Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 2(64), No. 2 - 2022, 167-172
https://doi.org/10.31926/but.mif.2022.2.64.2.12

INVO-k-CLEAN RINGS

Fatemeh RASHEDI[*,1]

Abstract

In this paper, we offer a new generalization of the invo-clean ring that is called invo k-clean ring. Let $2 \leq k \in \mathbb{N}$. Then a ring R is called invo k-clean if for each $a \in R$ there exist $v \in Inv(R)$ and $e \in P_k(R)$ such that $a = u + e$. We obtain some properties of invo k-clean rings.

2000 *Mathematics Subject Classification:* 16U60, 16U99.
Key words: invo-clean ring, invo k-clean ring.

1 Introduction

Let R be an associative ring with identity. As usual, $U(R)$ denotes the set of all units of R, $Inv(R)$ the subset of $U(R)$ consisting of all involutions of R, $Id(R)$ the set of all idempotents of R and $Nil(R)$ the set of all nilpotents of R. Traditionally, $J(R)$ stands for the Jacobson radical of R. Let $2 \leq k \in \mathbb{N}$. Then an element $e \in R$ is said to be k-potent if $e^k = e$. Assume that $P_k(R)$ is the set of k-potent elements of ring R. A ring R is said to be clean if for each $a \in R$ there exist $u \in U(R)$ and $e \in Id(R)$ such that $a = u + e$ [1, 7]. A ring R is said to be invo-clean if for each $a \in R$ there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $a = v + e$. If, in addition, $ve = ev$, R is said to be strongly invo-clean [2, 3, 4, 5]. In [3, Theorem 2.2] it is shown that, if R is an invo-clean ring and e is an idempotent, then the corner subring eRe is also invo-clean. In particular, if for any $n \in N$ the full matrix $n \times n$ ring $M_n(R)$ is invo-clean, then so is R. In [4, Corollary 2.16] it is shown that, if R is an invo-clean ring, then $J(R)$ is nil with index of nilpotence not exceeding 3. In [5, Corollary 3.2] it is proved that, a ring R of characteristic 2 is strongly invo-clean if and only if R is strongly nil-clean with index of nilpotence at most 2. In this paper, we introduce the notion of a invo k-clean ring as a new generalization of a invo-clean ring. Let $2 \leq k \in \mathbb{N}$. Then a ring R is said to be a invo k-clean if for each $a \in R$ there exist $v \in Inv(R)$ and $e \in P_k(R)$ such that $a = v + e$. We obtain an element-wise characterization of invo k-clean rings. The proofs of the obtained results in this article imitate those from the articles [2, 5].

[1*] *Corresponding author*, Department of Mathematics, Technical and Vocational University (TVU), Tehran, Iran, e-mail: f-rashedi@tvu.ac.ir

2 Main results

We start in this section with the following notion, which motivates the writing of this short note.

Definition 2.1. *Let R be a ring and $2 \leq k \in \mathbb{N}$. Then an element $a \in R$ is called invo k-clean if there exist $v \in Inv(R)$ and $e \in P_k(R)$ such that $a = v + e$. A ring R is called invo k-clean if every element of R is invo k-clean.*

Every idempotent element of a ring R is an invo k-clean element, because $e = (2e-1) + (1-e)$, $(2e-1)^2 = 1$ and $(1-e)^2 = 1-e$. Hence every Boolean ring is invo k-clean. Moreover, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}_8$ are invo k-clean.
A ring R is said to be invo-clean if every $a \in R$ can be written as $a = v + e$, where $v \in Inv(R)$ and $e \in Id(R)$ [2, 3, 4, 5]. It is clear that every invo-clean is an invo k-clean ring. However, invo k-clean rings are not invo-clean, in general. For example $\mathbb{Z}_5$ is invo k-clean but not invo-clean.
Let $2 \leq k \in \mathbb{N}$. A ring R is said to be k-clean if there exist $u \in U(R)$ and $e \in P_k(R)$ such that $a = u + e$ [8]. It is easy to see that every invo k-clean ring is k-clean.

