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ABOUT SOME SPEEDS OF CONVERGENCE TO THE CONSTANT OF EULER

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Abstract

The speed of convergence of the classical sequence which defines the constant of Euler (or Euler-Mascheroni), $\gamma = \lim_{n\to\infty} \gamma_n = 0,577215\ldots$, where $\gamma_n = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln n$, was intensively studied. In 1983 I established in [14] one of the first two sided estimates of this speed, namely $\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}$. Further several new sequences with a faster convergence are defined either by modifying the argument of the logarithm (De Temple, 1993, Negoi 1997, Ivan 2002) or by modifying the last term 1/n of the harmonic sum (Vernescu 1999). Now we give a systematic study of these speeds of convergence and especially of the last ones.

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1 Introduction

The classical convergent and decreasing sequence which defines, by its limit the constant of Euler γ, has the general term:

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$$

(this being denoted sometime by c_n). We also denote here, as usually, by H_n the n-th harmonic number, $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

The speed of convergence of $(\gamma_n)_n$ to its limit γ is described by the two-sided estimate:

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n} \qquad (1)$$

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that I obtained in 1983 (see [14]). Later, 1991 (see see [17]), R. M. Young found it again in a weaker form

$$\frac{1}{2n+2} < \gamma_n - \gamma < \frac{1}{2n}. \tag{1'}$$

The speed of convergence of the sequence $(\gamma_n)_n$ to γ is of order of $1/n$, denoted by $O(1/n)$, that is a slow speed. It is remembering and can be compared with the speed of the convergence of $\left(1+\frac{1}{n}\right)^n \underset{n\to\infty}{\to} \mathrm{e}$, that is described by the two-sided estimate

$$\frac{\mathrm{e}}{2n+2} < \mathrm{e} - \left(1+\frac{1}{n}\right)^n < \frac{\mathrm{e}}{2n+1}$$

(see Pólya and Szegö [11] and, for the shortest proof, see [15]). Therefore the numerical computation of e is made by using the alternative formula, $e = \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{1}{k!}$, with the speed of convergence of $O(1/(n!n))$. (Also see [13], [9] and [10].)

For this reason, other faster convergences to γ were searched. First the argument of the logarithm was changed, and then the harmonic number H_n.

So, D. W. De Temple showed in 1993 ([1]), that, if $R_n = H_n - \ln(n+1/2)$, then $(R_n)_n$ tends decreasing to γ with the speed of $O(1/n^2)$ and established the inequality

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}. \tag{2}$$

In 1997 ([8]) T. Negoi has proved that, if $T_n = H_n - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right)$, then $(T_n)_n$ tends increasing to γ with the speed of $O(1/n^3)$ and

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}. \tag{3}$$

I have defined in 1999 ([16]) a new faster convergence to γ by replacing in γ_n not the argument of the logarithm, but the last term of H_n, $1/n$ by $1/(2n)$. If:

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{2n} - \ln n,$$

then $(x_n)_n$ tends increasing to γ and we have

$$\frac{1}{12(n+1)^2} < \gamma - x_n < \frac{1}{12n^2}. \tag{4}$$

(This two-sided estimate (4) gave as a trivial consequence a refinement of (1), namely

$$\frac{1}{2n} - \frac{1}{12n^2} < \gamma_n - \gamma < \frac{1}{2n} - \frac{1}{12(n+1)^2} \tag{4'}$$

also see [7]).

Of course, all the inequalities (1), (2), (3), (4) give immediately the attached first iterated limits of the respective sequences

$$\lim_{n\to\infty} n(\gamma_n - \gamma) = \frac{1}{2},$$

$$\lim_{n\to\infty} n^2(R_n - \gamma) = \frac{1}{24},$$
$$\lim_{n\to\infty} n^3(\gamma - T_n) = \frac{1}{48},$$
$$\lim_{n\to\infty} n^2(\gamma - x_n) = \frac{1}{12}.$$

2 Main results

In his universitary textbook [3], on page 215, Professor Mircea Ivan considers the sequence $(x_n)_n$ of general term:

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln\sqrt{n(n+1)}$$

and proposes as an exercise to show that:

$$\lim_{n\to\infty} n^2(x_n - \gamma) = \frac{1}{6}. \tag{5}$$

(Let's note first that $x_n = (H_n - \ln n) + \left(\ln n - \ln\sqrt{n(n+1)}\right) = \gamma_n + \ln\sqrt{\frac{n+1}{n}} \underset{n\to\infty}{\to} \gamma$.)

The solution can be done, e.g., by using the lemma of Cesàro for the case $\frac{0}{0}$ and then moving on to the continuous real variable and using the differential calculus tools.)

