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A NOTE ON CSI-$\xi^{\perp}$-RIEMANNIAN SUBMERSIONS FROM KENMOTSU MANIFOLDS

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Abstract

The object of this article is to define and study the Clairaut semi-invariant $\xi^{\perp}$ -Riemannian submersions (Csi-$\xi^{\perp}$ -Riemannian submersions, In short) from Kenmotsu manifolds onto Riemannian manifolds. We obtain necessary and sufficient condition for a semi-invariant $\xi^{\perp}$-Riemannian submersion to be Csi-$\xi^{\perp}$-Riemannian submersion. We also work out on some fundamental differential geometric properties of these submersions. Moreover, we present consequent non-trivial example of such submersion.

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1 Introduction

In 1960s, the theory of Riemannian submersions between Riemannian manifolds were independently studied by O' Neill [20] and Grey [13]. Later, Watson [33] introduced the notion of almost Hermition submersions and studied geometric properties among fibers, base manifolds and total manifolds. After that, the notion of almost Hermitian submersions have been actively studied between different kinds of sub-classes (almost Hermitian manifolds and almost contact manifolds). The concept of anti-invariant submersion was first defined by Sahin [25] from almost Hermitian manifolds onto Riemannian manifolds. Later, he introduced semi-invariant submersion [28] from almost Hermitian manifolds onto Riemannian manifolds as a generalization of holomorphic submersions and anti-invariant

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submersion. Further, Different kinds of Riemannian submersions on different structures have been studied, such as: slant submersions [26], semi-slant submersions [21], hemi-slant Riemannian submersions [30], quasi-bi-slant submersions [22] (see also [23],[24], [27]) etc.

We note that Riemannian submersions have applications in physics, mechanics and robotics. Such as: Kaluza-Klein theory [10], Yang-Mills theory [11], Supergravity and superstring theories ([14], [15]) etc. Bedrossian and Spong [7] showed in the existence of a class of robotic chains having Riemannian curvature that is locally vanishing, once potential energy and friction phenomena are ignored. C. Altafini [4] commenced using the notion of Riemannian submersions for the modeling and control of redundant robotic chain and obtained that Riemannian submersion gives a close relationship between inverse kinematic in robotics and the pull back vectors.

On the other hand, in the theory surface of revolution, a well known Clairaut's theorem [8] states that for any geodesic $c(c : I_1 \subset R \to M$ on $M)$ on the revolution surface M the product $rsin\theta$ is constant with along c, where $\theta(s)$ be the angle between $c(s)$ and the meridian curve through $c(s)$, $s \in I_1$. It means, it is independent of s. In 1972, Bishop applied this idea to the Riemannian submersions and introduced the concept of Clairaut submersion as: a submersion $\pi : M \to N$ is said to be a Clairaut submersion if there is a function $r : M \to R^+$ such that for every geodesic, making an angle θ with the horizontal subspaces, $rsin\theta$ is constant [8]. Moreover, he gave a characterization of Clairaut submersion, studied the behaviour of geodesic and further obtain a generalization of Clairaut's theorem. Afterwards, this notion has been studied in Lorentzian spaces, timelike and spacelike spaces [18] (see also [31], [32]). In [3], Allison shown that such submersions have their applications in static spacetimes. In [12], the author used the notion of Clairaut submersion for obtaining decomposition theorems on Riemannian manifolds. Moreover, Clairaut submersions have been further generalized in [5]. Lee et al. [18] investigated new conditions for anti-invariant Riemannian submersions to be Clairaut when the total manifolds are Kahlerian. In 2017, Sahin introduced Clairaut Riemannian map [29] and studied it's geometric properties. Recently, S. Kumar et al. studied the Clairauts semi-invariant Riemannian maps from almost Hermitian manifolds in [16].

In 2013, Lee [17] initiate the notion of anti invariant $\xi^{\perp}-$Riemannian Submersions and investigate interesting geometric properties of these submersions. Akyol, Sari and Aksoy [1] introduce semi-invariant $\xi^{\perp}-$Riemannian Submersions as well as semi-slant $\xi^{\perp}-$Riemannian Submersions [2], as a generalization of anti invariant $\xi^{\perp}-$Riemannian Submersions. The authors obtain the geometry of the total space and the base space for the existence of such submersions.

In the present paper, we are interested in studying the idea of Csi-$\xi^{\perp}$-Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds. The paper is organized as follows: In the second section, we gather main notions and formulae for other sections. In the third section, we give the definition of Csi-$\xi^{\perp}$-Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds. We investigate differential geometric properties of such submersions. In the fourth section we present illustrative example of the Csi-$\xi^{\perp}$-Riemannian submersion from Kenmotsu manifold onto Riemannian manifold.

2 Preliminaries

An $(2m+1)$-dimensional differentiable manifold M_1 which admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a $1-$ form η such that

$$\phi^2 = -I + \eta \otimes \xi, \phi \circ \xi = 0, \eta \circ \xi = 0, \tag{1}$$

$$\eta(\xi) = 1, \tag{2}$$

where I denote the identity tensor. The manifold M_1 with an almost contact structure (ϕ, ξ, η) is called an almost contact manifold.

If there exists a Riemannian metric g_1 on an almost contact manifold M_1 satisfying the following conditions

$$g_1(\phi W_1, \phi W_2) = g_1(W_1, W_2) - \eta(W_1)\eta(W_2), g_1(\phi W_1, W_2) = -g_1(W_1, \phi W_2), \tag{3}$$

$$g_1(W_1, \xi) = \eta(W_1), \tag{4}$$

where W_1, W_2 are the vector fields on M_1, then structure (ϕ, ξ, η, g_1) is called almost contact metric structure and the manifold M_1 is called an almost contact metric manifold. An almost contact manifold M_1 with almost contact metric structure (ϕ, ξ, η, g_1) is denoted by $(M_1, \phi, \xi, \eta, g_1)$. The fundamental 2-form Φ is defined by $\Phi(W_1, W_2) = g_1(W_1, \phi W_2)$.