Lemma 2.2. *Let R be a ring in which every element is k-potent. Then R is invo k-clean.*

Proof. Assume that $a \in R$. Hence $a = 1 + (1-a)$, where $1 \in Inv(R)$ and $1 - a \in P_k(R)$. Therefore R is invo k-clean. □

Lemma 2.3. *Let $2 \leq k \in \mathbb{N}$ and R be a ring in which every element is involution. Then R is invo k-clean.*

Proof. Assume that $a \in R$. Hence $a = 1 + (1-a)$, where $1 \in P_k(R)$ and $1 - a \in Inv(R)$. Therefore R is invo k-clean. □

Recall that an element a of a ring R is said to be tripotent if the equality $a^3 = a$ holds. If each element of R is tripotent, the ring R is said to be tripotent [6].

Proposition 2.4. *Let $2 \leq k \in \mathbb{N}$. Then every tripotent ring is invo k-clean.*

Proof. Suppose that R is a tripotent ring and $a \in R$. So $a^3 = a$. Thus $1 - a^2 \in P_k(R)$ and $a^2 + a - 1 \in Inv(R)$. Since $a = (1-a^2) + (a^2 + a - 1)$, R is invo k-clean. □

The following example shows that the converse of Proposition 2.4 is not true in general.

Example 2.5. *Let $2 \leq k \in \mathbb{N}$ and R be the group ring BG, where $B \ncong \mathbb{Z}_2$ is a Boolean ring and G is a group consisting only of elements of order at most 2. Since $char(R) = 2$, $a^4 = a^2$ for every $a \in R$. Since $1 + a^2 \in P_k(R)$, $a^2 + a + 1 \in Inv(R)$ and $a = (1 + a^2) + (a^2 + a + 1)$, R is an invo k-clean. Since B contains non-trivial idempotents, $a^3 \neq a^2$ and $a^3 \neq a$ for every $a \in R$. Therefore R is not a tripotent ring.*

Lemma 2.6. *Let $2 \leq k \in \mathbb{N}$ and R be an invo k-clean ring. Then each homomorphic image of R is invo k-clean.*

Proof. Assume that $h : R \longrightarrow R'$ be a ring homomorphism and R be an invo k-clean ring. Let $a' \in h(R)$. Then $a' = h(a)$ for some $a \in R$. Since R is invo k-clean, there exist $v \in Inv(R)$ and $e \in P_k(R)$ such that $a = v + e$. Since h is a homomorphism, $h(a) = h(v) + h(e)$ and $h(e) = h(e^k) = (h(e))^k$. Hence $h(e) \in P_k(h(R))$.Then $h(v)^2 = h(v^2) = h(1) = 1$, and so $h(v) \in Inv(h(R))$. So $h(R)$ invo k-clean, as required. □

Proposition 2.7. *Let $2 \leq k \in \mathbb{N}$ and R be an invo k-clean ring. Then $30 \in Nil(R)$.*

Proof. Suppose that $3 = v + e$, where $v \in Inv(R)$ and $e \in P_k(R)$. Hence $v^2 = (3 - e)^2 = 1$, and so $8 = 5e$, whence $24 = 0$ and $3 \cdot 24 = 72 = 0$ by squaring both sides. Hence $30^3 = 27000 = 72 \cdot 375 = 0$. Therefore $30 \in Nil(R)$. □

By using the above proposition, we have the following result.

Corollary 2.8. *Let $2 \leq k \in \mathbb{N}$ and R be an invo k-clean ring. Then the following two equivalencies hold.*

(i) $5 \in U(R) \Longleftrightarrow 6 \in Nil(R)$.

(ii) $6 \in U(R) \Longleftrightarrow 5 \in Nil(R)$

Proof. These relations follow directly from the fact that $1 + Nil(R) \subseteq U(R)$ and that $30 = 5 \cdot 6 \in Nil(R)$ by Proposition 2.7. □

As an interesting consequence, we obtain the following one.

