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THE NORMING SETS OF $\mathcal{L}(^2l_1^2)$ and $\mathcal{L}_s(^2l_1^3)$

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Abstract

Let $n \in \mathbb{N}$. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$, where $\mathcal{L}(^nE)$ denotes the space of all continuous n-linear forms on E. For $T \in \mathcal{L}(^nE)$, we define

$$\text{Norm}(T) = \Big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T. We classify Norm(T) for every $T \in \mathcal{L}(^2l_1^2)$ or $\mathcal{L}_s(^2l_1^3)$, where $l_1^n = \mathbb{R}^n$ with the l_1-norm.

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1 Introduction

In 1961 Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, especially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E} |T(x_1, \cdots, x_n)|$. $\mathcal{L}_s(^nE)$ denote the

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closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$.

For $T \in \mathcal{L}(^nE)$, we define

$$\text{Norm}(T) = \Big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T. Notice that $(x_1, \ldots, x_n) \in \text{Norm}(T)$ if and only if $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$. Indeed, if $(x_1, \ldots, x_n) \in \text{Norm}(T)$, then

$$|T(\epsilon_1 x_1, \ldots, \epsilon_n x_n)| = |\epsilon_1 \cdots \epsilon_n T(x_1, \ldots, x_n)| = |T(x_1, \ldots, x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$. If $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$, then

$$(x_1, \ldots, x_n) = \Big(\epsilon_1(\epsilon_1 x_1), \ldots, \epsilon_n(\epsilon_n x_n)\Big) \in \text{Norm}(T).$$

The following examples show that it is possible that Norm(T) be empty or an infinite set.

Examples. (a) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2c_0).$$

We claim that Norm(T) $= \emptyset$. Obviously, $\|T\| = 1$. Assume that Norm(T) $\neq \emptyset$. Let $\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) \in \text{Norm}(T)$. Then,

$$1 = \Big|T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)\Big| \leq \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i|\ |y_i| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, Norm(T) $= \emptyset$.

(b) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = x_1 y_1 \in \mathcal{L}_s(^2c_0).$$

Then,

$$\begin{aligned} &\text{Norm}(T) \\ = \ &\Big\{\Big((\pm 1, x_2, x_3, \ldots),\ (\pm 1, y_2, y_3, \ldots)\Big) \in c_0 \times c_0 : |x_j| \leq 1,\ |y_j| \leq 1 \text{ for } j \geq 2\Big\}. \end{aligned}$$

A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^nE)$, we define

$$\text{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}.$$

Norm(P) is called the *norming set* of P. Notice that Norm$(P) = \emptyset$ or a finite set or an infinite set.

Kim [7] classified Norm(P) for every $P \in \mathcal{P}(^2l_\infty^2)$, where $l_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If Norm$(T) \neq \emptyset$, $T \in \mathcal{L}(^nE)$ is called a *norm attaining* n-linear form and if Norm$(P) \neq \emptyset$, $P \in \mathcal{P}(^nE)$ is called a *norm attaining* n-homogeneoue polynomial.(See [3])

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study Norm(T) for $T \in \mathcal{L}(^nE)$. For $m \in \mathbb{N}$, let $l_1^m := \mathbb{R}^m$ with the the l_1-norm and $l_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = l_1^m$ or l_∞^2 and $T \in \mathcal{L}(^nE)$, Norm$(T) \neq \emptyset$ since S_E is compact. Kim [6, 8, 9] classified Norm(T) for every $T \in \mathcal{L}_s(^2l_\infty^2), \mathcal{L}(^2l_\infty^2), \mathcal{L}(^2l_1^2)$ or $\mathcal{L}_s(^3l_1^2)$. Recently, Kim [10] classified Norm(T) for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$ $\|(x, y)\|_{h(w)} = \max\Big\{|y|, |x| + (1 - w)|y|\Big\}$.

In this paper, we classify Norm(T) for every $T \in \mathcal{L}(^2l_1^2)$ or $\mathcal{L}_s(^2l_1^3)$.

2 The norming sets of $\mathcal{L}(^2l_1^2)$

Theorem 1. *Let* $n, m \geq 2$. *Let* $T \in \mathcal{L}(^ml_1^n)$ *with*

$$T\Big(\Big(x_1^{(1)}, \ldots, x_n^{(1)}\Big), \ldots, \Big(x_1^{(m)}, \ldots, x_n^{(m)}\Big)\Big) = \sum_{1 \leq i_k \leq n,\ 1 \leq k \leq m} a_{i_1 \cdots i_m}\ x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

for some $a_{i_1 \cdots i_m} \in \mathbb{R}$. *Then*

$$\|T\| = \max\{|a_{i_1 \cdots i_m}| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\}.$$

Proof. Let $M := \max\{|a_{i_1 \cdots i_m}| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\}$. Let $\Big(x_1^{(k)}, \ldots, x_n^{(k)}\Big) \in$

$S_{l_1^n}$ for $1 \le k \le m$. It follows that

$$\begin{aligned} & \Big|T\Big((x_1^{(1)}, \ldots, x_n^{(1)}), \ldots, (x_1^{(m)}, \ldots, x_n^{(m)})\Big)\Big| \\ \le & \sum_{1\le i_k\le n,\ 1\le k\le m} |a_{i_1\cdots i_m}|\ |x_{i_1}^{(1)}|\cdots|x_{i_m}^{(m)}| \\ \le & M \sum_{1\le i_k\le n,\ 1\le k\le m} |x_{i_1}^{(1)}|\cdots|x_{i_m}^{(m)}| \\ = & M\Big(\sum_{1\le j\le n} |x_j^{(1)}|\Big)\cdots\Big(\sum_{1\le j\le n} |x_j^{(m)}|\Big) = M \\ = & \max\Big\{\Big|T(e_{i_1}, \ldots, e_{i_m})\Big| : 1 \le i_k \le n,\ 1 \le k \le m\Big\} \\ \le & \|T\|. \end{aligned}$$

Therefore, $\|T\| = M$. □

Theorem 2. ([10]) *Let $n, m \ge 2$. Let $T \in \mathcal{L}(^m l_1^n)$ be the same as in Theorem A. Suppose that $\Big(t_1^{(1)}, \ldots, t_n^{(1)}\Big), \ldots, \Big(t_1^{(m)}, \ldots, t_n^{(m)}\Big) \in \mathrm{Norm}(T)$. If $|a_{i_1'\cdots i_m'}| < \|T\|$ for $1 \le i_k' \le n,\ 1 \le k \le m$, then $t_{i_1'}^{(1)} \cdots t_{i_m'}^{(m)} = 0$.*

Proof. We present a different proof that given in [10].

Let $\delta > 0$ be such that $|a_{i_1'\cdots i_m'}| + \delta < \|T\|$. We define $T_\pm \in \mathcal{L}(^m l_1^n)$ be such that

$$\begin{aligned} & T_\pm\Big(\Big(x_1^{(1)}, \ldots, x_n^{(1)}\Big), \ldots, \Big(x_1^{(m)}, \ldots, x_n^{(m)}\Big)\Big) = \\ & T\Big(\Big(x_1^{(1)}, \ldots, x_n^{(1)}\Big), \ldots, \Big(x_1^{(m)}, \ldots, x_n^{(m)}\Big)\Big) \pm \delta x_{i_1'}^{(1)} \cdots x_{i_m'}^{(m)}. \end{aligned}$$

By Theorem 1, $\|T_\pm\| = 1$. It follows that

$$\begin{aligned} \|T\| \ \ge\ & \Big|T_\pm\Big(\Big(t_1^{(1)}, \ldots, t_n^{(1)}\Big), \ldots, \Big(t_1^{(m)}, \ldots, t_n^{(m)}\Big)\Big)\Big| \\ =\ & \Big|T\Big(\Big(t_1^{(1)}, \ldots, t_n^{(1)}\Big), \ldots, \Big(t_1^{(m)}, \ldots, t_n^{(m)}\Big)\Big)\Big| + \delta\Big|t_{i_1'}^{(1)} \cdots t_{i_m'}^{(m)}\Big| \\ =\ & \|T\| + \delta\Big|t_{i_1'}^{(1)} \cdots t_{i_m'}^{(m)}\Big|, \end{aligned}$$

which implies that $t_{i_1'}^{(1)} \cdots t_{i_m'}^{(m)} = 0$. □

Lemma 1. *Let $(x, y) \in S_{l_1^2}$.*

(1) If $|x \pm y| = 1$, then $|x| = 1$ or $|y| = 1$.

(2) Let $a, b \in \mathbb{R}$ be such that $0 < |x| < 1, |a| \le 1$ and $|b| \le 1$. If $1 = |xa + yb|$, then $|a| = |b| = 1$ and sign$(xy) =$ sign(ab).

Proof. (1). Without loss of generality we may assume that $|x| \ge |y|$. Since $|x| - |y| = 1 = |x| + |y|$, $|x| = 1, y = 0$.

(2). *Claim 1.* $|a| = |b| = 1$.

Assume the contrary. Then $|a| < 1$ or $|b| < 1$. Notice that $|x| > 0$ and $|y| > 0$. It follows that

$$1 = |xa + yb| \leq |x||a| + |y||b| < |x| + |y| = 1,$$

which is a contradiction. Thus the claim 1 holds.