An almost contact metric manifold M_1 is called a Kenmotsu manifold [9] if

$$(\nabla_{Z_1}\phi)Z_2 = g(\phi Z_1, Z_2)\xi - \eta(Z_2)\phi Z_1 \tag{5}$$

for any vector fields Z_1 and Z_2 on M_1, where ∇ is the Riemannian connection of the Riemannian metric g_1. If $(M_1, \phi, \xi, \eta, g_1)$ be a Kenmotsu manifold, then the following equation holds:

$$\nabla_{Z_1}\xi = Z_1 - \eta(Z_1)\xi. \tag{6}$$

Define O'Neill's tensors [20] $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_{Z_1}Z_2 = \mathcal{H}\nabla_{\mathcal{H}Z_1}\mathcal{V}Z_2 + \mathcal{V}\nabla_{\mathcal{H}Z_1}\mathcal{H}Z_2, \tag{7}$$

$$\mathcal{T}_{Z_1}Z_2 = \mathcal{H}\nabla_{\mathcal{V}Z_1}\mathcal{V}Z_2 + \mathcal{V}\nabla_{\mathcal{V}Z_1}\mathcal{H}Z_2 \tag{8}$$

for any vector fields Z_1, Z_2 on M_1, where ∇ is the Levi-Civita connection of g_1. It is easy to see that $\mathcal{T}_{Z_1}$ and $\mathcal{A}_{Z_1}$ are skew-symmetric operators on the tangent bundle of M_1 reversing the vertical and the horizontal distributions.

From equations (7) and (8), we have

$$\nabla_{Y_1}Y_2 = \mathcal{T}_{Y_1}Y_2 + \mathcal{V}\nabla_{Y_1}Y_2, \tag{9}$$

$$\nabla_{Y_1}W_1 = \mathcal{T}_{Y_1}W_1 + \mathcal{H}\nabla_{Y_1}W_1, \tag{10}$$

$$\nabla_{W_1} Y_1 = \mathcal{A}_{W_1} Y_1 + \mathcal{V} \nabla_{W_1} Y_1, \tag{11}$$

$$\nabla_{W_1} W_2 = \mathcal{H} \nabla_{W_1} W_2 + \mathcal{A}_{W_1} W_2 \tag{12}$$

for all $Y_1, Y_2 \in \Gamma(\ker \pi_*)$ and $W_1, W_2 \in \Gamma(\ker \pi_*)^\perp$, where $\mathcal{H} \nabla_{Y_1} W_1 = \mathcal{A}_{W_1} Y_1$, if W_1 is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

It is seen that for $p \in M_1$, $X_1 \in \mathcal{V}_q$ and $Y_1 \in \mathcal{H}_q$ the linear operators

$$\mathcal{A}_{Y_1},\ \mathcal{T}_{X_1} : T_p M_1 \to T_p M_1,$$

are skew-symmetric, i.e.

$$g_1(\mathcal{A}_{Y_1} Z_1, Z_2) = -g_1(Z_1, \mathcal{A}_{Y_1} Z_2) \text{ and } g_1(\mathcal{T}_{X_1} Z_1, Z_2) = -g_1(Z_1, \mathcal{T}_{X_1} Z_2) \tag{13}$$

for each $Z_1, Z_2 \in T_P M_1$. Since $\mathcal{T}_{X_1}$ is skew-symmetric, we observe that π has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

The differentiable map π between two Riemannian manifolds is totally geodesic if

$$(\nabla \pi_*)(U_1, U_2) = 0, \text{ for all } U_1, U_2 \in \Gamma(TM_1).$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths. A Riemannian submersion is a Riemannian submersion with totally umbilical fibers if [6]

$$\mathcal{T}_{Y_1} Y_2 = g_1(Y_1, Y_2) H \tag{14}$$

for all $Y_1, Y_2 \in \Gamma(\ker \pi_*)$, where H is the mean curvature vector field of fibers.

Let $\pi : (M_1, g_1) \to (M_2, g_2)$ is a smooth map between Riemannian manifolds. Then the differential map π_* of π can be observed a section of the bundle $Hom(TM_1, \pi^{-1}TM_2) \to M_1$, where $\pi^{-1}TM_2$ is the bundle which has fibers $(\pi^{-1}TM_2)_x = T_{\pi(x)} M_2$ has a connection ∇ induced from the Riemannian connection ∇^{M_1} and the pullback connection, where $x \in M_1$. Then the second fundamental form of π is given by

$$(\nabla \pi_*)(W_1, W_2) = \nabla^{\pi}_{W_1} \pi_*(W_2) - \pi_*(\nabla^{M_1}_{W_1} W_2) \tag{15}$$

for vector field $W_1, W_2 \in \Gamma(TM_1)$, where ∇^π is the pullback connection [6]. We know that the second fundamental form is symmetric.

Lemma 1. *[6] Let (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds. If $\pi : M_1 \to M_2$ Riemannian submersion between Riemannian manifolds, then for any horizontal vector fields U_1, U_2 and vertical vector fields V_1, V_2, we have*

(i) $(\nabla \pi_*)(U_1, U_2) = 0$,
(ii) $(\nabla \pi_*)(V_1, V_2) = -\pi_*(\mathcal{T}_{V_1} V_2) = -\pi_*(\nabla^{M_1}_{V_1} V_2)$,
(iii) $(\nabla \pi_*)(U_1, V_1) = -\pi_*(\nabla^{M_1}_{U_1} V_1) = -\pi_*((\mathcal{A}_{U_1} V_1)$.