Now, considering not only the previous limits, but also the two-sided estimates (1), (2), (3) and (4), we can pose the problem of finding a two-sided estimation for the sequence $(x_n)_n$ of Ivan.

Theorem 1. *For the sequence $(x_n)_n$, we have the two sided estimate*

$$\frac{1}{6(n+1)^2} < H_n - \ln\sqrt{n(n+1)} - \gamma < \frac{1}{6n^2}.$$

Proof. An elementary but somewhat laborious proof consist [as in the proofs of inequalities (1) – (4)] to decompose the double inequality into two inequalities, to isolate the constant γ, and use a monotonicity argument.

But using some tools „forte“, we obtain the result faster. Take into account the formulas

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \varepsilon, \text{ where } 0 < \varepsilon < \frac{1}{256n^6}$$

and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + o(x^7), \text{ where } |x| < 1.$$

Performing the calculation, we obtain

$$x_n = \gamma + \frac{1}{6n^2} - \frac{1}{6n^3} + \frac{2}{15n^4} - \frac{1}{10n^5} + o\left(\frac{1}{n^5}\right).$$

This shows that $x_n > \gamma$. Also, we obtain $x_n - \gamma < \frac{1}{6n^2}$ and, again after a little calculation, $x_n - \gamma > \frac{1}{6(n+1)^2}$. Q.E.D. □

3 A sequence which converges to γ constructed by using the harmonic mean

The argument of the logarithm in the sequence of De Temple is the arithmetic mean of the numbers n and $n+1$ and the argument of the logarithm in the sequence of Ivan is the geometric mean of the same numbers. Let's note simply by $a(n)$, $g(n)$ and $h(n)$ the arithmetic, geometric, respectively harmonic means of the numbers n and $n+1$, i. e. $a(n) = n + 1/2$, $g(n) = \sqrt{n(n+1)}$, $h(n) = 2n(n+1)/(2n+1)$.

We can consider now the sequence of general term

$$H_n - \ln h(n) = H_n - \ln \frac{2n(n+1)}{2n+1}.$$

Using similar tools as before, we obtain in a similar manner, that the sequence also converges to γ and we can prove the following

Theorem 2. *We have the limit and the double estimate below*

$$\lim_{n\to\infty} n^2(H_n - h(n) - \gamma) = \frac{1}{3}$$

and

$$\frac{1}{3(n+1)^2} < H_n - \ln \frac{2n(n+1)}{2n+1} - \gamma < \frac{1}{3n^2}.$$

4 A special discrete scale of convergences to γ and a concluding remark

The above results allows us to establish a a scale of increasingly refined convergences: $(\gamma_n)_n$ with the speed of convergence of $O(1/n)$, $(R_n)_n$and $(x_n)_n$ with the speed of $O(1/n^2)$and finally $(T_n)_n$ with the speed of $O(1/n^3)$.

Consider now the sequence of general term $H_n - \ln(n+1)$, the adjacent sequence of the classical Euler's first convergence $(\gamma_n)_n$,which converges increasing to γ and for which the two sided estimate holds

$$\frac{1}{2n+1} < \gamma - (H_n - \ln(n+1)) < \frac{1}{2n}$$

(see [2]). This gives immediately us the first iterated limit:

$$\lim_{n\to\infty} n\left(\gamma - (H_n - \ln(n+1))\right) = \frac{1}{2}.$$

and so, this convergence is also of speed of $O(1/n)$ as $(\gamma_n)_n$.

Let's consider the above result, but for a moment, without the sequence $(T_n)_n$. We can construct a special discrete finite scale of some sequences which tend to γ. Let the family of functions $\{\varphi : \mathbb{N}^* \to [n, n+1] \mid n \in \mathbb{N}\}$, be and consider for each of these functions the sequence defined by the formula $x_n(\varphi(n)) = H_n - \ln \varphi(n)$. It seems that when $\varphi(n)$ increases from n to $n + 1/2$, the speed of convergence becomes more and more refined and when $\varphi(n)$ decreases from $n+1$ to $n+1/2$, the speed of convergence also becomes more and more refined. But this finding is not entirely correct, because the best convergence speed is achieved not when $\varphi(n) = n+\frac{1}{2}$, but when $\varphi(n) = n+\frac{1}{2}+\frac{1}{24n}$! This can lead to an ,,open problem", namely to find the next term of the finite sequence of the functions φ, arguments of the logarithms, n, $n+\frac{1}{2}$, $n+\frac{1}{2}+\frac{1}{24n}$ which may be of the form $n+\frac{1}{2}+\frac{1}{24n}+\frac{\alpha}{n^2}$, where α is a real constant, so that convergence is the fastest, e.g. of order at least of $1/n^4$.

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