Claim 2. $\text{sign}(xy) = \text{sign}(ab)$.

Assume the contrary. Then $\text{sign}(xy) = -\text{sign}(ab)$. It follows that

$$\begin{aligned} 1 &= |xa + yb| = \Big|x \text{ sign}(a) + y \text{ sign}(b)\Big| \\ &= \Big|\text{sign}(y) \text{ sign}(a)(x \text{ sign}(a) + y \text{ sign}(b))\Big| \\ &= \Big||x| \text{ sign}(xy) + |y| \text{ sign}(ab)\Big| = \Big||x| - |y|\Big| \\ &\leq \max\{|x|, |y|\} < 1 \text{ (because } |x| < 1, |y| < 1), \end{aligned}$$

which is a contradiction. Thus the claim 2 holds. □

Let $T((x_1, y_1),\ (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2 l_1^2)$ for some $a, b, c, d \in \mathbb{R}$. For simplicity we denote $T = (a, b, c, d)$. By Theorem 1, $\|T\| = \max\Big\{|a|, |b|, |c|, |d|\Big\}$. Notice that if $\|T\| = 1$, then $|a| \leq 1, |b| \leq 1, |c| \leq 1, |d| \leq 1$.

Lemma 2. *Let* $T = (a, b, c, d) \in \mathcal{L}(^2 l_1^2)$ *for some* $a, b, c, d \in \mathbb{R}$. *Then, there exists* $T^{'} = (a^*, b^*, c^*, d^*) \in \mathcal{L}(^2 l_1^2)$ *such that* $a^*, b^*, c^*, d^* \in \{\pm a, \pm b, \pm c, \pm d\}$ *with* $a^* \geq |b^*|$ *and* $a^* \geq c^* \geq d^* \geq 0$ *and* $\|T\| = \|T^{'}\|$.

Proof. If $a < 0$, taking $-T$, we assume that $a \geq 0$. If $|b| > a$, we define $T_1 = (|b|, \text{sign}(b)a, \text{sign}(b)d, \text{sign}(b)c)$. Then, $\|T_1\| = \|T\|$. Hence, we may assume that $a \geq |b|$. If $c < 0$, we define $T_2 = (a, -b, -c, d)$. Then, $\|T_2\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq 0$. If $d < 0$, we define $T_3 = (a, -b, c, -d)$. Then, $\|T_3\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq 0, d \geq 0$. If $c < d$, we define $T_4 = (a, b, d, c)$. Then, $\|T_4\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq d \geq 0$. If $a < c, b \geq 0$, we define $T_5 = (c, d, a, b)$. Then, $\|T_5\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq d$. If $a < c, b < 0$, we define $T_6 = (c, -d, a, -b)$. Then, $\|T_6\| = \|T\|$. Hence, we may assume that $a \geq |b|$ and $a \geq c \geq d \geq 0$. Therefore, we can find a bilinear form $T^{'}$ which satisfies the conditions of the lemma. □

Let $n \geq 2$ and $\mathcal{W} \subseteq S_{l_1^n} \times S_{l_1^n}$. We denote

$$\text{Sym}(\mathcal{W}) \ := \ \Big\{(X, Y), (Y, X) : (X, Y) \in \mathcal{W}\Big\}.$$

We are in position to prove the main result in Section 2.

Theorem 3. *Let $T = (a, b, c, d) \in \mathcal{L}(^2 l_1^2)$ be such that $\|T\| = 1$ with $a \geq |b|$ and $a \geq c \geq d \geq 0$. Then we have the following:*

Case 1. *If $1 = a > |b|, 1 > c \geq d$, then*

$$\text{Norm}(T) = \{((\pm 1, 0),\ (\pm 1, 0))\}.$$

Case 2. $(1 = a = |b|, 1 > c \geq d)$ *or* $(1 = a > |b|, 1 = c > d)$

If $1 = a = |b|, 1 > c \geq d$, then

$$\text{Norm}(T) = \{((\pm 1, 0),\ (\pm 1, 0)), ((0, \pm 1),\ (0, \pm 1))\}.$$

If $1 = a > |b|, 1 = c > d$, then

$$\text{Norm}(T) = \{((\pm 1, 0),\ \pm (t, 1-t)) : 0 \leq t \leq 1\}.$$

Case 3. $(1 = a = c = d > |b|)$ *or* $(1 = a = |b|, 1 = c > d)$

If $1 = a = |b|, 1 = c > d$, then

$$\text{Norm}(T) = \{(\pm(1, 0),\ \pm (t, 1-t)), (\pm(t, b(1-t)), \pm(0, 1)) : 0 \leq t \leq 1\}.$$

If $1 = a = c = d > |b|$, then

$$\text{Norm}(T) = \text{Sym}\Big((\pm(1, 0),\ \pm (t, 1-t)) : 0 \leq t \leq 1\}\Big).$$

Case 4. $1 = a = |b| = c = d$

If $b = 1$, then

$$\text{Norm}(T) = \{(\pm(t, 1-t),\ \pm (s, 1-s)) : 0 \leq t, s \leq 1\}.$$

If $b = -1$, then

$$\text{Norm}(T) = \text{Sym}\Big(\{(\pm(0, 1),\ \pm (t, t-1)), (\pm(1, 0),\ \pm (t, 1-t)) : 0 \leq t \leq 1\}\Big).$$

Proof. Let $\Big((x_1, y_1), (x_2, y_2)\Big) \in \text{Norm}(T)$. Without loss of generality we may assume that $x_j \geq 0$ for all $j = 1, 2$. Since $1 = \|T\| = \max\{a, |b|, c, d\}$, $a = 1, |b| = 1, c = 1$ or $d = 1$. Thus, we consider four cases.

Case 1. $1 = a > |b|, 1 > c \geq d$

Claim. $\text{Norm}(T) = \{((\pm 1, 0),\ (\pm 1, 0))\}$.

By Theorem 2, $b = c = d = 0$. Thus

$$1 = |T((x_1, y_1), (x_2, y_2))| = |x_1||x_2|,$$

which shows that $\Big((x_1, y_1), (x_2, y_2)\Big) = ((1, 0),\ (1, 0))$. Thus the claim holds.

Case 2. $(1 = a = |b|, 1 > c \geq d)$ or $(1 = a > |b|, 1 = c > d)$

Suppose that $1 = a = |b|, 1 > c \geq d$.

Claim. $\text{Norm}(T) = \{((\pm 1, 0),\ (\pm 1, 0)), ((0, \pm 1),\ (0, \pm 1))\}$.

By Theorem 2, $c = d = 0$. Thus

$$1 \ = \ |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 + by_1y_2| = |x_1||x_2| + |y_1||y_2|.$$

We will show that $x_1 = 0$ or 1. Suppose not. Then $0 < x_1 < 1$. By Lemma 1(2), $|x_2| = |y_2| = 1$, which is a contradiction. Thus $x_1 = 0$ or 1.

If $x_1 = 0$, then $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big((0, \pm 1), (0, \pm 1)\Big)$.

If $x_1 = 1$, then $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big((1, 0), (1, 0)\Big)$. Thus the claim holds.

Suppose that $1 = a > |b|, 1 = c > d$.

Claim. $\text{Norm}(T) = \{((\pm 1, 0),\ \pm (t, 1 - t)) : 0 \leq t \leq 1\}$.

By Theorem 2, $b = d = 0$. Thus

$$1 \ = \ |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 + x_1y_2| = |x_1||x_2 + y_2|.$$

Thus $x_1 = |x_2 + y_2| = 1$. Thus $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big((1, 0), (t, 1 - t)\Big)$ for some $0 \leq t \leq 1$. Thus the claim holds.

Case 3. $(1 = a = c = d > |b|)$ or $(1 = a = |b|, 1 = c > d)$

Suppose that $1 = a = c = d > |b|$.

Claim. $\text{Norm}(T) = \{(\pm(1, 0),\ \pm (t, 1 - t)), (\pm(t, 1 - t), \pm(1, 0)) : 0 \leq t \leq 1\}$.

By Theorem 2, $b = 0$. Thus

$$1 \ = \ |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 + x_1y_2 + x_2y_1| = |x_1||x_2 + y_2| + |y_1||x_2|.$$

If $x_1 = 0$, then $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big((0, \pm 1), (1, 0)\Big)$.

If $x_1 = 1$, then $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big((1, 0), \pm(t, 1 - t)\Big)$ for some $0 \leq t \leq 1$.

Let $0 < x_1 < 1$. By Lemma 2(2), $|x_2 + y_2| = |x_2| = 1$, so $x_2 = 1, y_2 = 0, |x_1 + y_1| = 1$. Thus $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big(\pm(t, 1 - t), (1, 0)\Big)$ for some $0 \leq t \leq 1$. Thus the claim holds.

Suppose that $1 = a = |b|, 1 = c > d$.

Claim. If $|b| = 1$, then $\text{Norm}(T) = \{(\pm(1, 0),\ \pm (t, 1 - t)), (\pm(t, b(1 - t)), \pm(0, 1)) : 0 \leq t \leq 1\}$.