Now, we recall following definitions for later use:

Definition 1. *[27] Let π be a Riemannian submersion from an almost Hermitian manifold (M_1, J, g_1) onto a Riemannian manifold (M_2, g_2). Then, we say that π is an invariant Riemannian submersion if the vertical distribution is invariant with respect to the complex structure J, i.e.,*

$$J(\ker \pi_*) = \ker \pi_*.$$

Definition 2. *[19] Let M_1 be an almost contact manifold with Riemannian metric g_1 and let M_2 be a Riemannian manifold with Riemannian metric g_2. Suppose that there exists a Riemannian submersion $\pi : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ such that $\phi(\ker \pi_*) \subseteq (\ker \pi_*)^{\perp}$. Then we say that π is an anti-invariant Riemannian submersion.*

Definition 3. *[27] Let M_1 be an almost contact manifold with Riemannian metric g_1 and let M_2 be a Riemannian manifold with Riemannian metric g_2. Then we say that π is a semi-invariant Riemannian submersion if there is a distribution $D_1 \subseteq \ker \pi_*$ such that*

$$\ker \pi_* = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subseteq (\ker \pi_*)^{\perp},$$

where D_2 is orthogonal complementary to D_1 in $\ker \pi_*$.

Let μ denotes the complementary orthogonal subbundle to $\phi(\ker \pi_*)$ in $(\ker \pi_*)^{\perp}$. Then, we have

$$(\ker \pi_*)^{\perp} = \phi(D_2) \oplus \mu.$$

Obviously μ is an invariant subbundle of $(\ker \pi_*)^{\perp}$ with respect to the contact structure ϕ.

Let π be a semi-invariant submersion from an almost contact metric manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). First of all, from definition, we have $\phi(\ker \pi_*)^{\perp} \cap (\ker \pi_*) \neq \{0\}$. We denote the complementary orthonormal distribution to $\phi(D_2)$ in $(\ker \pi_*)^{\perp}$ by μ. Then we have

$$(\ker \pi_*)^{\perp} = \phi(\ker \pi_*) \oplus \mu.$$

It is informal to know that μ is an invariant distribution of $(\ker \pi_*)^{\perp}$, under the endomorphism ϕ.

Definition 4. *[17] Let $\pi : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a Riemannian submersion from an almost contact metric manifold onto a Riemannian manifold. Suppose that there exists a Riemannian submersion π such that ξ is normal to $(\ker \pi_*)$ and $(\ker \pi_*)$ is anti-invariant with respect to ϕ i.e., $\phi(\ker \pi_*) \subset (\ker \pi_*)^{\perp}$. Then we say that π is an anti-invariant $\xi^{\perp}$-Riemannian submersion.*

3 Csi-$\xi^{\perp}$-Riemannian submersions from a Kenmotsu manifolds

In this subsection, we define and study Csi-$\xi^{\perp}$ -Riemannian submersion from Kenmotsu manifolds onto Riemannian manifolds. After giving a new necessary and sufficient condition for such submersions to be Clairaut. We also obtain some fundamental results for this kind of $\xi^{\perp}$-Riemannian submersions.

Definition 5. *[1] Let $\pi : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a Riemannian submersion from an almost contact metric manifold onto a Riemannian manifold. A Riemannian submersion π is called a semi-invariant $\xi^{\perp}$-Riemannian submersion if there is a distribution $D_1 \subset (\ker \pi_*)$ such that*

$$(\ker \pi_*) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subset (\ker \pi_*)^{\perp},$$

where D_2 is orthogonal complementary to D_1 in $(\ker \pi_*)$.

In the theory of Riemannian submersions, Bishop [8] introduces the notion of Clairaut submersion in the following way:

Definition 6. *A Riemannian submersion $\pi : (M_1, g_1) \to (M_2, g_2)$ is called a Clairaut submersion if there exists a positive function r on M_1, such that, for any geodesic α on M_1, the function $(r \circ \alpha) \sin \theta$ is constant, where, for any t, $\theta(t)$ is the angle between $\dot{\alpha}$ and the horizontal space at $\alpha(t)$.*

He also gave the following necessary and sufficient condition for a Riemannian submersion to be a Clairaut submersion as follows:

Theorem 1. *[8] Let $\pi : (M_1, g_1) \to (M_2, g_2)$ be a Riemannian submersion with connected fibers. Then, π is a Clairaut Riemannian submersion with $r = e^f$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla f$, where ∇f is the gradient of the function f with respect to g_1.*

Definition 7. *A semi-invariant $\xi^{\perp}$-Riemannian submersion π from a Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2) is called Csi-$\xi^{\perp}$-Riemannian submersion if it satisfies the condition of Clairaut Riemannian submersion i.e. the π is a Clairaut Riemannian submersion with $r = e^f$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla f$ is the gradient of the function f with respect to g_1.*

Now, using definition (7), we have

$$(\ker \pi_*) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subseteq (\ker \pi_*)^{\perp}.$$

Thus for any $V_1 \in (\ker \pi_*)$, we put

$$V_1 = PV_1 + QV_1, \tag{16}$$

where $PV_1 \in \Gamma(D_1)$ and $QV_1 \in \Gamma(D_2)$.

In addition, for $Y_1 \in (\ker \pi_*)$, we get

$$\phi Y_1 = \psi Y_1 + \omega Y_1, \tag{17}$$

where $\phi Y_1 \in \Gamma(D_1)$ and $\omega Y_1 \in \Gamma(\phi D_2)$.

The horizontal distribution $\Gamma(\ker \pi_*)^\perp$ is decomposed as

$$\Gamma(\ker \pi_*)^\perp = \phi(D_2) \oplus \mu.$$

Here μ is an invariant distribution of ϕ and contains ξ.
Also for $X_2 \in \Gamma(\ker \pi_*)^\perp$, we have

$$\phi X_2 = BX_2 + CX_2, \tag{18}$$

where $BX_2 \in \Gamma(D_2)$ and $CX_2 \in \Gamma(\mu)$.

