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OPTIMAL INEQUALITIES FOR SUBMANIFOLDS OF RIEMANN MANIFOLDS OF NEARLY QUASI-CONSTANT CURVATURE WITH QUARTER-SYMMETRIC CONNECTION

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Abstract

In this paper, we obtain Chen inequalities for submanifolds in Riemannian manifolds of nearly quasi-constant curvature with a special kind of quarter-symmetric connection and discuss the equality case of the inequalities. We also obtain some Casorati inequalities for submanifolds in Riemannian manifolds of nearly quasi-constant curvature with the quarter-symmetric connection.

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1 Introduction

A challenging question concerning the existence of minimal immersions into a Euclidean space of arbitrary dimension was raised by Chern [8]. To answer the question, Chen[6] obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established a sharp inequality for a submanifold in a real space form using the scalar curvature and the sectional curvature (intrinsic invariants) and squared mean curvature (extrinsic invariant). The inequalities in this direction are known

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as Chen inequalities [4, 5, 6]. Afterwards, distinguished geometers studied similar problems for different submanifolds in various ambient spaces with different connections; see, for example, [20, 25, 32, 33, 35].

Hayden [15] introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. Nakao [21] studied submanifolds of a Riemannian manifold with semi-symmetric metric connections. Agashe and Chafle[1, 2] introduced the notion of a semi-symmetric non-metric connection and studied some of its properties and submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection. In [20, 25], Mihai and Özgür studied Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection and a semi-symmetric non-metric connection, respectively. The concept of "quarter-symmetric" connection was originally introduced by S. Golab[12]. Recently, in [26], the authors investigated Einstein warped products and multiply warped products with a quarter-symmetric connection. In 2019, Yang[30] obtained Chen inequalities for submanifolds of complex space forms and Sasakian space forms with quarter-symmetric connections.

Chen and yano[7] introduced the generalized notion of real space forms to quasi constant curvature manifolds. De and Gazi[9] extended the quasi constant curvature to nearly quasi-constant curvature manifolds. Özgür[23] studied Chen inequalities for submanifolds of Riemannian manifolds of quasi-constant curvature. Özgür and De[24] generalize these inequalities to submanifolds of Riemannian manifolds of nearly quasi-constant curvature. In the same way, some other basic inequalities were investigated for submanifolds of Riemannian manifolds of quasi-constant curvature and nearly quasi-constant curvature[32, 33, 34, 35]

The Casorati curvature(extrinsic invariant) of a submanifold of a Riemannian manifold introduced by Casorati defined as the normalized square length of the second fundamental form [3]. The notion of Casorati curvature gives a better intuition of the curvature compared to Gaussian curvature. The concept of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold [13]. The geometrical meaning and the importance of the Casorati curvature discussed by some distinguished geometers [10, 11, 17, 28, 29]. Therefore it attracts the attention of geometers to obtain the optimal inequalities for the Casorati curvatures of the submanifolds of different ambient spaces [18, 19].

In this paper, we obtain Chen's inequalities for submanifolds of Riemannian manifolds of nearly quasi-constant curvature with quarter-symmetric connection. The chronology of the paper is as follows. In Section 2, we give a brief introduction about the quarter-symmetric connection. In Section 3, we establish first Chen inequality for submanifolds of Riemannian manifolds of nearly quasi-constant curvature endowed with the quarter-symmetric connection and in the last section, we obtain some inequalities for generalized normalized δ-Casorati curvatures for submanifolds of Riemannian manifolds of nearly quasi-constant curvatures.

2 Preliminaries

Chen and K. Yano[7] introduced the notion of quasi-constant curvature. A Riemannian manifold $(\widetilde{M}, g)$ is called a Riemannian manifold of quasi-constant curvature if its curvature tensor $\widetilde{\widetilde{R}}$ satisfies the condition

$$\begin{aligned} \widetilde{\widetilde{R}}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad +q[g(X,W)\pi(Y)\pi(Z) - g(X,Z)\pi(Y)\pi(W) \\ &\quad +g(Y,Z)\pi(X)\pi(W) - g(Y,W)\pi(X)A(Z)], \end{aligned}$$

where p, q are scalar functions and π is a 1-form given by

$$g(X,P) = \pi(X),$$

P is a fixed unit vector field. It is straightforward to see that if $q = 0$, then $(\widetilde{M}, g)$ reduces to a Riemannian manifold of constant curvature.