By Theorem 2, $d = 0$. Thus

$$1 \ = \ |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 + by_1y_2 + x_1y_2| = |x_1||x_2 + y_2| + |y_1||y_2|.$$

If $x_1 = 0$, then $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big((0, \pm 1), (1, \pm 0)\Big)$.

If $x_1 = 1$, then $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big((1, 0), \pm(t, 1-t)\Big)$ for some $0 \le t \le 1$.

Let $0 < x_1 < 1$. By Lemma 1(2), $|x_2 + y_2| = |y_2| = 1$, so $x_2 = 0, y_2 = \pm 1, |x_1 + by_1| = 1$. Thus $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big(\pm(t, b(1-t)), (0, \pm 1)\Big)$ for some $0 \le t \le 1$. Thus the claim holds.

Case 4. $1 = a = |b| = c = d$

It is obvious that if $b = 1$, then

$$\text{Norm}(T) = \{(\pm(t, 1-t), \ \pm(s, 1-s)) : 0 \le t, s \le 1\}.$$

Suppose that $b = -1$.

Claim.

$$\text{Norm}(T) = \text{Sym}\Big(\{(\pm(0, 1), \ \pm(t, t-1)), (\pm(1, 0), \ \pm(t, 1-t)) : 0 \le t \le 1\}\Big).$$

Notice that

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 - y_1y_2 + x_1y_2 + x_2y_1| \\ &= |x_1|\ |x_2 + y_2| + |y_1|\ |x_2 - y_2|. \end{aligned}$$

If $x_1 = 0$ or 1, then $|x_2 - y_2| = 1$ and $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big((0, \pm 1), \pm(t, t-1)\Big)$ for some $0 \le t \le 1$.

Let $0 < x_1 < 1$. By Lemma 1(2), $|x_2 \pm y_2| = 1$, so $x_2 = 0$ or 1.

If $x_2 = 0$, then $|x_1 - y_1| = 1$ and $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big(\pm(t, t-1), (0, \pm 1)\Big)$ for some $0 \le t \le 1$.

If $x_2 = 1$, then $|x_1 + y_1| = 1$ and $\Big((x_1, y_1), (x_2, y_2)\Big) = \Big(\pm(t, 1-t), (1, 0)\Big)$ for some $0 \le t \le 1$. Thus the claim holds. We complete the proof. □

Corollary 1. *Let* $T, S \in \mathcal{L}(^2 l_1^2)$ *with* $\|T\| = \|S\|$. *If* $\text{Norm}(T) = \text{Norm}(S)$, *then* $T = \pm S$.

Proof. It follows from Theorem 3. □

3 The norming sets of $\mathcal{L}_s(^2 l_1^3)$

Let $T((x_1, y_1, z_1), \ (x_2, y_2, z_2)) = ax_1x_2 + by_1y_2 + cz_1z_2 + d(x_1y_2 + x_2y_1) + e(x_1z_2 + x_2z_1) + f(y_1z_2 + y_2z_1) \in \mathcal{L}_s(^2 l_1^3)$ for some $a, b, c, d, e, f \in \mathbb{R}$. For simplicity we denote $T = (a, b, c, d, e, f)$. By Theorem 1, $\|T\| = \max\Big\{|a|, |b|, |c|, |d|, |e|, |f|\Big\}$. Notice that if $\|T\| = 1$, then $|a| \le 1, |b| \le 1, |c| \le 1, |d| \le 1, |e| \le 1$ and $|f| \le 1$.

Theorem 4. *Let* $T((x_1,y_1,z_1),\ (x_2,y_2,z_2)) = ax_1x_2 + by_1y_2 + cz_1z_2 + d(x_1y_2 + x_2y_1) + e(x_1z_2 + x_2z_1) + f(y_1z_2 + y_2z_1) \in \mathcal{L}_s(^2l_1^3)$ *for some* $a,b,c,d,e,f \in \mathbb{R}$. *Then there are* $a^*,b^*,c^*,d^*,e^*,f^* \in \{\pm a, \pm b, \pm c, \pm d, \pm e, \pm f\}$ *such that* $a^* \geq |b^*| \geq |c^*| \geq 0, d^* \geq 0, e^* \geq 0$ *and* $\|T\| = \|(a^*,b^*,c^*,d^*,e^*,f^*)\|$.

Proof. If $|b| > |a|$, we let $T_1((x_1,y_1,z_1),\ (x_2,y_2,z_2)) = T((y_1,x_1,z_1),\ (y_2,x_2,z_2))$. Notice that $\|T_1\| = \|T\|$. Hence, we may assume that $|a| \geq |b|$. If $|c| > |a|$, we consider $T_2((x_1,y_1,z_1),\ (x_2,y_2,z_2)) = T((y_1,z_1,x_1),\ (y_2,z_2,x_2))$. Notice that $\|T_2\| = \|T\|$. Thus, we may assume that $|a| \geq |b| \geq |c| \geq 0$. If $a < 0$, we let $T_3 = -T$. Notice that $\|T_3\| = \|T\|$ and $a \geq |b| \geq |c| \geq 0$. If $d < 0$, we let $T_4((x_1,y_1,z_1),\ (x_2,y_2,z_2)) = T((-x_1,y_1,z_1),\ (-x_2,y_2,z_2))$. Notice that $\|T_4\| = \|T\|$. Thus we may assume that $a \geq |b| \geq |c| \geq 0$ and $d \geq 0$. If $e < 0$, we let $T_5((x_1,y_1,z_1),\ (x_2,y_2,z_2)) = T((x_1,y_1,-z_1),\ (x_2,y_2,-z_2))$. Notice that $\|T_5\| = \|T\|$. Thus we may assume that $a \geq |b| \geq |c| \geq 0, d \geq 0, e \geq 0$. We complete the proof. □

Lemma 3. *Let* $n \geq 2$ *and* $a_k^{(j)} \in \mathbb{R}$ *with* $|a_k^{(j)}| \leq 1$ *for* $j,k = 1,\ldots,n$. *Let* $f_j(t_1,\ldots,t_n) = \sum_{1\leq k\leq n} a_k^{(j)} t_k$ *for* $j = 1,\ldots,n$. *Let* $(t_1^{(1)},\ldots,t_n^{(1)}),\ (t_1^{(2)},\ldots,t_n^{(2)}) \in S_{l_1^n}$ *be such that*

$$1 = \Big|\sum_{1\leq j\leq n} t_j^{(1)} f_j(t_1^{(2)},\ldots,t_n^{(2)})\Big|.$$

Then $(|f_j(t_1^{(2)},\ldots,t_n^{(2)})| = 1$ *for all* $j = 1,\ldots,n)$ *or* (*there is* $1 \leq j_0 \leq n$ *such that* $t_{j_0}^{(1)} = 0$).

Proof. Suppose that there is $1 \leq j_0 \leq n$ such that $|f_{j_0}(t_1^{(2)},\ldots,t_n^{(2)})| < 1$. We claim that $t_{j_0}^{(1)} = 0$. Suppose not. It follows that

$$\begin{aligned}
1 &= \Big|\sum_{1\leq j\leq n} t_j^{(1)} f_j(t_1^{(2)},\ldots,t_n^{(2)})\Big| \\
&\leq |t_{j_0}^{(1)}|\ |f_{j_0}(t_1^{(2)},\ldots,t_n^{(2)})| + \sum_{1\leq j\neq j_0\leq n} |t_j^{(1)}|\ |f_j(t_1^{(2)},\ldots,t_n^{(2)})| \\
&< |t_{j_0}^{(1)}| + \sum_{1\leq j\neq j_0\leq n} |t_j^{(1)}|\ |f_j(t_1^{(2)},\ldots,t_n^{(2)})| \\
&\leq |t_{j_0}^{(1)}| + \sum_{1\leq j\neq j_0\leq n} |t_j^{(1)}| \Big(\sum_{1\leq k\leq n} |a_k^{(j)}|\ |t_k^{(2)}|\Big) \\
&\leq |t_{j_0}^{(1)}| + \sum_{1\leq j\neq j_0\leq n} |t_j^{(1)}| = 1,
\end{aligned}$$

which is a contradiction. Thus, $t_{j_0}^{(1)} = 0$. □

Lemma 4. *Let* $T((x_1,y_1,z_1),\ (x_2,y_2,z_2)) = x_1x_2 + by_1y_2 + cz_1z_2 + (x_1y_2 + x_2y_1) + (x_1z_2 + x_2z_1) - (y_1z_2 + y_2z_1) \in \mathcal{L}_s(^2l_1^3)$ *for some* $|b| = |c| = 1$. *Let* $\Big((x_1,y_1,z_1),\ (x_2,y_2,z_2)\Big) \in \mathrm{Norm}(T)$. *Then* $x_1y_1z_1 = 0$ *or* $x_2y_2z_2 = 0$.

Proof. Assume the contrary. Then $x_j y_j z_j \neq 0$ for all $j = 1, 2$.

It follows that

$$\begin{aligned}(*)\quad 1 &= |T((x_1, y_1, z_1),\ (x_2, y_2, z_2))| \\ &= |x_1(x_2 + y_2 + z_2) + y_1(x_2 + by_2 - z_2) + z_1(x_2 - y_2 + cz_2)| \\ &= |x_1|\ |x_2 + y_2 + z_2| + |y_1|\ |x_2 + by_2 - z_2| + |z_1|\ |x_2 - y_2 + cz_2|.\end{aligned}$$

Since $x_1y_1z_1 \neq 0$ and $(x_1, y_1, z_1) \in S_{l_1^3}$, $0 < |x_1| < 1, 0 < |y_1| < 1$ and $0 < |z_1| < 1$.