Lemma 2. *Let π be a semi-invariant $\xi^\perp$-Riemannian submersion from a Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then, we get*

$$\mathcal{V}\nabla_{Y_1}\psi Y_2 + \mathcal{T}_{Y_1}\omega Y_2 = B\mathcal{T}_{Y_1}Y_2 + \psi\mathcal{V}\nabla_{Y_1}Y_2, \tag{19}$$

$$\mathcal{T}_{Y_1}\psi Y_2 + \mathcal{H}\nabla_{Y_1}\omega Y_2 = C\mathcal{T}_{Y_1}Y_2 + \omega\mathcal{V}\nabla_{Y_1}Y_2 + g_1(\psi Y_1, Y_2)\xi, \tag{20}$$

$$\mathcal{V}\nabla_{V_1}BV_2 + \mathcal{A}_{V_1}CV_2 + \eta(V_2)BV_1 = B\mathcal{H}\nabla_{V_1}V_2 + \psi\mathcal{A}_{V_1}V_2, \tag{21}$$

$$\mathcal{A}_{V_1}BV_2 + \mathcal{H}\nabla_{V_1}CV_2 + \eta(V_2)CV_1 = C\mathcal{H}\nabla_{V_1}V_2 + \omega\mathcal{A}_{V_1}V_2 + g_1(CV_1, V_2)\xi, \tag{22}$$

$$\mathcal{V}\nabla_{Y_1}BV_1 + \mathcal{T}_{Y_1}CV_1 + \eta(V_1)\psi Y_1 = \psi\mathcal{T}_{Y_1}V_1 + B\mathcal{H}\nabla_{Y_1}V_1, \tag{23}$$

$$\mathcal{T}_{Y_1}BV_1 + \mathcal{H}\nabla_{Y_1}CV_1 + \eta(V_1)\omega Y_1 = \omega\mathcal{T}_{Y_1}V_1 + C\mathcal{H}\nabla_{Y_1}V_1 + g_1(\omega Y_1, V_1)\xi, \tag{24}$$

$$\mathcal{V}\nabla_{V_1}\psi Y_1 + \mathcal{A}_{V_1}\omega Y_1 = B\mathcal{A}_{V_1}Y_1 + \psi\mathcal{V}\nabla_{V_1}Y_1, \tag{25}$$

$$\mathcal{A}_{V_1}\psi Y_1 + \mathcal{H}\nabla_{V_1}\omega Y_1 = C\mathcal{A}_{V_1}Y_1 + \omega\mathcal{V}\nabla_{V_1}Y_1 + g_1(BV_1, Y_1)\xi, \tag{26}$$

where $Y_1, Y_2 \in \Gamma(\ker \pi_*)$ and $V_1, V_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (5), (9), (10), (11), (12), (17) and (18), we get Lemma 2. □

Lemma 3. *Let π be a semi-invariant $\xi^\perp$-Riemannian submersion from a Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). If $\alpha : I_2 \subset R \to M_1$ is a regular curve and $Z_1(t)$ and $Z_2(t)$ are the vertical and horizontal components of the tangent vector field $\dot\alpha = E$ of $\alpha(t)$, respectively, then α is a geodesic if and only if along α the following equations hold:*

$$\begin{aligned}
&\mathcal{V}\nabla_{\dot\alpha}\psi Z_1 + \mathcal{V}\nabla_{\dot\alpha}BZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2 \\
&-\eta(Z_2)(\psi Z_1 + BZ_1) = 0, \\
&\mathcal{H}\nabla_{\dot\alpha}\omega Z_1 + \mathcal{H}\nabla_{\dot\alpha}CZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2 + g_1(\phi\dot\alpha, \dot\alpha)\xi \\
&-\eta(Z_2)(\omega Z_1 + CZ_1) = 0.
\end{aligned}$$

Proof. Let $\alpha : I_2 \to M_1$ be a regular curve on M_1. Since $Z_1(t)$ and $Z_2(t)$ are the vertical and horizontal parts of the tangent vector field $\dot{\alpha}(t)$, i.e., $\dot{\alpha}(t) = Z_1(t) + Z_2(t)$. From equations $(5), (9), (10), (11), (12), (17)$ and (18), we get

$$\begin{aligned}
& \phi\nabla_{\dot{\alpha}}\dot{\alpha} \\
= \; & \nabla_{\dot{\alpha}}\phi\dot{\alpha} - (\nabla_{\dot{\alpha}}\phi)\dot{\alpha}, \\
= \; & \nabla_{Z_1}\psi Z_1 + \nabla_{Z_1}\omega Z_1 + \nabla_{Z_1}BZ_2 + \nabla_{Z_1}CZ_2 + \nabla_{Z_2}\psi Z_1 + \nabla_{Z_2}\omega Z_1 + \\
& \nabla_{Z_2}BZ_2 + \nabla_{Z_2}CZ_2 - g_1(\phi\dot{\alpha}, \dot{\alpha})\xi + \eta(Z_2)(\psi Z_1 + BZ_1) + \eta(Z_2)(\omega Z_1 + CZ_1), \\
= \; & \mathcal{T}_{Z_1}\psi Z_1 + \mathcal{V}\nabla_{Z_1}\psi Z_1 + \mathcal{T}_{Z_1}\omega Z_1 + \mathcal{H}\nabla_{Z_1}\omega Z_1 + \mathcal{T}_{Z_1}BZ_2 + \mathcal{V}\nabla_{Z_1}BZ_2 + \\
& \mathcal{T}_{Z_1}CZ_2 + \mathcal{H}\nabla_{Z_1}CZ_2 + \mathcal{A}_{Z_2}\psi Z_1 + \mathcal{V}\nabla_{Z_2}\psi Z_1 + \mathcal{H}\nabla_{Z_2}\omega Z_1 + \mathcal{A}_{Z_2}\omega Z_1 + \\
& \mathcal{A}_{Z_2}BZ_2 + \mathcal{V}\nabla_{Z_2}BZ_2 + \mathcal{H}\nabla_{Z_2}CZ_2 + \mathcal{A}_{Z_2}CZ_2 - g_1(\phi\dot{\alpha}, \dot{\alpha})\xi + \\
& \eta(Z_2)(\psi Z_1 + BZ_1) + \eta(Z_2)(\omega Z_1 + CZ_1).
\end{aligned}$$