For $n > 2$, a non-flat Riemannian manifold $(\widetilde{M}, g)$ is said to be a quasi-Einstein manifold if its Ricci tensor satisfies the condition

$$S(X,Y) = pg(X,Y) + q\pi(X)\pi(Y),$$

where p, q are scalar functions and π is 1-form acting same as above. It can be easily verified that every Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

In 2009, Gazi and De [9] generalized the notion of Riemannian manifold of quasi-constant curvature to Riemannian manifold of nearly quasi-constant and the curvature tensor satisfies

$$\begin{aligned} \widetilde{\widetilde{R}}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad +q[g(X,W)B(Y,Z) - g(X,Z)B(Y,W) \\ &\quad +g(Y,Z)B(X,W) - g(Y,W)B(X,Z)], \end{aligned} \tag{1}$$

where p, q are scalar functions and B is a non-vanishing $(0,2)$ type symmetric tensor.

For $n > 2$, a non-flat Riemannian manifold $(\widetilde{M}, g)$ is said to be a nearly quasi-Einstein manifold if its Ricci tensor satisfy

$$S(X,Y) = pg(X,Y) + qB(X,Y).$$

It can be easily verified that every Riemannian manifold of nearly quasi-constant curvature is a nearly quasi-Einstein manifold.

We know that the outer product of two convariant vectors is a covariant $(0,2)$ tensor, but converse in not true. Hence nearly quasi-constant Riemannian manifolds act as a bigger class of Riemannian manifolds in the sense that every Riemannian manifold of quasi-constant curvature is nearly quasi-constant Riemannian manifold, but there are plenty of examples where the converse is not true.

Example 1. *([9]) Let* $(\mathbb{R}^4, g)$ *be a Riemannian manifold with the metric* g *defined as follows*

$$ds^2 = (x^4)^{\frac{4}{5}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

This is a nearly quasi-constant Riemannian manifold but not a quasi-constant Riemannian manifold.

Let $\widetilde{M}$ be an $(n+m)$-dimensional Riemannian manifold with nearly quasi-constant curvature with Riemannian metric g and $\overline{\widetilde{\nabla}}$ be the Levi-Civita connection on $\widetilde{M}$. Let $\overline{\nabla}$ be a linear connection defined by

$$\overline{\nabla}_X Y = \overline{\widetilde{\nabla}}_X Y + \Lambda_1 \pi(Y) X - \Lambda_2 g(X,Y) P, \tag{2}$$

for X, Y on $\widetilde{M}$, Λ_1, Λ_2 are real constants and P the vector field on $\widetilde{M}$ such that $\pi(X) = g(X,P)$, where π are one form. If $\overline{\nabla} g = 0$, then $\overline{\nabla}$ is known as quarter -symmetric metric connection and if $\overline{\nabla} g \neq 0$, then $\overline{\nabla}$ is known as quarter -symmetric non-metric connection.

The special cases of (2) can be obtained as
(i) when $\Lambda_1 = \Lambda_2 = 1$, then the above connection reduces to semi-symmetric metric connection.
(ii) when $\Lambda_1 = 1$ and $\Lambda_2 = 0$, then the above connection reduces to semi-symmetric non metric connection.

The curvature tensor with respect to $\overline{\nabla}$ is defined as

$$\overline{R}(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z. \tag{3}$$

Similarly, we can define the curvature tensor with respect to $\overline{\widetilde{\nabla}}$.

Now, using (2), the curvature tensor takes the following form[30]

$$\begin{aligned} \overline{R}(X,Y,Z,W) = {} & \overline{\widetilde{R}}(X,Y,Z,W) + \Lambda_1 \alpha(X,Z) g(Y,W) - \Lambda_1 \alpha(Y,Z) g(X,W) \\ & + \Lambda_2 g(X,Z) \alpha(Y,W) - \Lambda_2 g(Y,Z) \alpha(X,W) \\ & + \Lambda_2 (\Lambda_1 - \Lambda_2) g(X,Z) \beta(Y,W) \\ & - \Lambda_2 (\Lambda_1 - \Lambda_2) g(Y,Z) \beta(X,W), \end{aligned} \tag{4}$$

where

$$\alpha(X,Y) = (\overline{\widetilde{\nabla}}_X \pi)(Y) - \Lambda_1 \pi(X)\pi(Y) + \frac{\Lambda_2}{2} g(X,Y)\pi(P),$$

and

$$\beta(X,Y) = \frac{\pi(P)}{2} g(X,Y) + \pi(X)\pi(Y)$$

are $(0,2)$-tensors. For simplicity, we denote by $\mathrm{tr}(\alpha) = a$ and $\mathrm{tr}(\beta) = b$.