By Lemma 3, $|x_2 + y_2 + z_2| = |x_2 + by_2 - z_2| = |x_2 - y_2 + cz_2| = 1$.

If $b = 1$, then by Lemma 1(1), $|x_2 + y_2| = 1$ or $|z_2| = 1$.

Suppose that $|x_2 + y_2| = 1$. Then $z_2 = 0$. Thus $x_2y_2z_2 = 0$. This is a contradiction.

Let $|z_2| = 1$. Then $x_2 = y_2 = 0$. Thus $x_2y_2z_2 = 0$. This is a contradiction.

If $b = -1$, then by Lemma 1(1), $|y_2 + z_2| = 1$ or $|x_2| = 1$. Thus $x_2y_2z_2 = 0$. This is a contradiction. We complete the proof. □

Notice that for $T = (a, b, c, d, e, f) \in \mathcal{L}_s(^2 l_1^3)$,

$$\begin{aligned}&\mathrm{Norm}((a, c, b, e, d, f)) \\ =\ &\Big\{\Big((x_1, z_1, y_1), (x_2, z_2, y_2)\Big) : \Big((x_1, y_1, z_1), (x_2, y_2, z_2)\Big) \in \mathrm{Norm}(T)\Big\}.\end{aligned}$$

We are in position to prove the main result in Section 3.

Theorem 5. *Let* $T((x_1, y_1, z_1),\ (x_2, y_2, z_2)) = ax_1x_2 + by_1y_2 + cz_1z_2 + d(x_1y_2 + x_2y_1) + e(x_1z_2 + x_2z_1) + f(y_1z_2 + y_2z_1) \in \mathcal{L}_s(^2 l_1^3)$ *be such that* $\|T\| = 1$ *for some* $a \geq |b| \geq |c| \geq 0, d \geq 0, e \geq 0, f \in [-1, 1]$. *Then we have the following six cases: let* $S = \{a, b, c, d, e, f\}$ *and* $M = \{x \in S :\ |x| = 1\}$.

Case 1. $|M| = 1$.

If $a = 1$ *and* $|b| < 1, d < 1, e < 1, |f| < 1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1, 0, 0), \pm(1, 0, 0)\Big)\Big\}.$$

If $d = 1$ *and* $a < 1, e < 1, |f| < 1$, *then*

$$\mathrm{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1, 0, 0), \pm(0, 1, 0)\Big)\Big\}\Big).$$

If $e = 1$ *and* $a < 1, d < 1, |f| < 1$, *then*

$$\mathrm{Norm}(T) = Sym\Big\{\Big(\pm(1, 0, 0), \pm(0, 0, 1)\Big)\Big\}.$$

If $|f| = 1$ *and* $a < 1, d < 1, e < 1$, *then*

$$\mathrm{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0, 1, 0), \pm(0, 0, 1)\Big)\Big\}\Big).$$

Case 2. $|M| = 2$.

If $a = |b| = 1$ *and* $|c| < 1, d < 1, e < 1, |f| < 1,$ *then*

$$\text{Norm}(T) = \Big\{\Big(\pm(1,0,0), \pm(1,0,0)\Big), \Big(\pm(0,1,0), \pm(0,1,0)\Big)\Big\}.$$

If $a = d = 1$ *and* $|b| < 1, e < 1, |f| < 1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(t, 1-t, 0)\Big) : 0 \le t \le 1\Big\}\Big).$$

If $a = e = 1$ *and* $|b| < 1, d < 1, |f| < 1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(t, 0, 1-t)\Big) : 0 \le t \le 1\Big\}\Big).$$

If $a = |f| = 1$ *and* $|b| < 1, d < 1, e < 1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(1,0,0)\Big), \Big(\pm(0,1,0), \pm(0,0,1)\Big)\Big\}\Big).$$

If $d = e = 1$ *and* $a < 1, |f| < 1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(0, t, 1-t)\Big) : 0 \le t \le 1\Big\}\Big).$$

If $d = |f| = 1$ *and* $a < 1, e < 1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0,1,0), \pm(t, 0, f(1-t))\Big) : 0 \le t \le 1\Big\}\Big).$$

If $e = |f| = 1$ *and* $a < 1, d < 1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0,0,1), \pm(t, f(1-t), 0)\Big) : 0 \le t \le 1\Big\}\Big).$$

Case 3. $|M| = 3$.

If $a = |b| = |c| = 1$ *and* $d < 1, e < 1, |f| < 1,$ *then*

$$\begin{aligned}\text{Norm}(T) = &\Big\{\Big(\pm(1,0,0), \pm(1,0,0)\Big), \Big(\pm(0,1,0), \pm(0,1,0)\Big), \\ &\Big(\pm(0,0,1), \pm(0,0,1)\Big)\Big\}.\end{aligned}$$

If $a = |b| = d = 1$ *and* $|c| < 1, e < 1, |f| < 1,$ *then*

$$\text{Norm}(T) = Sym\Big\{\Big(\pm(u, b(1-u), 0), \pm(v, 1-v, 0)\Big) : 0 \le u, v \le 1\Big\}.$$

If $a = |b| = e = 1$ *and* $|c| < 1, d < 1, |f| < 1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0,1,0), \pm(0,1,0)\Big), \Big(\pm(1,0,0), \pm(t, 0, 1-t)\Big) : 0 \le t \le 1\Big\}\Big).$$

If $a=|b|=|f|=1$ *and* $|c|<1, d<1, e<1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(1,0,0)\Big), \Big(\pm(0,1,0), \pm(0,bt,f(1-t))\Big) : 0\le t\le 1\Big\}\Big).$$

If $a=d=e=1$ *and* $|b|<1, |f|<1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big) : |r+s+t| = |r|+|s|+|t| = 1\Big\}\Big).$$

If $a=d=|f|=1$ *and* $|b|<1, e<1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(t,1-t,0)\Big), \Big(\pm(0,1,0), \pm(t,0,f(1-t))\Big) : 0\le t\le 1\Big\}\Big).$$

If $a=e=|f|=1$ *and* $|b|<1, d<1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(t,0,1-t)\Big), \Big(\pm(0,0,1), \pm(t,f(1-t),0)\Big) : 0\le t\le 1\Big\}\Big).$$

If $d=e=|f|=1$ *and* $a<1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(0,t,1-t)\Big), \Big(\pm(0,1,0), \pm(t,0,f(1-t))\Big), \Big(\pm(0,0,1), \pm(t,f(1-t),0)\Big) : 0\le t\le 1\Big\}\Big).$$

Case 4. $|M|=4$.

If $a=b=|c|=d=1$ *and* $e<1, |f|<1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0,0,1), \pm(0,0,1)\Big), \Big(\pm(t,1-t,0), \pm(s,1-s,0)\Big) : 0\le t,s\le 1\Big\}\Big).$$

If $a=-b=|c|=d=1$ *and* $e<1, |f|<1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0,0,1), \pm(0,0,1)\Big), \Big(\pm(1,0,0), \pm(t,1-t,0)\Big), \Big(\pm(0,1,0), \pm(t,-(1-t),0)\Big) : 0\le t\le 1\Big\}\Big).$$

If $a=b=|c|=e=1$ *and* $d<1, |f|<1,$ *then*

$$\text{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0,1,0), \pm(0,1,0)\Big), \Big(\pm(t,0,1-t), \pm(s,0,c(1-s))\Big) : 0\le t,s\le 1\Big\}\Big).$$

If $a = -b = c = e = 1$ *and* $d < 1, |f| < 1,$ *then*

$$\mathrm{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0,1,0), \pm(0,1,0)\Big), \Big(\pm(t,0,1-t), \pm(s,0,1-s)\Big) : 0 \le t, s \le 1\Big\}\Big).$$

If $a = -b = -c = e = 1$ *and* $d < 1, |f| < 1,$ *then*

$$\mathrm{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0,1,0), \pm(0,1,0)\Big), \Big(\pm(1,0,0), \pm(t,0,1-t)\Big), \Big(\pm(0,0,1), \pm(t,0,-(1-t))\Big) : 0 \le t \le 1\Big\}\Big).$$

If $a = b = |c| = |f| = 1$ *and* $d < 1, e < 1,$ *then*

$$\mathrm{Norm}(T) Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(1,0,0)\Big), \Big(\pm(0,t,cf(1-t)), \pm(0,s,cf(1-s))\Big) : 0 \le t, s \le 1\Big\}\Big).$$

If $a = b = -c = |f| = 1$ *and* $d < 1, e < 1,$ *then*

$$\mathrm{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(1,0,0)\Big), \Big(\pm(0,1,0), \pm(0,t,f(1-t))\Big), \Big(\pm(0,0,1), \pm(0,t,-f(1-t))\Big) : 0 \le t \le 1\Big\}\Big).$$

