\|_\infty$.
Hypothesis (H_2) is satisfied with $\mu_0(t) = u_0$. Also, (H_3) holds from the definition of the function δ.

Simple computations show that all conditions of Theorem 2 are satisfied. Hence, we get the existence of solutions and the uniqueness for problem (7).

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