Let M^n be an n-dimensional submanifold of an $(n+m)$-dimensional Riemannian manifolds with nearly quasi-constant curvature $\widetilde{M}$. On the submanifold M, we consider the induced quarter-symmetric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\widetilde{\nabla}$. Let R and $\widetilde{R}$ be the curvature tensors of ∇ and $\widetilde{\nabla}$. Decomposing the vector field P on M uniquely into its tangent and normal components $P^\top$ and $P^\perp$, respectively, then we have $P = P^\top + P^\perp$. The Gauss formulas with respect to ∇ and $\widetilde{\nabla}$ can be written as:

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \qquad X, Y \in \Gamma(TM),$$

$$\widetilde{\overline{\nabla}}_X Y = \widetilde{\nabla}_X Y + \widetilde{\sigma}(X,Y), \qquad X, Y \in \Gamma(TM),$$

where $\widetilde{\sigma}$ is the second fundamental form of M in $\widetilde{M}$ and

$$\sigma(X,Y) = \widetilde{\sigma}(X,Y) - \Lambda_2 g(X,Y)P^\perp.$$

In $\widetilde{M}^{n+m}$ we can choose a local orthonormal frame $\{E_1, \cdots, E_n, E_{n+1}, \cdots, E_{n+m}\}$ such that, restricting to M, $\{E_1, \cdots, E_n\}$ are tangent to M^n. We write $\sigma_{ij}^r = g(\sigma(E_i, E_j), E_r)$. The squared length of σ is $||\sigma||^2 = \sum_{i,j=1}^n g(\sigma(E_i, E_j), \sigma(E_i, E_j))$ and the mean curvature vector of M associated to ∇ is $H = \frac{1}{n}\sum_{i=1}^n \sigma(E_i, E_i)$. Similarly, the mean curvature vector of M associated to $\widetilde{\nabla}$ is $\widetilde{H} = \frac{1}{n}\sum_{i=1}^n \widetilde{\sigma}(E_i, E_i)$. Let $\widetilde{M}^{n+m}$ be an $(n+m)$-dimensional Riemannian manifolds of nearly quasi-constant curvature endowed with a quarter-symmetric connection satisfying (2). The curvature tensor $\widetilde{\overline{R}}$ with respect to the Levi-Civita connection $\widetilde{\overline{\nabla}}$ on $\widetilde{M}^{n+m}$ is expressed by

$$\begin{aligned}\widetilde{\overline{R}}(X,Y,Z,W) &= p\Big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\Big\} \\ &\quad + q\Big\{g(X,W)B(Y,Z) - g(X,Z)B(Y,W) \\ &\quad + g(Y,Z)B(X,W) - g(Y,W)B(X,Z)\Big\}. \qquad (5)\end{aligned}$$

By (2) and (5), we get

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= p\Big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\Big\} \\ &\quad + q\Big\{g(X,W)B(Y,Z) - g(X,Z)B(Y,W) + g(Y,Z)B(X,W) \\ &\quad - g(Y,W)B(X,Z)\Big\} + \lambda_1\alpha(X,Z)g(Y,W) - \lambda_1\alpha(Y,Z)g(X,W) \\ &\quad + \lambda_2 g(X,Z)\alpha(Y,W) - \lambda_2 g(Y,Z)\alpha(X,W) \\ &\quad + \lambda_2(\lambda_1 - \lambda_2)g(X,Z)\beta(Y,W) \\ &\quad - \lambda_2(\lambda_1 - \lambda_2)g(Y,Z)\beta(X,W). \qquad (6)\end{aligned}$$

Similar to [30], we have the Gauss equation

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(\sigma(X,W),\sigma(Y,Z)) + g(\sigma(Y,W),\sigma(X,Z)) \\ + (\lambda_1 - \lambda_2)g(\sigma(Y,Z),P)g(X,W) + (\lambda_2 - \lambda_1)g(\sigma(X,Z),P)g(Y,W). \tag{7}$$