If $a = -b = c = |f| = 1$ *and* $d < 1, e < 1,$ *then*

$$\mathrm{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(1,0,0)\Big), \Big(\pm(0,0,1), \pm(0,t,f(1-t))\Big), \Big(\pm(0,1,0), \pm(0,t,-f(1-t))\Big) : 0 \le t \le 1\Big\}\Big).$$

If $a = b = d = e = 1$ *and* $|c| < 1, |f| < 1,$ *then*

$$\mathrm{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(u,1-u,0), \pm(v,1-v,0)\Big) : |r+s+t| = |r|+|s|+|t| = 1, 0 \le u, v \le 1\Big\}\Big).$$

If $a = -b = d = e = 1$ *and* $|c| < 1, |f| < 1,$ *then*

$$\mathrm{Norm}(T) = Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(0,1,0), \pm(u,-(1-u),0)\Big) : |r+s+t| = |r|+|s|+|t| = 1, 0 \le u \le 1\Big\}\Big).$$

If $a = b = d = |f| = 1$ *and* $|c| < 1, e < 1,$ *then*

$$\mathrm{Norm}(T) = Sym\Big(\Big\{\Big(\pm(0,1,0), \pm(r,s,ft)\Big), \Big(\pm(1,0,0), \pm(u,1-u,0)\Big) : |r+s+t| = |r|+|s|+|t| = 1, 0 \le u \le 1\Big\}\Big).$$

If $a=-b=d=|f|=1$ *and* $|c|<1, e<1,$ *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,1,0),\pm(r,-s,ft)\Big),\Big(\pm(u,1-u,0),\pm(v,1-v,0)\Big):\\ &\quad |r+s+t|=|r|+|s|+|t|=1, 0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=|b|=e=|f|=1$ *and* $|c|<1, d<1,$ *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(u,0,1-u)\Big),\Big(\pm(0,1,0),\pm(0,bu,f(1-u))\Big),\\ &\quad \Big(\pm(0,0,1),\pm(u,f(1-u),0)\Big): 0\le u\le 1\Big\}\Big).\end{aligned}$$

If $a=d=e=f=1$ *and* $|b|<1,$ *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(u,0,1-u),\pm(v,1-v,0)\Big):\\ &\quad |r+s+t|=|r|+|s|+|t|=1, 0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=d=e=-f=1$ *and* $|b|<1,$ *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,1,0),\pm(u,0,-(1-u))\Big),\\ &\quad \Big(\pm(0,0,1),\pm(u,-(1-u),0)\Big): |r+s+t|=|r|+|s|+|t|=1,\\ &\quad 0\le u\le 1\Big\}\Big).\end{aligned}$$

Case 5. $|M|=5$.

If $a=b=c=d=e=1$ *and* $|f|<1,$ *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(u,1-u,0),\pm(v,1-v,0)\Big),\\ &\quad \Big(\pm(u,0,1-u),\pm(v,0,1-v)\Big): |r+s+t|=|r|+|s|+|t|=1,\\ &\quad 0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=-b=-c=d=e=1$ *and* $|f|<1,$ *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,1,0),\pm(u,-(1-u),0)\Big),\\ &\quad \Big(\pm(0,0,1),\pm(u,0,-(1-u))\Big): |r+s+t|=|r|+|s|+|t|=1,\\ &\quad 0\le u\le 1\Big\}\Big).\end{aligned}$$

If $a=b=-c=d=e=1$ *and* $|f|<1,$ *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,0,1),\pm(u,0,-(1-u))\Big),\\ &\quad \Big(\pm(u,1-u,0),\pm(v,1-v,0)\Big): |r+s+t|=|r|+|s|+|t|=1,\\ &\quad 0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=-b=c=d=e=1$ *and* $|f|<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,1,0),\pm(u,-(1-u),0)\Big),\\ &\Big(\pm(u,0,1-u),\pm(v,0,1-v)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=b=d=e=f=1$ *and* $|c|<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(u,1-u,0),\pm(r,s,t)\Big),\Big(\pm(0,0,1),\pm(u,1-u,0)\Big):\\ &|r+s+t|=|r|+|s|+|t|=1,0\le u\le 1\Big\}\Big).\end{aligned}$$

If $a=-b=d=e=f=1$ *and* $|c|<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,1,0),\pm(r,-s,t)\Big),\\ &\Big(\pm(0,0,1),\pm(u,1-u,0)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\le u\le 1\Big\}\Big).\end{aligned}$$

If $a=b=d=e=-f=1$ *and* $|c|<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,1,0),\pm(r,s,-t)\Big),\\ &\Big(\pm(0,0,1),\pm(u,-(1-u),0)\Big),\Big(\pm(u,1-u,0),\pm(v,1-v,0)\Big):\\ &|r+s+t|=|r|+|s|+|t|=1,0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=-b=d=e=-f=1$ *and* $|c|<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,1,0),\pm(r,-s,-t)\Big),\\ &\Big(\pm(0,u,1-u),\pm(v,-(1-v),0)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=b=c=d=f=1$ *and* $e<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,1,0),\pm(r,s,t)\Big),\Big(\pm(0,u,1-u),\pm(0,v,1-v)\Big),\\ &\Big(\pm(u,1-u,0),\pm(v,1-v,0)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=b=c=d=-f=1$ *and* $e<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,1,0),\pm(r,s,-t)\Big),\Big(\pm(u,1-u,0),\pm(v,1-v,0)\Big),\\ &\Big(\pm(0,u,-(1-u)),\pm(0,v,-(1-v))\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=-b=-c=d=f=1$ *and* $e<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,1,0),\pm(r,-s,t)\Big),\Big(\pm(1,0,0),\pm(u,1-u,0)\Big),\\ &\quad \Big(\pm(0,u,-(1-u)),\pm(0,v,-(1-v))\Big):\\ &\qquad |r+s+t|=|r|+|s|+|t|=1, 0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=-b=-c=d=-f=1$ *and* $e<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,1,0),\pm(r,-s,-t)\Big),\Big(\pm(1,0,0),\pm(u,1-u,0)\Big),\\ &\quad \Big(\pm(0,u,1-u),\pm(0,v,1-v)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &\quad 0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=b=-c=d=f=1$ *and* $e<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,1,0),\pm(r,s,t)\Big),\Big(\pm(0,0,1),\pm(0,u,-(1-u))\Big),\\ &\quad \Big(\pm(u,1-u,0),\pm(v,1-v,0)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &\quad 0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=b=-c=d=-f=1$ *and* $e<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,1,0),\pm(r,s,-t)\Big),\Big(\pm(0,0,1),\pm(0,u,1-u)\Big),\\ &\quad \Big(\pm(u,1-u,0),\pm(v,1-v,0)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &\quad 0\le u,v\le 1\Big\}\Big).\end{aligned}$$

If $a=-b=c=d=f=1$ *and* $e<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,1,0),\pm(r,-s,t)\Big),\Big(\pm(1,0,0),\pm(u,1-u,0)\Big),\\ &\quad \Big(\pm(0,0,1),\pm(0,u,1-u)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &\quad 0\le u\le 1\Big\}\Big).\end{aligned}$$

If $a=-b=c=d=-f=1$ *and* $e<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,1,0),\pm(r,-s,-t)\Big),\Big(\pm(1,0,0),\pm(u,1-u,0)\Big),\\ &\quad \Big(\pm(0,0,1),\pm(0,u,-(1-u))\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &\quad 0\le u\le 1\Big\}\Big).\end{aligned}$$

If $a = b = c = e = f = 1$ *and* $d < 1$, *then*

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,0,1), \pm(r,s,t)\Big), \Big(\pm(1,0,0), \pm(u,0,1-u)\Big),\\ &\quad \Big(\pm(0,u,1-u), \pm(0,v,1-v)\Big) : |r+s+t| = |r|+|s|+|t| = 1,\\ &\quad 0 \le u, v \le 1\Big\}\Big).\end{aligned}$$

If $a = b = c = e = -f = 1$ *and* $d < 1$, *then*

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,0,1), \pm(r,-s,t)\Big), \Big(\pm(1,0,0), \pm(u,0,1-u)\Big),\\ &\quad \Big(\pm(0,u,-(1-u)), \pm(0,v,-(1-v))\Big) :\\ &\qquad |r+s+t| = |r|+|s|+|t| = 1, 0 \le u, v \le 1\Big\}\Big).\end{aligned}$$

If $a = -b = -c = e = f = 1$ *and* $d < 1$, *then*

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,0,1), \pm(r,s,-t)\Big), \Big(\pm(1,0,0), \pm(u,0,1-u)\Big),\\ &\quad \Big(\pm(0,u,-(1-u)), \pm(0,v,-(1-v))\Big) :\\ &\qquad |r+s+t| = |r|+|s|+|t| = 1, 0 \le u, v \le 1\Big\}\Big).\end{aligned}$$

If $a = -b = -c = e = -f = 1$ *and* $d < 1$, *then*

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,0,1), \pm(r,-s,-t)\Big), \Big(\pm(1,0,0), \pm(u,0,1-u)\Big),\\ &\quad \Big(\pm(0,u,1-u), \pm(0,v,1-v)\Big) : |r+s+t| = |r|+|s|+|t| = 1,\\ &\quad 0 \le u, v \le 1\Big\}\Big).\end{aligned}$$