Corollary 2.9. *Let $2 \leq k \in \mathbb{N}$, R be an invo k-clean ring such that $U(R) \cap P_k(R) = \{1\}$. Then $J(R)$ is nil with index of nilpotence at most 3.*

Proof. Suppose that $a \in J(R)$ and $a = v + e$ for some $v \in Inv(R)$ and $e \in P_k(R)$. Hence $a - v = e \in U(R) \cap P_k(R) = \{1\}$, and so $e = 1$. Then $a = v + 1$, $a^2 = 2(v + 1) = 2a$ and $a^3 = 2a^2$. Hence $a^3 = 4a$. Replacing a by $2a$ in the $a^2 = 2a$ and $a^3 = 4a$. Then $4a^2 = 4a$ and $8a^3 = 8a$. Since $8a^3 = 8a$, $8a(1 - a^2) = 0$. Since $1 - a^2 \in 1 + J(R) \subseteq U(R)$, $8a = 0$. Multiplying both sides of $a^2 = 2a$ by 4, $4a^2 = 8a$. Then $4a^2 = 4a = 8a = 0$. Therefore $a^3 = 4a = 0$, and so $J(R)$ is nil with index of nilpotence at most 3. □

Lemma 2.10. *Let $2 \leq k \in \mathbb{N}$ and R be a ring with $u \in U(R)$ and $e \in P_k(R)$ such that $u^2 e = eu^2 = e$ and $u = e + p$, where $p \in Nil(R)$. Then $e = 1$.*

Proof. Suppose $u = e+p$ for some $e \in P_k(R)$ and $p \in Nil(R)$ with $p^m = 0$, $m \in \mathbb{N}$. Hence $u^2 = e + ep + pe + p^2$, and so $u^2 e = e = e + epe + pe + p^2 e$. Then $(p + p^2)e = -epe$. Similarly, $eu^2 = e$ insures that $e(p + p^2) = -epe$. Thus e commutes with the nilpotent $(p + p^2)^n = [p(1 + p)]^n = p^n(1 + p)^n$ for all $n \in \mathbb{N}$. Therefore the same is valid for u.

Furthermore, $u-(p+p^2)=e-p^2$ with $u-(p+p^2)=u_2=e-p^2$ being a unit, one sees that $u_2-(2p^3+p^4)=e-(p^2+2p^3+p^4)=e-(p+p^2)^2$. Putting $u_3=u_2+(p+p^2)^2$, we observe that u_3 is a unit since u_2 commutes with $(p+p^2)^2$ and that $u_3=e+p^3(2+p)$. In the same manner $u_4=u_3-2(p+p^2)^3=e-p^4(5+6p+2p^2)$, $u_5=u_4+5(p+p^2)^4=e+p^5(14+28p+20p^2+5p^3)$ and $u_6=u_5-14(p+p^2)^5=e-p^6b$, where $a=f(p)$ is a function (polynomial) of p. Repeating the same procedure m-times, we will find a unit u_m such that $u_m=e+p^m\cdot b=e$ for some element b depending on p; $b=-1=-p^0$ provided $m=2$. Thus $e=1$. □

By using the above lemma, we have the following result.

Corollary 2.11. *Let $2\leq k\in\mathbb{N}$ and R be a ring with $v\in Inv(R)$ and $e\in P_k(R)$ such that $v=e+q$, where $p\in Nil(R)$. Then $e=1$.*

Proof. It follows from Lemma 2.10. □

Proposition 2.12. *Let $2\leq k\in\mathbb{N}$ and R be an invo k-clean ring with $2\in U(R)$. Then $Nil(R)=\{0\}$.*

Proof. Suppose that $p\in Nil(R)$ and $pv+e$, where $v\in Inv(R)$ and $e\in P_k(R)$. Thus $-v=-p+e$, where $-v\in Inv(R)$ and $-p\in Nil(R)$. By Corollary 2.11, $e=1$. Then $p=v+1$, and so $p^2=2+2v=2(1+v)=2p$. Thus $p(2-p)=0$. Since $2-p\in U(R)$, $p=0$. Therefore $Nil(R)=\{0\}$. □

Adding the condition for a lack of non-trivial k-potents in the ring, we derive the following.