Let Π be a 2-plane section at a point $p \in M$ and spanned by orthonormal basis E_1 and E_2 i:e $\Pi = span\{E_1, E_2\}$. As $R(X,Y,Z,W) \neq R(X,Y,W,Z)$, we can define the sectional curvature $K(\Pi)$ of M with respect to induced connection ∇ as

$$K(\Pi) = \frac{1}{2}\Big[R(E_1,E_2,E_2,E_1) - R(E_1,E_2,E_1,E_2)\Big], \tag{8}$$

where $K(\Pi)$ is independent choice of the orthonormal basis E_1, E_2 . If $\{E_1,\dots,E_n\}$ and $\{E_{n+1},\dots,E_{n+m}\}$ are orthonormal basis of T_pM and $T_p^{\perp}M$ at any $p \in M$, then the scalar curvature τ at that point is given by

$$\tau(p) = \sum_{1\leq i<j\leq n} K(E_i \wedge E_j).$$

The normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

The norm of the squared mean curvature of the submanifold is defined by

$$\|H\|^2 = \frac{1}{n^2}\sum_{\gamma=n+1}^{n+m}\left(\sum_{i=1}^{n}\sigma_{ii}^{\gamma}\right)^2,$$

and the squared norm of second fundamental form h is denoted by $\mathcal{C}$ defined as

$$\mathcal{C} = \frac{1}{n}\sum_{\gamma=n+1}^{n+m}\sum_{i,j=1}^{n}(\sigma_{ij}^{\gamma})^2,$$

known as Casorati curvature of the submanifold.

If we suppose that L is an s-dimensional subspace of TM, $s \geq 2$, and $\{E_1, E_2, \dots, E_s\}$ is an orthonormal basis of L. then the scalar curvature of the s-plane section L is given as

$$\tau(L) = \sum_{1\leq\gamma<\beta\leq s} K(E_\gamma \wedge E_\beta)$$

and the Casorati curvature $\mathcal{C}$ of the subspace L is as follows

$$\mathcal{C}(L) = \frac{1}{s}\sum_{\gamma=n+1}^{n+m}\sum_{i,j=1}^{s}(\sigma_{ij}^{\gamma})^2.$$

A point $p \in M$ is said to be an *invariantly quasi-umbilical point* if there exist m mutually orthogonal unit normal vectors $\xi_{n+1}, \dots, \xi_{n+m}$ such that the shape operators with respect to all directions ξ_γ have an eigenvalue of multiplicity $n-1$ and that for each ξ_γ the distinguished Eigen direction is the same. The submanifold is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point.

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\widetilde{\delta}_c(n-1)$ are defined as

$$[\delta_c(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n} inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\} \quad (9)$$

and

$$[\widetilde{\delta}_c(n-1)]_p = 2\mathcal{C}_p + \frac{2n-1}{2n} sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\}. \quad (10)$$

For a positive real number $t \neq n(n-1)$, the generalized normalized δ-Casorati curvatures $\delta_c(t; n-1)$ and $\widetilde{\delta}_c(t; n-1)$ are given as

$$\begin{aligned} &[\delta_c(t;n-1)]_p \\ = \;& t\mathcal{C}_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt} inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\}, \end{aligned}$$

if $0 < t < n^2 - n$, and

$$\begin{aligned} &[\widetilde{\delta}_c(t;n-1)]_p \\ = \;& t\mathcal{C}_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt} sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\}, \end{aligned}$$

if $t > n^2 - n$.

Now, we recall the following lemmas, which plays an important role for the proof of the main results.

Lemma 1. *[33] Let $g(a_1, a_2, ..., a_n)$ $(n \geq 3)$ be a function in $\mathbf{R}^n$ defined by*

$$g(a_1, a_2, ..., a_n) = (a_1 + a_2)\sum_{i=3}^{n} a_i + \sum_{3 \leq i < j \leq n} a_i a_j.$$

If $a_1 + a_2 + ... + a_n = (n-1)\epsilon$, we have

$$g(a_1, a_2, ..., a_n) \leq \frac{(n-1)(n-2)}{2}\epsilon^2.$$

The equality sign holds if and only if $a_1 + a_2 = a_3 = ... = a_n = \epsilon$.