If $a = b = -c = e = f = 1$ *and* $d < 1$, *then*

$$\begin{aligned}\text{Norm}(T &= Sym\Big(\Big\{\Big(\pm(0,0,1), \pm(r,s,-t)\Big), \Big(\pm(0,1,0), \pm(0,u,-(1-u))\Big),\\ &\quad \Big(\pm(1,0,0), \pm(u,0,1-u)\Big) : |r+s+t| = |r|+|s|+|t| = 1,\\ &\quad 0 \le u \le 1\Big\}\Big).\end{aligned}$$

If $a = b = -c = e = -f = 1$ *and* $d < 1$, *then*

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,0,1), \pm(r,-s,-t)\Big), \Big(\pm(0,1,0), \pm(0,u,-(1-u))\Big),\\ &\quad \Big(\pm(1,0,0), \pm(u,0,1-u)\Big) : |r+s+t| = |r|+|s|+|t| = 1,\\ &\quad 0 \le u \le 1\Big\}\Big).\end{aligned}$$

If $a=-b=c=e=f=1$ and $d<1$, then

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,0,1),\pm(r,s,t)\Big),\Big(\pm(0,1,0),\pm(0,u,-(1-u))\Big),\\ &\Big(\pm(1,0,0),\pm(u,0,1-u)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\leq u\leq 1\Big\}\Big).\end{aligned}$$

If $a=-b=c=e=-f=1$ and $d<1$, then

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(0,0,1),\pm(r,-s,t)\Big),\Big(\pm(0,1,0),\pm(0,u,1-u)\Big),\\ &\Big(\pm(1,0,0),\pm(u,0,1-u)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\leq u\leq 1\Big\}\Big).\end{aligned}$$

Case 6. $|M|=6$.

Suppose that $f=1$.
If $a=b=c=d=e=1$, then

$$\begin{aligned}\text{Norm}(T) &= \Big\{\Big(\pm(r,s,t),\pm(r^{'},s^{'},t^{'})\Big):|r+s+t|=|r|+|s|+|t|\\ &=|r^{'}+s^{'}+t^{'}||r^{'}|+|s^{'}|+|t^{'}|=1\Big\}.\end{aligned}$$

If $a=-b=-c=d=e=1$, then

$$\begin{aligned}\text{Norm}(T) &= Sym\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,1,0),\pm(r,-s,t)\Big),\\ &\Big(\pm(0,0,1),\pm(r,s,-t)\Big),\Big(\pm(0,u,-(1-u)),\pm(0,v,-(1-v))\Big),\\ &\Big(\pm(u,1-u,0),\pm(v,0,1-v)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\leq u,v\leq 1\Big\}.\end{aligned}$$

If $a=b=-c=d=e=1$, then

$$\begin{aligned}\text{Norm}(T) &= Sym\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,0,1),\pm(r,s,-t)\Big),\\ &\Big(\pm(u,1-u,0),\pm(r,s,t)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\leq u\leq 1\Big\}.\end{aligned}$$

If $a=-b=c=d=e=1$, then

$$\begin{aligned}\text{Norm}(T) &= Sym\Big\{\Big(\pm(1,0,0),\pm(r,s,t)\Big),\Big(\pm(0,1,0),\pm(r,-s,t)\Big),\\ &\Big(\pm(u,0,1-u),\pm(r,s,t)\Big):|r+s+t|=|r|+|s|+|t|=1,\\ &0\leq u\leq 1\Big\}.\end{aligned}$$

Suppose that $f = -1$.

If $a = b = c = d = e = 1$, *then*

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(0,1,0), \pm(r,s,-t)\Big),\\ &\Big(\pm(0,0,1), \pm(r,-s,t)\Big), \Big(\pm(0,u,-(1-u)), \pm(0,v,-(1-v))\Big),\\ &\Big(\pm(u,0,1-u), \pm(v,0,1-v)\Big), \Big(\pm(u,1-u,0), \pm(v,1-v,0)\Big) :\\ &|r+s+t| = |r|+|s|+|t| = 1, 0 \le u,v \le 1\Big\}\Big).\end{aligned}$$

If $a = -b = -c = d = e = 1$, *then*

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(0,u,1-u), \pm(r,-s,-t)\Big),\\ &\Big(\pm(u,0,-(1-u)), \pm(0,v,1-v)\Big) : |r+s+t| = |r|+|s|+|t|\\ &= 1, 0 \le u,v \le 1\Big\}\Big).\end{aligned}$$

If $a = b = -c = d = e = 1$, *then*

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(0,1,0), \pm(r,s,-t)\Big),\\ &\Big(\pm(0,0,1), \pm(r,-s,-t)\Big), \Big(\pm(u,0,-(1-u)), \pm(0,v,1-v)\Big),\\ &\Big(\pm(u,1-u,0), \pm(v,1-v,0)\Big) : |r+s+t| = |r|+|s|+|t| = 1,\\ &0 \le u,v \le 1\Big\}\Big).\end{aligned}$$

If $a = -b = c = d = e = 1$, *then*

$$\begin{aligned}\text{Norm}(T) &= Sym\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(0,0,1), \pm(r,-s,t)\Big),\\ &\Big(\pm(0,1,0), \pm(r,-s,-t)\Big), \Big(\pm(u,-(1-u),0), \pm(0,1-v,v)\Big),\\ &\Big(\pm(u,0,1-u), \pm(v,0,1-v)\Big) : |r+s+t| = |r|+|s|+|t| = 1,\\ &0 \le u,v \le 1\Big\}\Big).\end{aligned}$$

Proof. We will give the proof only for one of these cases because the proofs of the others are similar.

Let $((x_1,y_1,z_1),\ (x_2,y_2,z_2)) \in \text{Norm}(T)$. Without loss of generality we may assume that $x_j \ge 0$ for all $j = 1,2$.

Case 1. $|M| = 1$.

We claim that if $|f| = 1$ and $a < 1, b < 1, c < 1, d < 1, e < 1$, then

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\Big(\pm(0,1,0), \pm(0,0,1)\Big)\Big\}\Big).$$

By Theorem 2,

$$1=|T((x_1,y_1,z_1),\ (x_2,y_2,z_2))|=|f(y_1z_2{+}y_2z_1)|=|y_1z_2{+}y_2z_1|=|y_1|\ |z_2|{+}|y_2|\ |z_1|.$$

Claim. $|y_1|=0$ or 1.

Suppose not. Then $0<|y_1|<1$. By Lemma 1(2), $|z_1|=|z_2|=1$. Thus $1=|x_1|+|y_1|+|z_1|>1$, a contradiction. Thus the claim holds.

If $|y_1|=0$, then $|y_2|\ |z_1|=1$, so $|z_1|=|y_2|=1$. Thus $(x_1,y_1,z_1)=(0,0,\pm1)$, $(x_2,y_2,z_2)=(0,\pm1,0)$. Since T is symmetric,

$$\text{Norm}(T)\ =\ \text{Sym}\Big(\Big\{\Big(\pm(0,1,0),\pm(0,0,1)\Big)\Big\}\Big).$$

Case 2. $|M|=2$.

We claim that If $a=|b|=1$ and $|c|<1,d<1,e<1,|f|<1$, then

$$\text{Norm}(T)\ =\ \Big\{\Big(\pm(1,0,0),\pm(1,0,0)\Big),\Big(\pm(0,1,0),\pm(0,1,0)\Big)\Big\}.$$

By Theorem 2,

$$0=z_1z_2,\ 1=|T((x_1,y_1,z_1),\ (x_2,y_2,z_2))|=|x_1x_2+by_1y_2|=|x_1|\ |x_2|+|y_1|\ |y_2|.$$

Let $z_1=0$.

Claim. $x_1=0$ or 1.

Suppose not. Then $0<x_1<1$. Hence $0<|y_1|<1$. By Lemma 1(2), $|x_2|=|y_2|=1$. This is a contradiction. Thus the claim holds.

If $x_1=0$, then $|y_1|\ |y_2|=1$. Thus $(x_1,y_1,z_1)=(0,\pm1,0),(x_2,y_2,z_2)=(0,\pm1,0)$. If $x_1=1$, then $x_2=1$. Thus $(x_1,y_1,z_1)=(1,0,0)=(x_2,y_2,z_2)=(1,0,0)$.

Let $z_2=0$.

Claim. $x_2=0$ or 1.

Suppose not. Then $0<x_2<1$. Hence $0<|y_2|<1$. By Lemma 1(2), $|x_1|=|y_1|=1$. This is a contradiction. Thus the claim holds.

If $x_2=0$, then $|y_1|\ |y_2|=1$. Thus $(x_1,y_1,z_1)=(0,\pm1,0),(x_2,y_2,z_2)=(0,\pm1,0)$. If $x_2=1$, then $x_1=1$. Thus $(x_1,y_1,z_1)=(1,0,0)=(x_2,y_2,z_2)$. Thus,

$$\text{Norm}(T)\ =\ \Big\{\Big(\pm(1,0,0),\pm(1,0,0)\Big),\Big(\pm(0,1,0),\pm(0,1,0)\Big)\Big\}.$$

Case 3. $|M|=3$.