Taking the vertical and horizontal components in above equation, we have

$$\begin{aligned}
\mathcal{V}\phi\nabla_{\dot{\alpha}}\dot{\alpha} \; &= \; \mathcal{V}\nabla_{\dot{\alpha}}\psi Z_1 + \mathcal{V}\nabla_{\dot{\alpha}}BZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2 \\
& \quad + \eta(Z_2)(\psi Z_1 + BZ_1), \\
\mathcal{H}\phi\nabla_{\dot{\alpha}}\dot{\alpha} \; &= \; \mathcal{H}\nabla_{\dot{\alpha}}\omega Z_1 + \mathcal{H}\nabla_{\dot{\alpha}}CZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2 \\
& \quad - g_1(\phi\dot{\alpha}, \dot{\alpha})\xi + \eta(Z_2)(\omega Z_1 + CZ_1),
\end{aligned}$$

Now, α is a geodesic on M_1 if and only if $\mathcal{V}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = 0$ and $\mathcal{H}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = 0$, which completes the proof. □

Theorem 2. *Let π be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then π is a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ if and only if*

$$\begin{aligned}
& (g_1(\nabla f, V_2) + \eta(V_2)|)||V_1||^2 \\
= \; & -g_1(\mathcal{V}\nabla_{\dot{\alpha}}BV_2, \psi V_1) - g_1(\mathcal{H}\nabla_{\dot{\alpha}}CV_2, \omega V_1) - g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) - \\
& g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1)
\end{aligned}$$

where $\alpha : I_2 \to M_1$ is a geodesic on M_1 and V_1, V_2 are vertical and horizontal components of $\dot{\alpha}(t)$.

Proof. Let $\alpha : I_2 \to M_1$ be a geodesic on M_1 with $V_1(t) = \mathcal{V}\dot{\alpha}(t)$ and $V_2(t) = \mathcal{H}\dot{\alpha}(t)$. Let $\theta(t)$ denote the angle in $[0, \pi]$ between $\dot{\alpha}(t)$ and $V_2(t)$. Assuming $\nu = ||\dot{\alpha}(t)||,^2$ then we get

$$g_1(V_1(t), V_1(t)) = \nu \sin^2\theta(t), \tag{27}$$

$$g_1(V_2(t), V_2(t)) = \nu \cos^2\theta(t). \tag{28}$$

Now, differentiating (27), we get

$$\frac{d}{dt}g_1(V_1(t), V_1(t)) = 2\nu\cos\theta(t)\sin\theta(t)\frac{d\theta}{dt}.$$

Using equation (3), we get

$$g_1(\phi\nabla_{\dot{\alpha}}V_1, \phi V_1) = \nu\cos\theta(t)\sin\theta(t)\frac{d\theta}{dt}. \tag{29}$$

Now, using equation (5), we get

$$\nabla_{\dot{\alpha}}\phi V_1 = \phi\nabla_{\dot{\alpha}}V_1 + g_1(\phi\dot{\alpha}, V_1)\xi,$$

$$\begin{aligned} & g_1(\phi\nabla_{\dot{\alpha}}V_1, \phi V_1) \\ = \;& g_1(\nabla_{\dot{\alpha}}\phi V_1, \phi V_1), \\ = \;& g_1(\mathcal{V}\nabla_{\dot{\alpha}}\psi V_1, \psi V_1) + g_1(\mathcal{H}\nabla_{\dot{\alpha}}\omega V_1, \omega V_1) + g_1((\mathcal{A}_{V_2} + \mathcal{T}_{V_1})\psi V_1, \omega V_1) + \\ & g_1((\mathcal{A}_{V_2} + \mathcal{T}_{V_1})\omega V_1, \psi V_1). \end{aligned}$$

Using Lemma 3 and (29), in above equation, we get

$$\begin{aligned} & g_1(\phi\nabla_{\dot{\alpha}}V_1, \phi V_1) \\ = \;& -g_1(\mathcal{V}\nabla_{\dot{\alpha}}BV_2, \psi V_1) - g_1(\mathcal{H}\nabla_{\dot{\alpha}}CV_2, \omega V_1) - g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) - \\ & g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1) - \eta(V_2)g_1(V_1, V_1). \end{aligned} \tag{30}$$

From equations (29) and (30), we have

$$\begin{aligned} & 2\nu\cos\theta(t)\sin\theta(t)\frac{d\theta}{dt} \\ = \;& -g_1(\mathcal{V}\nabla_{\dot{\alpha}}BV_2, \psi V_1) - g_1(\mathcal{H}\nabla_{\dot{\alpha}}CV_2, \omega V_1) - g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) - \\ & g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1) - \eta(V_2)g_1(V_1, V_1). \end{aligned} \tag{31}$$

Moreover, π is a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ if and only if $\frac{d}{dt}(e^{f\circ\alpha}\sin\theta) = 0$, i.e., $e^{f\circ\alpha}(\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{df}{dt}) = 0$. By multiplying this with non-zero factor $\upsilon\sin\theta$, we have

$$\begin{aligned} -\upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= \upsilon\sin^2\theta\frac{df}{dt}, \\ \upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_1(V_1, V_1)\frac{df}{dt}, \\ \upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_1(\nabla f, \dot{\alpha})||V_1||^2, \\ \upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_1(\nabla f, V_2)||V_1||^2. \end{aligned} \tag{32}$$