Lemma 2. *[33] Let $g(a_1, a_2, ..., a_n)$ be a function in $\mathbf{R}^n$ defined by*

$$g(a_1, a_2, ..., a_n) = a_1 \sum_{i=2}^{n} a_i$$

If $a_1 + a_2 + ... + a_n = 2\epsilon$, we have

$$g(a_1, a_2, ..., a_n) \leq \epsilon^2.$$

The equality sign holds if and only if $a_1 = a_2 + a_3 + ... + a_n = \epsilon$.

Oprea[22] gives new direction to prove the Chen inequalities using optimization techniques. For a submanifold (M, g) of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and $\mathcal{F} : M \to \mathbf{R}$ be a differential function. If we have a constrained problem

$$min_{x \in M} \mathcal{F}(x) \tag{11}$$

then the following result holds

Lemma 3. *[22] Let $x_\circ \in M$ is the solution of the problem 11, then*
(i) $(grad(\mathcal{F}))(x_\circ) \in T^{\perp}_{x_\circ} M$
(ii) the bilinear form
$\mathcal{B} : T_{x_\circ} M \times T_{x_\circ} M \to \mathbf{R}$
$\mathcal{B}(X, Y) = Hess_{\mathcal{F}}(X, Y) + \tilde{g}(\sigma(X, Y), (grad(\mathcal{F}))(x_\circ))$
is positive semi-definie, where σ is the second fundamental form of M in $\widetilde{M}$ and grad($\mathcal{F}$) if the gradient of g.

3 Chen inequalities

Theorem 1. *Let M is an n-dimensional submanifold of an $(n+m)$-dimensional Riemannian manifolds with nearly quasi-constant curvature $\widetilde{M}$ endowed with a connection $\overline{\nabla}$, then*

$$\begin{aligned}\tau(p) - K(\Pi) \quad \leq \quad & \frac{(n+1)(n-2)}{2} p + q(n-2) trB + trB|_{\pi^\perp} - \frac{(\Lambda_1 + \Lambda_2)}{2}(n-1)a \\ & - \frac{\Lambda_2(\Lambda_1 - \Lambda_2)}{2}(n-1)b - \frac{(\Lambda_1 + \Lambda_2)}{2}(n-1)n\pi(H) \\ & + \frac{\Lambda_1 + \Lambda_2}{2} tr(\alpha \mid_\Pi) + \frac{\Lambda_2(\Lambda_1 - \Lambda_2)}{2} tr(\beta \mid_\Pi) \\ & + \frac{\Lambda_1 - \Lambda_2}{2} g(tr(\sigma \mid_\Pi), P) + \frac{n^2(n-2)}{2(n-1)} \|H\|^2.\end{aligned}$$

Proof. Let $p \in M$ and $\{E_1, \cdots, E_n\}$ and $\{E_{n+1}, \cdots, E_{n+m}\}$ be orthonormal basis of T_pM and $T^{\perp}_p M$ respectively. For $X = W = E_i$, $Y = Z = E_j$, $i \neq j$ by (2.11), we have

$$\begin{aligned} & p\Big\{g(E_j, E_j)g(E_i, E_i) - g(E_i, E_j)g(E_j, E_i)\Big\} + q\Big\{g(E_i, E_i)B(E_j, E_j) \\ & -g(E_i, E_j)B(E_j, E_i) + g(E_j, E_j)B(E_i, E_i) - g(E_j, E_i)B(E_i, E_j)\Big\} \\ & +\Lambda_1\alpha(E_i, E_j)g(E_j, E_i) - \Lambda_1\alpha(E_j, E_j)g(E_i, E_i) + \Lambda_2\alpha(E_i, E_j)g(E_j, E_i) \\ & -\Lambda_2\alpha(E_j, E_j)g(E_i, E_i) + \Lambda_2(\Lambda_1 - \Lambda_2)\beta(E_i, E_j)g(E_j, E_i) \\ & -\Lambda_2(\Lambda_1 - \Lambda_2)\beta(E_j, E_j)g(E_i, E_i) \\ = \quad & R(E_i, E_j, E_j, E_i) - g(\sigma(E_i, E_i), \sigma(E_j, E_j)) + g(\sigma(E_j, E_i), \sigma(E_i, E_j)) + (\Lambda_1 \\ & -\Lambda_2)g(\sigma(E_j, E_j), P)g(E_i, E_i) + (\Lambda_2 - \Lambda_1)g(\sigma(E_i, E_j), P)g(E_j, E_i). \end{aligned} \tag{12}$$