We claim that if $a=|c|=|f|=1$ and $|b|<1,d<1,e<1$, then

$$\begin{aligned}\text{Norm}(T)\ =\ &\text{Sym}\Big(\Big\{\Big(\pm(1,0,0),\pm(1,0,0)\Big),\Big(\pm(0,0,1),\pm(0,ft,c(1-t)\Big):\\&0\le t\le1\Big\}\Big).\end{aligned}$$

By Theorem 2, $0 = y_1y_2$ and

$$\begin{aligned} 1 &= |T((x_1, y_1, z_1),\ (x_2, y_2, z_2))| = |x_1x_2 + cz_1z_2 + f(y_1z_2 + y_2z_1)| \\ &= |x_1|\ |x_2| + |y_1|\ |z_2| + |z_1|\ |fy_2 + cz_2|. \end{aligned}$$

Let $y_1 = 0$. Then

$$1 = |x_1|\ |x_2| + |z_1|\ |fy_2 + cz_2|.$$

We claim that $x_1 = 0, 1$. Suppose not. By Lemma 1(2), $|x_2| = |fy_2 + cz_2| = 1$. This is a contradiction.

If $x_1 = 0$, then $|z_1| = |x_2+fy_2| = 1$. Thus $(x_1, y_1, z_1) = (0, 0, \pm 1)$, $(x_2, y_2, z_2) = (0, ft, c(1-t))$ for some $0 \le t \le 1$. If $x_1 = 1$, then $|z_1| = |x_2 + z_2| = 1$. Thus $(x_1, y_1, z_1) = (1, 0, 0) = (x_2, y_2, z_2)$.

Let $y_2 = 0$.

Then

$$1 = |x_2|\ |x_1| + |z_2|\ |fy_1 + cz_1|.$$

We claim that $x_2 = 0, 1$. Suppose not. By Lemma 1(2), $|x_1| = |fy_1 + cz_1| = 1$. This is a contradiction.

If $x_2 = 0$, then $(x_1, y_1, z_1) = (0, ft, c(1-t))$, $(x_2, y_2, z_2) = (0, 0, \pm 1)$ for some $0 \le t \le 1$. If $x_2 = 1$, then $(x_1, y_1, z_1) = (1, 0, 0) = (x_2, y_2, z_2)$. Thus,

$$\begin{aligned} \text{Norm}(T) &= \text{Sym}\Big(\Big\{\Big(\pm(1,0,0), \pm(1,0,0)\Big), \Big(\pm(0,0,1), \pm(0, ft, c(1-t)\Big) : \\ &\quad 0 \le t \le 1\Big\}\Big). \end{aligned}$$

Case 4. $|M| = 4$.

We claim that if $a = b = |c| = d = 1$ and $e < 1, |f| < 1$, then

$$\begin{aligned} \text{Norm}(T) &= \text{Sym}\Big(\Big\{\Big(\pm(0,0,1), \pm(0,0,1)\Big), \Big(\pm(t, 1-t, 0), \pm(s, 1-s, 0)\Big) : \\ &\quad 0 \le t, s \le 1\Big\}\Big). \end{aligned}$$

By Theorem 2, $0 = x_1z_2 = x_2z_1 = y_1z_2 = y_2z_1$ and

$$1 = |T((x_1, y_1, z_1),\ (x_2, y_2, z_2))| = |x_1x_2 + y_1y_2 + cz_1z_2 + (x_1y_2 + x_2y_1)|.$$

Let $y_1 = 0$. Then

$$1 = |x_1x_2 + cz_1z_2 + x_1y_2| = |x_1|\ |x_2 + y_2| + |z_1|\ |z_2|.$$

Note that $x_1 = 0, 1$ or $0 < x_1 < 1$. We claim that $x_1 = 0, 1$. Suppose not. By Lemma 1(2), $|x_2 + y_2| = |z_2| = 1$, which is impossible because $(x_2, y_2, z_2) \in S_{l_1^3}$. Thus $x_1 = 0, 1$.

If $x_1 = 0$, then $|z_1| = |z_2| = 1$. Thus $(x_1, y_1, z_1) = (0, 0, \pm 1)$, $(x_2, y_2, z_2) = (0, 0, \pm 1)$. If $x_1 = 1$, then $|z_1| = |x_2 + y_2| = 1$. Thus $(x_1, y_1, z_1) = (1, 0, 0)$, $(x_2, y_2, z_2) = (t, 1-t, 0)$ for some $0 \le t \le 1$.

Let $z_2 = 0$. Then

$$1 = |x_1x_2 + y_1y_2 + cz_1z_2 + (x_1y_2 + x_2y_1)| = |x_1 + y_1|\ |x_2 + y_2|.$$

Thus $|x_1 + y_1| = |x_2 + y_2| = 1$. Hence, $(x_1, y_1, z_1) = (t, 1 - t, 0),\ (x_2, y_2, z_2) = (s, 1 - s, 0)$ for some $0 \le t, s \le 1$.

Thus,

$$\begin{aligned}\text{Norm}(T) &= \text{Sym}\Big(\Big\{\Big(\pm(0,0,1), \pm(0,0,1)\Big), \Big(\pm(t, 1-t, 0), \pm(s, 1-s, 0)\Big) : \\ &\quad 0 \le t, s \le 1\Big\}\Big).\end{aligned}$$

Case 5. $|M| = 5$.

We claim that if $a = b = c = d = e = 1$ and $|f| < 1$, then

$$\begin{aligned}\text{Norm}(T) &= \text{Sym}\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(u, 1-u, 0), \pm(v, 1-v, 0)\Big), \\ &\quad \Big(\pm(u, 0, 1-u), \pm(v, 0, 1-v)\Big) : |r+s+t| = |r| + |s| + |t| = 1, \\ &\quad 0 \le u, v \le 1\Big\}\Big).\end{aligned}$$

By Theorem 2, $0 = y_1z_2 = y_2z_1$ and

$$1 = |T((x_1, y_1, z_1),\ (x_2, y_2, z_2))| = |x_1x_2 + y_1y_2 + z_1z_2 + (x_1y_2 + x_2y_1) + (x_1z_2 + x_2z_1)|.$$

Let $y_1 = 0$. Then

$$1 = |x_1x_2 + z_1z_2 + x_1y_2 + x_1z_2 + x_2z_{)}| = |x_1|\ |x_2 + y_2 + z_2| + |z_1|\ |x_2 + z_2|.$$

Note that $x_1 = 0, 1$ or $0 < x_1 < 1$.

If $x_1 = 0$, then $|z_1| = |x_2 + z_2| = 1$. Thus $(x_1, y_1, z_1) = (0, 0, \pm 1, 0),\ (x_2, y_2, z_2) = (u, 0, 1-u)$ for some $0 \le u \le 1$. If $x_1 = 1$, then $|x_2 + y_2 + z_2| = 1$. Thus $(x_1, y_1, z_1) = (1, 0, 0),\ (x_2, y_2, z_2) = (r, s, t)$ for some $r, s, t \in \mathbb{R}$ with $|r+s+t| = |r| + |s| + |t| = 1$. If $0 < x_1 < 1$, by Lemma 1(2), $|x_2 + y_2 + z_2| = |x_2 + z_2| = 1$. Thus $y_2 = 0$ and $1 = |x_1 + y_1|\ |x_2 + y_2|$. Thus $(x_1, y_1, z_1) = (u, 0, 1 - u),\ (x_2, y_2, z_2) = (v, 0, 1 - v)$ for some $0 \le u, v \le 1$.

Let $z_2 = 0$. Then

$$1 = |x_1x_2 + y_1y_2 + (x_1y_2 + x_2y_1) + x_2z_1| = |x_1|\ |x_2 + y_2| + |y_1|\ |x_2 + y_2| + |z_1|\ |x_2|.$$

Note that $x_1 = 0, 1$ or $0 < x_1 < 1$.

Suppose that $x_1 = 0$.

Then

$$1 = |x_1x_2 + y_1y_2 + (x_1y_2 + x_2y_1) + x_2z_1| = |y_1|\ |x_2 + y_2| + |z_1|\ |x_2|.$$

If $y_1 = 0$, then $(x_1, y_1, z_1) = (x_2, y_2, z_2) = (0, 0, 1)$. If $|y_1| = 1$, then $(x_1, y_1, z_1) = (0, 1, 0)$ and $(x_2, y_2, z_2) = (u, 1 - u, 0)$ for some $0 \le u \le 1$. If $0 < |y_1| < 1$, then

$|x_2| = |x_2 + y_2| = 1$ and so $(x_1, y_1, z_1) = (1, 0, 0)$. and $(x_2, y_2, z_2) = (r, s, t)$ for some $r, s, t \in \mathbb{R}$ with $|r + s + t| = |r| + |s| + |t| = 1$.

Suppose that $x_1 = 1$.

Then $(x_1, y_1, z_1) = (1, 0, 0),\ (x_2, y_2, z_2) = (u, 1 - u, 0)$ for some $0 \le u \le 1$.

Suppose that $0 < x_1 < 1$.

If $y_1 = 0$, then $|x_2| = |x_1+z_1| = 1$. Thus $(x_1, y_1, z_1) = (u, 0, 1-u),\ (x_2, y_2, z_2) = (1, 0, 0)$ for some $0 \le u \le 1$. Let $0 < |y_1| < 1$. If $z-1 = 0$, then $1 = |x_1+y_1|\,|x_2+y_2|$. Thus $(x_1, y_1, z_1) = (u, 1 - u, 0),\ (x_2, y_2, z_2) = (v, 1 - v, 0)$ for some $0 \le u, v \le 1$.