Theorem 2.13. *Let $2\leq k\in\mathbb{N}$ and R be an invo k-clean ring with $P_k(R)=\{0,1\}$ and $2\in U(R)$. Then $R\cong\mathbb{Z}_3$.*

Proof. Suppose that $a\in R$. Then $a=v+1$ or $a=v$, where $v\in Inv(R)$. Since $\frac{1-v}{2}$ is a k-potent element of R, whence $\frac{1-v}{2}=1$ or $\frac{1-v}{2}=0$. Thus $v=-1$ or $v=1$. Hence $R=\{0,-1,1,2\}$. But it must be that $2=-1$, because only $2\cdot(-1)=1$ or $2\cdot 2=1$ is possible. Therefore $3=0$, and so $R=\{0,1,2\}$. Then $R\cong\mathbb{Z}_3$. □

Definition 2.14. *Let R be a ring and $2\leq k\in\mathbb{N}$. Then an element $a\in R$ is called strongly invo k-clean if there exist $v\in Inv(R)$ and $e\in P_k(R)$ of R such that $a=v+e$ and $ve=ev$. A ring R is called strongly invo k-clean if every element of R is strongly invo k-clean.*

Lemma 2.15. *Let R be a strongly invo k-clean ring. Then each homomorphic image of R is strongly invo k-clean.*

Proof. Assume that $h:R\longrightarrow R'$ be a ring homomorphism and R be a strongly invo k-clean ring. By Lemma 2.6 $h(R)$ is invo k-clean. Also $h(v)h(e)=h(ve)=h(ev)=h(e)h(v)$. Hence $h(R)$ is strongly invo k-clean. □

Definition 2.16. *A ring R is said to satisfy k-invo property $k \geq 2$ if $1+v+v^2+\cdots+v^{k-2} \in Inv(R)$ for every $v \in Inv(R)$.*

Theorem 2.17. *Let $2 \leq k \in \mathbb{N}$ be a prime number, R be a strongly invo k-clean ring with k-invo property and $Char(R) = k$. Then k-potents lift modulo every ideal of R.*

Proof. Suppose that $2 \leq k \in \mathbb{N}$ be a prime number, R be a strongly invo k-clean ring with k-invo property and $Char(R) = k$. Assume that I is an ideal of R such that $a - a^k \in I$. Since R is strongly invo k-clean, $a = v + e$, where $v \in Inv(R)$, $e \in P_k(R)$ and $ve = ev$. Since $a - a^k \in I$, $v + e - (v + e)^k \in I$. Since $ve = ev$,

$$v + e - (v^k + kv^{k-1}e + \frac{k(k-1)}{2!}v^{k-2}e^2 + \cdots + kve^{k-1} + e^k) \in I.$$

Since $Char(R) = k$ and $e^k = e$, $v - v^k \in I$. Now, assume that $f = 1 + e$. Since $Char(R) = k$ and $e^k = e$,

$$f^k = (1+e)^k = 1 + ke + \frac{k(k-1)}{2!}e^2 + \cdots + ke^{k-1} + e^k = 1 + e = f.$$

Thus $f \in P_k(R)$. Again, $f - a = 1 + e - (v + e) = 1 - v$. Since I is an ideal, $v \in Inv(R)$ and $v - v^k \in I$, $1 - v^k \in I$. Thus

$$(1-v)(1+v+v^2+\cdots+v^{k-2}) \in I.$$

Since R has k-invo-property, $1 - v \in I$, and so $f - a \in I$. Therefore k-potents lift modulo every ideal of R. □

Lemma 2.18. *Let I be an ideal of a ring R with $I \subseteq J(R)$. Then $v+I \in Inv(R/I)$ if and only if $v \in Inv(R)$.*