Taking the summation over i and j and simplifying, we have

$$\begin{aligned}\tau &= \left[\frac{n(n-1)p}{2}+(n-1)qtr(B)\right]-\frac{(\Lambda_1+\Lambda_2)}{2}(n-1)a-\frac{\Lambda_2(\Lambda_1-\Lambda_2)}{2}(n-1)b\\ &\quad -\frac{(\Lambda_1-\Lambda_2)}{2}(n-1)n\pi(H)+\sum_{n+1}^{n+m}\sum_{1\le i,j=n}[\sigma^r_{ii}\sigma^r_{jj}-(\sigma^r_{ij})^2] \qquad (13)\end{aligned}$$

From (6) and (8), we have

$$\begin{aligned}K(\Pi) &= \Big(p+q(tr(B|_\pi)+tr(B|_{\pi^\perp}))\Big)-\frac{\Lambda_1+\Lambda_2}{2}tr(\alpha\mid_\Pi)-\frac{\Lambda_2(\Lambda_1-\Lambda_2)}{2}tr(\beta\mid_\Pi)\\ &\quad -\frac{\Lambda_1-\Lambda_2}{2}g(tr(\sigma\mid_\Pi),P)+\sum_{r=n+1}^{n+m}[\sigma^r_{11}\sigma^r_{22}-(\sigma^r_{12})^2]. \qquad (14)\end{aligned}$$

Subtracting (13) and (14), we get

$$\begin{aligned}\tau(p)-K(\Pi) &= \frac{(n+1)(n-2)}{2}p+q(n-2)trB+trB|_{\Pi^\perp}-\frac{(\Lambda_1+\Lambda_2)}{2}(n-1)a\\ &\quad -\frac{\Lambda_2(\Lambda_1-\Lambda_2)}{2}(n-1)b-\frac{(\Lambda_1+\Lambda_2)}{2}(n-1)n\pi(H)\\ &\quad +\frac{\Lambda_1+\Lambda_2}{2}tr(\alpha\mid_\Pi)\\ &\quad +\frac{\Lambda_2(\Lambda_1-\Lambda_2)}{2}tr(\beta\mid_\Pi)+\frac{\Lambda_1-\Lambda_2}{2}g(tr(\sigma\mid_\Pi),P)\\ &\quad +\sum_{r=n+1}^{n+m}[\sum_{1\le i<j\le n}\sigma^r_{ii}\sigma^r_{jj}-\sigma^r_{11}\sigma^r_{22}-\sum_{1\le i<j\le n}(\sigma^r_{ij})^2+(\sigma^r_{12})^2]. \qquad (15)\end{aligned}$$

By Lemma 1, we have

$$\sum_{r=n+1}^{n+m}\left[\sum_{1\le i<j\le n}\sigma^r_{ii}\sigma^r_{jj}-\sigma^r_{11}\sigma^r_{22}-\sum_{1\le i<j\le n}(\sigma^r_{ij})^2+(\sigma^r_{12})^2\right]\le\frac{n^2(n-2)}{2(n-1)}\|H\|^2. \qquad (16)$$

By (15) and (16), we get the desired result. □

Corollary 1. *If P is a tangent vector field on M, then $H=\widetilde{H}$. In this case, the inequality in Theorem 1 becomes*

$$\begin{aligned}\tau(p)-K(\Pi) &\le \frac{(n+1)(n-2)}{2}p+q(n-2)trB+trB|_{\pi^\perp}-\frac{(\Lambda_1+\Lambda_2)}{2}(n-1)a\\ &\quad -\frac{\Lambda_2(\Lambda_1-\Lambda_2)}{2}(n-1)b-\frac{(\Lambda_1+\Lambda_2)}{2}(n-1)n\pi(H)\\ &\quad +\frac{\Lambda_1+\Lambda_2}{2}tr(\alpha\mid_\Pi)+\frac{\Lambda_2(\Lambda_1-\Lambda_2)}{2}tr(\beta\mid_\Pi)\\ &\quad +\frac{\Lambda_1-\Lambda_2}{2}g(tr(\sigma\mid_\Pi),P)+\frac{n^2(n-2)}{2(n-1)}\|\widetilde{H}\|^2. \qquad (17)\end{aligned}$$