Thus,

$$\begin{aligned}\mathrm{Norm}(T) \;=\; & \mathrm{Sym}\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(u,1-u,0), \pm(v,1-v,0)\Big),\\ & \Big(\pm(u,0,1-u), \pm(v,0,1-v)\Big) : |r+s+t| = |r|+|s|+|t| = 1,\\ & 0 \le u, v \le 1\Big\}\Big).\end{aligned}$$

Case 6. $|M| = 6$.

We claim that if $a = b = c = d = e = -f = 1$, then

$$\begin{aligned}\mathrm{Norm}(T) \;=\; & \mathrm{Sym}\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(0,1,0), \pm(r,s,-t)\Big),\\ & \Big(\pm(0,0,1), \pm(r,-s,t)\Big), \Big(\pm(0,u,-(1-u)), \pm(0,v,-(1-v))\Big),\\ & \Big(\pm(u,0,1-u), \pm(v,0,1-v)\Big), \Big(\pm(u,1-u,0), \pm(v,1-v,0)\Big) :\\ & |r+s+t| = |r|+|s|+|t| = 1, 0 \le u, v \le 1\Big\}\Big).\end{aligned}$$

By Lemma 4, $x_1y_1z_1 = 0$ or $x_2y_2z_2 = 0$. Since T is symmetric, we may assume that $x_1y_1z_1 = 0$. Then $x_1 = 0, y_1 = 0$ or $z_1 = 0$.

Suppose that $x_1 = 0$.

It follows that

$$\begin{aligned}1 &= |T((x_1, y_1, z_1),\ (x_2, y_2, z_2))|\\ &= |y_1(x_2 + y_2 - z_2) + z_1(x_2 - y_2 + z_2)|\\ &= |y_1|\,|x_2 + y_2 - z_2| + |z_1|\,|x_2 - y_2 + z_2|.\end{aligned}$$

Notice that $|y_1| = 0, 1$ or $0 < |y_1| < 1$. If $|y_1| = 0$, then $|z_1| = |x_2 - y_2 + z_2| = 1$. Thus $(x_1, y_1, z_1) = (0, 0, \pm 1),\ (x_2, y_2, z_2) = (r, -s, t)$ for some $r, s, t \in \mathbb{R}$ with $|r + s + t| = |r| + |s| + |t| = 1$. If $|y_1| = 1$, then $|x_2 + y_2 - z_2| = 1$. Thus $(x_1, y_1, z_1) = (0, \pm 1, 0),\ (x_2, y_2, z_2) = (r, s, -t)$ for some $r, s, t \in \mathbb{R}$ with $|r + s + t| = |r| + |s| + |t| = 1$. If $0 < |y_1| < 1$, then by Lemma 1(2), $|x_2 + y_2 - z_2| = |x_2 - y_2 + z_2| = 1$. By Lemma 1(1), $x_2 = 1$ or $|y_2 - z_2| = 1$. If $x_2 = 1$, then $(x_1, y_1, z_1) = \pm(0, t, 1-t),\ (x_2, y_2, z_2) = (1, 0, 0)$ for some $0 \le t \le 1$. If $|y_2 - z_2| = 1$, then $x_2 = 0$ and $|y_1 - z_1|\ |y_2 - z_2| = 1$, so $|y_1 - z_1| = |y_2 - z_2| = 1$. Thus $x_1 = x_2 = 0$ and $(x_1, y_1, z_1) = \pm(0, t, -(1 - t)),\ (x_2, y_2, z_2) = \pm(0, s, -(1 - s))$ for some $0 \le t, s \le 1$.

Suppose that $y_1 = 0$.

It follows that

$$\begin{aligned} 1 &= |T((x_1, y_1, z_1),\ (x_2, y_2, z_2))| \\ &= |x_1(x_2 + y_2 + z_2) + z_1(x_2 - y_2 + z_2)| \\ &= |x_1|\ |x_2 + y_2 + z_2| + |z_1|\ |x_2 - y_2 + z_2|. \end{aligned}$$

Notice that $x_1 = 0, 1$ or $0 < |x_1| < 1$. If $x_1 = 0$, then $|z_1| = |x_2 - y_2 + z_2| = 1$. Thus $(x_1, y_1, z_1) = (0, 0, \pm 1),\ (x_2, y_2, z_2) = (r, -s, t)$ for some $r, s, t \in \mathbb{R}$ with $|r+s+t| = |r|+|s|+|t| = 1$. If $x_1 = 1$, then $|x_2+y_2+z_2| = 1$. Thus $(x_1, y_1, z_1) = (\pm 1, 0, 0),\ (x_2, y_2, z_2) = (r, s, t)$ for some $r, s, t \in \mathbb{R}$ with $|r+s+t| = |r|+|s|+|t| = 1$. If $0 < x_1 < 1$, then by Lemma 1(2), $|x_2 + y_2 + z_2| = |x_2 - y_2 + z_2| = 1$. By Lemma 1(1), $|y_2| = 1$ or $|x_2 + z_2| = 1$. If $|y_2| = 1$, then $|x_1 - z_1| = 1$ and so $(x_1, y_1, z_1) = \pm(t, 0, -(1-t)),\ (x_2, y_2, z_2) = (0, \pm 1, 0)$ for some $0 \le t \le 1$. If $|y_2 + z_2| = 1$, then $y_2 = 0$ and $|x_1 + z_1|\ |x_2 + z_2| = 1$. Thus $y_1 = y_2 = 0$ and $(x_1, y_1, z_1) = \pm(t, 0, 1-t),\ (x_2, y_2, z_2) = \pm(s, 0, 1-s)$ for some $0 \le t, s \le 1$.

Suppose that $z_1 = 0$.

It follows that

$$\begin{aligned} 1 &= |T((x_1, y_1, z_1),\ (x_2, y_2, z_2))| \\ &= |x_1(x_2 + y_2 + z_2) + y_1(x_2 + y_2 - z_2)| \\ &= |x_1|\ |x_2 + y_2 + z_2| + |y_1|\ |x_2 + y_2 - z_2|. \end{aligned}$$

Notice that $x_1 = 0, 1$ or $0 < x_1 < 1$. If $x_1 = 0$, then $|y_1| = |x_2 + y_2 - z_2| = 1$. Thus $(x_1, y_1, z_1) = (0, \pm 1, 0),\ (x_2, y_2, z_2) = (r, s, -t)$ for some $r, s, t \in \mathbb{R}$ with $|r+s+t| = |r|+|s|+|t| = 1$. If $x_1 = 1$, then $|x_2+y_2+z_2| = 1$. Thus $(x_1, y_1, z_1) = (1, 0, 0),\ (x_2, y_2, z_2) = (r, s, t)$ for some $r, s, t \in \mathbb{R}$ with $|r+s+t| = |r|+|s|+|t| = 1$. If $0 < x_1 < 1$, then by Lemma 1(2), $|x_2 + y_2 + z_2| = |x_2 + y_2 - z_2| = 1$. By Lemma 1(1), $|z_2| = 1$ or $|x_2 + y_2| = 1$. If $|z_2| = 1$, then $|x_1 - y_1| = 1$ and so $(x_1, y_1, z_1) = \pm(t, -(1-t), 0),\ (x_2, y_2, z_2) = (0, 0, \pm 1)$ for some $0 \le t \le 1$. If $|x_2 + y_2| = 1$, then $z_2 = 0$ and $|x_1 + y_1|\ |x_2 + y_2| = 1$, so $|x_1 + y_1| = 1$. Thus $(x_1, y_1, z_1) = \pm(t, 1-t, 0),\ (x_2, y_2, z_2) = \pm(s, 1-s, 0)$ for some $0 \le t, s \le 1$. Therefore,

$$\begin{aligned} \mathrm{Norm}(T) &= \mathrm{Sym}\Big(\Big\{\Big(\pm(1,0,0), \pm(r,s,t)\Big), \Big(\pm(0,1,0), \pm(r,s,-t)\Big), \\ &\quad \Big(\pm(0,0,1), \pm(r,-s,t)\Big), \Big(\pm(0,v,-(1-v)), \pm(0,w,-(1-w))\Big), \\ &\quad \Big(\pm(v,0,1-v), \pm(w,0,1-w)\Big), \Big(\pm(v,1-v,0), \pm(w,1-w,0)\Big) : \\ &\quad |r+s+t| = |r|+|s|+|t| = 1, 0 \le u, v, w \le 1\Big\}\Big). \end{aligned}$$

We complete the proof. □

Corollary 2. *Let* $T, S \in \mathcal{L}_s({}^2 l_1^3)$ *with* $\|T\| = \|S\|$. *If* $\mathrm{Norm}(T) = \mathrm{Norm}(S)$, *then* $T = \pm S$.

Proof. It follows from Theorem 5. □

Question. *Let E be a real Banach space. Let $T, S \in \mathcal{L}(^2E)$ be such that $\|T\| = \|S\|$ and* $\text{Norm}(T) = \text{Norm}(S)$. *Is it true that $T = \pm S$?*

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