Proof. Clearly. □

Definition 2.19. *Let I be an ideal of a ring R. We say that R has modulo commutative property if for any two elements $a+I, b+I \in R/I$, $(a+I)(b+I) = (b+I)(a+I)$ implies that $ab = ba$ in R.*

Theorem 2.20. *Let $2 \leq k \in \mathbb{N}$ be a prime number, R be a ring satisfying k-invo property, modulo commutative property and $Char(R) = k$. If I is an ideal of R such that $I \subseteq J(R)$. Then R is strongly invo k-clean if and only if R/I is strongly invo k-clean and k-potents lift modulo I.*

Proof. Assume that R is a strongly invo k-clean ring satisfying k-invo property, modulo commutative property and $Char(R) = k$. Let I be an ideal of R such that $I \subseteq J(R)$. Then $R.I$ is strongly invo k-clean, by Lemma 2.15. Also by Theorem 2.17, k-potents lift modulo I. Conversely, let R/I be strongly invo k-clean and k-potents lift modulo I. Suppose that $a \in R$. Thus $a + I \in R/I$. Since R/I is strongly invo k-clean, $a + I = (v + I) + (e + I)$, where $v + I \in Inv(R/I)$, $e + I \in P_k(R/I)$ and $(v + I)(e + I) = (e + I)(v + I)$. Since $I \subseteq J(R)$, by Lemm 2.18 $v \in Inv(R)$. Now $e + I \in P_k(R/I)$ which implies that $e - e^m \in I$. Since

k-potents lift modulo I, there exists an k-potent $f \in R$ such that $f - e \in I$, and so $f + I = e + I$. Since $a + I = (v + I) + (f + I)$, $(v + I) = (a - f) + I$. Hence $(a - f) + I \in Inv(R/I)$. Then $a - f \in Inv(R)$, by Lemma 2.18. Therefore $a = (a - f) + f$, where $a - f \in Inv(R)$ and $f \in P_k(R)$. Since $(v + I)(e + I) = (e + I)(v + I)$, $((a - f) + I)(f + I) = (f + I)((a - f) + I)$. Since R satisfies the modulo commutative property, $(a - f)f = f(a - f))$. Therefore R is strongly invo k-clean. □

Here we shall formulate two questions of interest.

Problem 2.21. *When is a matrix ring invo k-clean?*

Problem 2.22. *Let R be a ring and* $e \in P_k(R)$ *such that the subrings* $e^{k-1}Re^{k-1}$ *and* $(1 - e^{k-1})R(1 - e^{k-1})$ *are invo k-clean. Is R also invo k-clean?*

Acknowledgement

The author would like to thank the referee for the careful reading of the manuscript and the constructive suggestion made.

References

[1] Anderson, D.D. and Camillo, V.P., *Commutative rings whose elements are a sum of a unit and an idempotent*, *Comm. Algebra* **30** (2002), 3327-3336.

[2] Danchev, P.V., *Quasi invo-clean rings*, Bull. Transilv. Univ. Bra¸sov Ser. III **63** (2021), 71-80.

[3] Danchev, P.V., *Corners of invo-clean unital rings*, Pure Mathematical Sciences **7** (2018), 27-31.

[4] Danchev, P.V., *Invo-clean unital rings*, Commun. Korean Math. Soc. **32** (2017), 19–27.

[5] Danchev, P.V., *Weakly invo-clean unital rings*, Afr. Mat. **28** (2017), 1285–1295.

[6] Hirano, Y. and Tominaga, H., *Rings in which every element is the sum of two idempotents*, Bull. Austral. Math. Soc. **37** (1988) 161–164.

[7] Nicholson, W.K., *Lifting idempotents and exchange rings*. Trans. Amer. Math. Soc. **229** (1977), 269-278.

[8] Purkait, S., Dutta, T.K. and Kar, S., *On m-clean and strongly m-clean rings*, Comm. Algebra **48** (2019) 218-227.