Thus, from equations (31) and (32), we have

$$\begin{aligned} & (g_1(\nabla f, V_2) - \eta(V_2)|)||V_1||^2 \\ = \;& g_1(\mathcal{V}\nabla_{\dot{\alpha}}BV_2, \psi V_1) + g_1(\mathcal{H}\nabla_{\dot{\alpha}}CV_2, \omega V_1) + g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})CV_2, \psi V_1) + \\ & g_1((\mathcal{T}_{V_1} + \mathcal{A}_{V_2})BV_2, \omega V_1). \end{aligned}$$

Hence the theorem (2) is proved. □

Corollary 1. *Let π be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then, we get*

$$g_1(\nabla f, \xi) = 1.$$

Theorem 3. *Let π be a Csi-$\xi^{\perp}$-Riemannian submersion from a Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2) with $r = e^f$. Then, we get*

$$\mathcal{A}_{\phi U_1} \phi V_1 = V_1(f) U_1 \tag{33}$$

for $V_1 \in \Gamma(\mu)$ and $U_1 \in \Gamma(D_2)$, such that ϕU_1 is basic.

Proof. Let π be Csi-$\xi^{\perp}$-Riemannian submersion from a Kenmotsu manifold onto a Riemannian manifold. For $Z_1, Z_2 \in \Gamma(D_2)$, using equation (14) and Theorem 1, we get

$$\mathfrak{T}_{Z_1} Z_2 = -g_1(Z_1, Z_2) gradf. \tag{34}$$

Taking inner product in equation (34) with ϕU_1, we have

$$g_1(\mathfrak{T}_{Z_1} Z_2, \phi U_1) = -g_1(Z_1, Z_2) g_1(gradf, \phi U_1), \tag{35}$$

for all $U_1 \in \Gamma(D_2)$.

From equations $(9), (34)$ and (35), we obtain

$$g_1(\nabla_{Z_1} \phi Z_2, U_1) = g_1(Z_1, Z_2) g_1(gradf, \phi U_1).$$

Since ∇ is metric connection, using equations (14) and (35) in above equation, we get

$$g_1(Z_1, U_1) g_1(gradf, \phi Z_2) = g_1(Z_1, Z_2) g_1(gradf, \phi U_1). \tag{36}$$

Taking $U_1 = Z_2$ and interchanging the role of Z_1 and Z_2, we obtain

$$g_1(Z_2, Z_2) g_1(gradf, \phi Z_1) = g_1(Z_1, Z_2) g_1(gradf, \phi Z_2). \tag{37}$$

Using equation (37) with $U_1 = Z_1$ in (36), we have

$$g_1(gradf, J Z_1) = \frac{(g_1(Z_1, Z_2))^2}{||Z_1||^2 ||Z_2||^2} g_1(gradf, \phi Z_1). \tag{38}$$

If $gradf \in \Gamma(\phi(D_2))$, then equation (38) and the condition of equality in the Schwarz inequality implies that either f is constant on $\phi(D_2)$ or the fibers are one dimensional.

On the other hand, using equations (3) and (5), we get

$$g_1(\phi \nabla_{Z_1} U_1, \phi V_1) = g_1(\nabla_{Z_1} \phi U_1, \phi V_1)$$

for $V_1 \in \Gamma(\mu)$ and $V_1 \neq \xi$. Now, using equation (3), we obtain

$$g_1(\nabla_{Z_1} \phi U_1, \phi V_1) = g_1(\nabla_{Z_1} U_1, V_1).$$

Using equations (9) and (14) in above equation, we get

$$g_1(\nabla_{Z_1}\phi U_1, \phi V_1) = -g_1(Z_1, U_1)g_1(gradf, V_1).$$

Since ϕU_1 is basic and using the fact that $\mathcal{H}\nabla_{Z_1}\phi U_1 = \mathcal{A}_{\phi U_1}Z_1$, we get

$$\begin{aligned} g_1(\nabla_{Z_1}\phi U_1, \phi V_1) &= -g_1(Z_1, U_1)g_1(gradf, V_1), \\ g_1(\mathcal{A}_{\phi U_1}Z_1, \phi V_1) &= -g_1(Z_1, U_1)g_1(gradf, V_1), \\ g_1(\mathcal{A}_{\phi U_1}\phi V_1, Z_1) &= g_1(Z_1, U_1)g_1(gradf, V_1) \\ g_1(\mathcal{A}_{\phi U_1}\phi V_1, Z_1) &= g_1(Z_1, U_1)g_1(\nabla f, V_1). \end{aligned} \tag{39}$$

Since $\mathcal{A}_{\phi U_1}\phi V_1$ and U_1 are vertical and ∇f is horizontal, we obtain equation (33). □

Theorem 4. *Let π be a Csi-$\xi^\perp$-Riemannian submersion from a Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2) with $r = e^f$ and* $\dim(D_2) > 1$. *Then, the fibers of π are totally geodesic or the semi-invariant distribution D_2 one-dimensional.*

Proof. Let π be a Csi-$\xi^\perp$-Riemannian submersion from a Kenmotsu manifold onto a Riemannian manifold. From Theorem (1), fibers are totally umbilical with mean curvature vector field $H = -gradf$, we have

$$\begin{aligned} -g_1(\nabla_{Z_1}X_1, Z_2) &= g_1(\nabla_{Z_1}Z_2, X_1), \\ -g_1(\nabla_{Z_1}X_1, Z_2) &= -g_1(Z_1, Z_2)g_1(gradf, X_1), \end{aligned}$$

for all $Z_1, Z_2 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker \pi_*)^\perp$.