Corollary 2. *If P is a tangent vector field on M, then $\sigma = \widetilde{\sigma}$. In this case, the equality case of (17) holds at a point $p \in M$ if and only if, with respect to a suitable orthonormal basis $\{E_a\}$ at p, the shape operators $A_r = A_{E_r}$ take the following forms:*

$$A_{n+1} = \begin{pmatrix} \sigma_{11}^{n+1} & 0 & 0 & \cdots & 0 \\ 0 & \sigma_{22}^{n+1} & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{11}^{n+1} + \sigma_{22}^{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \sigma_{11}^{n+1} + \sigma_{22}^{n+1} \end{pmatrix}$$

and

$$A_r = \begin{pmatrix} \sigma_{11}^{r} & \sigma_{12}^{r} & 0 & \cdots & 0 \\ \sigma_{12}^{r} & -\sigma_{11}^{r} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, r = n+2, \cdots, n+m.$$

4 Inequalities for generalized normalized δ-Casorati curvatures

Theorem 2. *Let M is an n-dimensional submanifold of a Riemannian manifolds with nearly quasi-constant curvature $\widetilde{M}$ of dimension $(n+m)$ endowed with a connection $\overline{\nabla}$, then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\begin{aligned} \rho \;\leq\; & \frac{\delta_c(t;n-1)}{n(n-1)} + \big[p + \frac{2q}{n} tr(B)\big] \\ & - \frac{(\Lambda_1 + \Lambda_2)}{n} a - \frac{\Lambda_2(\Lambda_1 - \Lambda_2)}{n} b - (\Lambda_1 - \Lambda_2)\pi(H), \end{aligned} \tag{18}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(t; n-1)$ satisfies

$$\begin{aligned} \rho \;\leq\; & \frac{\tilde{\delta}_c(t;n-1)}{n(n-1)} + \big[p + \frac{2q}{n} tr(B)\big] \\ & - \frac{(\Lambda_1 + \Lambda_2)}{n} a - \frac{\Lambda_2(\Lambda_1 - \Lambda_2)}{n} b - (\Lambda_1 - \Lambda_2)\pi(H), \end{aligned} \tag{19}$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (18) and (19) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{M}$, such that with respect to suitable tangent orthonormal frame $\{E_1, \ldots, E_n\}$ and normal orthonormal frame $\{E_{n+1}, \ldots, E_{n+m}\}$, the shape operator $A_r \equiv A_{E_\gamma}$,

$\gamma \in \{n+1, \ldots, n+m\}$, *take the following form*

$$A_{n+1} = \begin{pmatrix} \sigma_{11}^{\gamma} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \sigma_{22}^{\gamma} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \sigma_{33}^{\gamma} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \sigma_{n-1n-1}^{\gamma} & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{t}\sigma_{nn}^{\gamma} \end{pmatrix}, \tag{20}$$

$$A_{n+2} = \cdots = A_{n+m} = 0.$$

Proof. Let $\{E_1, E_2, \ldots, E_n\}$ and $\{E_{n+1}, E_{n+2}, \ldots, E_{n+m}\}$ be the orthonormal bases of T_pM and $T_p^{\perp}M$ respectively at a point $p \in M$. Using (13), we have

$$\begin{aligned} 2\tau &= \Big[n(n-1)p + 2(n-1)qtr(B)\Big] - (\Lambda_1 + \Lambda_2)(n-1)a - \Lambda_2(\Lambda_1 - \Lambda_2)(n-1)b \\ &\quad -(\Lambda_1 - \Lambda_2)(n-1)n\pi(H) + n^2\|H\|^2 - n\mathcal{C} \end{aligned} \tag{21}$$