Using equation (3) in above equation, we get

$$g_1(\nabla_{X_1}\phi Z_1, \phi Z_2) = g_1(\phi Z_1, \phi Z_2)g_1(gradf, X_1). \tag{40}$$

Since π is semi-invariant Riemannian map and using equation (15), we have

$$g_2(\pi_*(\nabla_{X_1}\phi Z_1), \pi_*(\phi Z_2)) = g_2(\pi_*(\phi Z_1), \pi_*(\phi Z_2))g_1(gradf, X_1). \tag{41}$$

From (15) in (41), we obtain

$$g_2(\overset{\pi}{\nabla}_{X_1}\pi_*(\phi Z_1), \pi_*(\phi Z_2)) = g_2(\pi_*(\phi Z_1), \pi_*(\phi Z_2))g_1(gradf, X_1), \tag{42}$$

which implies $\overset{\pi}{\nabla}_{X_1}\pi_*(\phi Z_1) = X_1(f)\pi_*(\phi Z_1)$, for all $Z_1 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker \pi_*)^\perp$, hence the proof. □

Theorem 5. *Let π be a Csi-$\xi^\perp$-Riemannian submersion with $r = e^f$ from a Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). If $\mathcal{T}$ is not equal to zero identically, then the invariant distribution D_1 cannot defined a totally geodesic foliation on M_1.*

Proof. For $V_1, V_2 \in \Gamma(D_1)$ and $Z_1 \in \Gamma(D_2)$, using equations (3), (9) and (14), we get

$$\begin{aligned} g_1(\nabla_{V_1} V_2, Z_1) &= g_1(\nabla_{V_1} \phi V_2, \phi Z_1), \\ &= g_1(\mathcal{T}_{V_1} \phi V_2, \phi Z_1), \\ &= -g_1(V_1, \phi V_2) g_1(gradf, \phi Z_1). \end{aligned}$$

Thus, the assertion can be seen from above equation and the fact that $gradf \in \phi(D_2)$. □

Theorem 6. *Let π be a Csi-$\xi^\perp$-Riemannian submersion with $r = e^f$ from a Kenmotsu manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then, the fibers of π are totally geodesic or the anti-invariant distribution D_2 one-dimensional.*

Proof. If the fibers of π are totally geodesic, it is obvious. For second one, since π is a Clairaut proper semi-invariant submersion, then either $\dim(D_2) = 1$ or $\dim(D_2) > 1$. If $\dim(D_2) > 1$, then we can choose $X_1, X_2 \in \Gamma(D_2)$ such that $\{X_1, X_2\}$ is orthonormal. From equations (9), (17) and (18), we get

$$\begin{aligned} \mathcal{T}_{X_1} \phi X_2 + \mathcal{H} \nabla_{X_1} \phi X_2 &= \nabla_{X_1} \phi X_2, \\ \mathcal{T}_{X_1} \phi X_2 + \mathcal{H} \nabla_{X_1} \phi X_2 &= B \mathcal{T}_{X_1} X_2 + C \mathcal{T}_{X_1} X_2 + \psi \mathcal{V} \nabla_{X_1} X_2 + \omega \mathcal{V} \nabla_{X_1} X_2. \end{aligned}$$

Taking inner product above equation with X_1, we obtain

$$g_1(\mathcal{T}_{X_1} \phi X_2, X_1) = g_1(B \mathcal{T}_{X_1} X_2, X_1) + g_1(\psi \mathcal{V} \nabla_{X_1} X_2, X_1). \tag{43}$$

From equation (3), (9) and (14), we have

$$g_1(\mathcal{T}_{X_1} X_1, \phi X_2) = -g_1(\mathcal{T}_{X_1} \phi X_2, X_1) = -g_1(gradf, \phi X_2) = g_1(\mathcal{T}_{X_1} X_2, \phi X_1). \tag{44}$$

From above equation, we obtain

$$\begin{aligned} g_1(gradf, \phi X_2) &= g_1(\mathcal{T}_{X_1} X_2, \phi X_1), \\ g_1(gradf, \phi X_2) &= g_1(X_1, X_2) g_1(gradf, \phi X_1), \\ g_1(gradf, \phi X_2) &= 0. \end{aligned}$$

So, we get

$$gradf \perp \phi(D_2).$$

Therefore, the dimension of D_2 must be one. □

Example 1. *Let $\mathbb{R}^7$ be a 7-dimensional Euclidean space given by the following:*

$$\mathbb{R}^7 = \{(x_1, x_2, x_3, y_1, y_2, y_3, z) \in R^7 | (x_1, x_2, x_3, y_1, y_2, y_3) \neq (0,0,0,0,0,0),\ z \neq 0\}.$$

We define the Kenmotsu structure (ϕ, ξ, η, g_1) on $\mathbb{R}^7$ given by the following:

$$\xi = \frac{\partial}{\partial z}, \eta = dz,$$

$$g_1 = \begin{bmatrix} e^{2z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{2z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A ϕ-basis for this structure can be given by $\{E_1 = e^{-z}\frac{\partial}{\partial y_1}, E_2 = e^{-z}\frac{\partial}{\partial y_2}, E_3 = e^{-z}\frac{\partial}{\partial y_3}, E_4 = e^{-z}\frac{\partial}{\partial x_1}, E_5 = e^{-z}\frac{\partial}{\partial x_2}, E_6 = e^{-z}\frac{\partial}{\partial x_3}, E_7 = \xi = \frac{\partial}{\partial z}\}$.

Let M_2 be $\{(u_1, u_2, u_3) \in R^3 | u_3 = z \neq 0\}$. We choose the Riemannian metric g_2 on M_2 in the following form:

$$g_2 = \begin{bmatrix} 4e^{2z} & 0 & 0 \\ 0 & 4e^{2z} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, we define the map $\pi : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ by the following:

$$\pi(x_1, x_2, x_3, y_1, y_2, y_3, z) = (\frac{x_2 + y_3}{\sqrt{2}}, \frac{x_3 + y_2}{\sqrt{2}}, z).$$

By direct calculations, we have

$$\begin{aligned} \ker \pi_* &= span\{X_1 = E_1, X_2 = \frac{1}{\sqrt{2}}(E_5 - E_3), X_3 = E_4, X_4 = \frac{1}{\sqrt{2}}(E_6 - E_2)\}, \\ D_1 &= span\{X_1 = E_1, X_3 = E_4\}, \\ D_2 &= span\{X_2 = \frac{1}{\sqrt{2}}(E_5 - E_3), X_4 = \frac{1}{\sqrt{2}}(E_6 - E_2)\}, \\ (\ker \pi_*)^{\perp} &= span\{V_1 = \frac{1}{\sqrt{2}}(E_5 + E_3), V_2 = \frac{1}{\sqrt{2}}(E_6 + E_2), V_3 = \xi\}. \end{aligned}$$

After some computations, one can see that $\pi_*(V_1) = \frac{1}{2}e^{-z}\frac{\partial}{\partial u_1}, \pi_*(V_2) = \frac{1}{2}e^{-z}\frac{\partial}{\partial u_2}$, $\pi_*(V_3) = \frac{\partial}{\partial u_3}$ and $g_1(V_i, V_j) = g_2(\pi_* V_i, \pi_* V_j)$ for all $V_i, V_j \in \Gamma(\ker \pi_*)^{\perp}$, $i, j = 1, 2, 3$. Thus π is semi-invariant $\xi^{\perp}$-Riemannian submersion. Moreover it is easy to see that $\phi X_1 = -X_3, \phi X_2 = V_1, \phi X_3 = X_1, \phi X_4 = -V_2$.

Now, we will find smooth function f on $\mathbb{R}^7$ satisfying $\mathcal{T}_X X = g_1(X, X)\nabla f$, for all $X \in \Gamma(\ker \pi_*)$.

Using the given Kenmotsu structure, we find

$$\begin{aligned} \nabla_{E_1} E_1 &= \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = \nabla_{E_4} E_4 = \nabla_{E_5} E_5 = \nabla_{E_6} E_6 = -\frac{\partial}{\partial z}, \quad (45) \\ \nabla_{E_1} E_2 &= \nabla_{E_1} E_3 = \nabla_{E_1} E_4 = \nabla_{E_2} E_1 = \nabla_{E_2} E_3 = \nabla_{E_2} E_4 = 0, \\ \nabla_{E_3} E_1 &= \nabla_{E_3} E_2 = \nabla_{E_3} E_4 = \nabla_{E_4} E_1 = \nabla_{E_4} E_2 = \nabla_{E_4} E_3 = 0. \end{aligned}$$

Therefore

$$\nabla_{X_1}X_1 = \nabla_{X_2}X_2 = \nabla_{X_3}X_3 = \nabla_{X_4}X_4 = -\frac{\partial}{\partial z}. \tag{46}$$

Now, we have

$$\mathcal{T}_X X = \mathcal{T}_{\lambda_1X_1+\lambda_2X_2+\lambda_3X_3+\lambda_4X_4}\lambda_1X_1 + \lambda_2X_2 + \lambda_3X_3 + \lambda_4X_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in R,$$

$$\begin{aligned} \mathcal{T}_X X &= \lambda_1^2\mathcal{T}_{X_1}X_1 + \lambda_2^2\mathcal{T}_{X_2}X_2 + \lambda_3^2\mathcal{T}_{X_3}X_3 + \lambda_4^2\mathcal{T}_{X_4}X_4 + \\ &\quad 2\lambda_1\lambda_2\mathcal{T}_{X_1}X_2 + 2\lambda_1\lambda_3\mathcal{T}_{X_1}X_3 + 2\lambda_1\lambda_4\mathcal{T}_{X_1}X_4 + \\ &\quad 2\lambda_2\lambda_3\mathcal{T}_{X_2}X_3 + 2\lambda_2\lambda_4\mathcal{T}_{X_2}X_4 + 2\lambda_3\lambda_4\mathcal{T}_{X_3}X_4. \end{aligned} \tag{47}$$

Using equations (14) and (46), we obtain

$$\begin{aligned} \mathcal{T}_{X_1}X_1 &= -\frac{\partial}{\partial z}, \mathcal{T}_{X_2}X_2 = -\frac{\partial}{\partial z}, \mathcal{T}_{X_3}X_3 = -\frac{\partial}{\partial z}, \\ \mathcal{T}_{X_4}X_4 &= -\frac{\partial}{\partial z}, \mathcal{T}_{X_1}X_2 = 0, \mathcal{T}_{X_1}X_3 = 0, \mathcal{T}_{X_1}X_4 = 0, \\ \mathcal{T}_{X_2}X_3 &= 0, \mathcal{T}_{X_2}X_4 = 0, \mathcal{T}_{X_3}X_4 = 0. \end{aligned} \tag{48}$$

Next, using equations (47) and (48), we get

$$\mathcal{T}_X X = -(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_3^2 + \lambda_4^2)\frac{\partial}{\partial z}. \tag{49}$$

Since $X = \lambda_1X_1 + \lambda_2X_2 + \lambda_3X_3 + \lambda_4X_4$, so $g_1(\lambda_1X_1 + \lambda_2X_2 + \lambda_3X_3 + \lambda_4X_4, \lambda_1X_1 + \lambda_2X_2 + \lambda_3X_3 + \lambda_4X_4) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2$. For any smooth function f on R^7, the gradient of f with respect to the metric g_1 is given by $\nabla f = e^{-2Z}\frac{\partial f}{\partial y_1}\frac{\partial}{\partial y_1} + e^{-2Z}\frac{\partial f}{\partial y_2}\frac{\partial}{\partial y_2} + e^{-2Z}\frac{\partial f}{\partial y_3}\frac{\partial}{\partial y_3} + e^{-2Z}\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + e^{-2Z}\frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_2} + e^{-2Z}\frac{\partial f}{\partial x_3}\frac{\partial}{\partial x_3} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}$. Hence $\nabla f = \frac{\partial}{\partial z}$ for the function $f = z$. Then it is easy to see that $\mathcal{T}_X X = -g_1(X, X)\nabla f$, thus by Theorem (1), π is a Csi-$\xi^\perp$-Riemannian submersion.

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