Consider a polynomial $\mathcal{Q}$ in the components of second fundamental form σ defined as

$$\begin{aligned} \mathcal{Q} &= t\mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) - 2\tau(p) + \big[n(n-1)p + 2(n-1)qtr(B)\big] \\ &\quad -(\Lambda_1 + \Lambda_2)(n-1)a - \Lambda_2(\Lambda_1 - \Lambda_2)(n-1)b - (\Lambda_1 - \Lambda_2)(n-1)n\pi(H) \end{aligned}$$

where L is hyperplane of tangent space at a point p. We assume that L is spanned by $E_1, E_2, \ldots, E_{n-1}$ and $\mathcal{Q}$ has an expression of the form

$$\begin{aligned} \mathcal{Q} &= \frac{t}{n}\sum_{\gamma=n+1}^{n+m}\sum_{i,j=1}^{n}(\sigma_{ij}^{\gamma})^2 + \frac{(n+t)(n^2-n-t)}{nt}\sum_{\gamma=n+1}^{n+m}\sum_{i,j=1}^{n-1}(\sigma_{ij}^{\gamma})^2 \\ &\quad -2\tau(p) + \big[n(n-1)p + 2(n-1)qtr(B)\big] \\ &\quad -(\Lambda_1 + \Lambda_2)(n-1)a - \Lambda_2(\Lambda_1 - \Lambda_2)(n-1)b - (\Lambda_1 - \Lambda_2)(n-1)n\pi(H) \end{aligned} \tag{22}$$

From (21) and (22), we arrive at

$$\begin{aligned} \mathcal{Q} &= \sum_{\gamma=n+1}^{n+m}\sum_{i=1}^{n-1}\left[\left(\frac{(n^2+nr-n-2t)}{t}\right)(\sigma_{ii}^{\gamma})^2 + \frac{2(n+t)}{n}(\sigma_{in}^{\gamma})^2\right] \\ &\quad + \sum_{\gamma=n+1}^{m}\left[2\left(\frac{2(n+t)(n-1)}{t}\right)\sum_{(i<j)=1}^{n}(\sigma_{ij}^{\gamma})^2 - 2\sum_{(i<j)=1}^{n}\sigma_{ii}^{\gamma}\sigma_{jj}^{\gamma} + \frac{t}{n}(\sigma_{nn}^{\gamma})^2\right] \\ &\geq \sum_{\gamma=n+1}^{n+m}\sum_{i=1}^{n-1}\left[\left(\frac{(n^2+n(r-1)-2t)}{t}\right)(\sigma_{ii}^{\gamma})^2 - 2\sum_{(i<j)=1}^{n}\sigma_{ii}^{\gamma}\sigma_{jj}^{\gamma} + \frac{t}{n}(\sigma_{nn}^{\gamma})^2\right]. \end{aligned} \tag{23}$$

For $t = n+1, \ldots, n+m$, lets us have a quadratic form $\mathcal{F}_{\gamma} : \mathbf{R}^n \to \mathbf{R}$ defined as

$$\mathcal{F}_{\gamma}(\sigma_{11}^{\gamma}, \ldots, \sigma_{nn}^{\gamma}) = \sum_{i=1}^{n-1}\frac{n^2+n(r-1)-2r}{r}(\sigma_{ii}^{\gamma})^2 - 2\sum_{(i<j)=1}^{n}\sigma_{ii}^{\gamma}\sigma_{jj}^{\gamma} + \frac{t}{n}(\sigma_{nn}^{\gamma})^2$$

and the optimization problem

$$\begin{aligned} &\min \ \ \mathcal{F}_\gamma \\ &\text{subject to} \ \ G : \sigma_{11}^\gamma + \cdots + \sigma_{nn}^\gamma = c^\gamma \end{aligned}$$

where c is a real constant. The partial derivatives of g_γ are

$$\begin{cases} \frac{\partial \mathcal{F}_\gamma}{\partial \sigma_{ii}^\gamma} = \frac{2(n+t)(n-1)}{t}\sigma_{ii}^\gamma - 2\sum_{l=1}^n \sigma_{ll}^\gamma \\ \frac{\partial \mathcal{F}_\gamma}{\partial \sigma_{nn}^\gamma} = \frac{2t}{n}\sigma_{nn}^\gamma - 2\sum_{l=1}^{n-1} \sigma_{ll}^\gamma \end{cases} \tag{24}$$

where $i = \{1, 2, \ldots, n-1\}, i \neq j$, and $\gamma \in \{n+1, \ldots, n+m\}$.

The vector $grad\mathcal{F}_\gamma$ is normal at G for the optimal $(\sigma_{11}^\gamma, \ldots, \sigma_{nn}^\gamma)$ of the problem. that is, it is collinear with the vector $(1, 1, \ldots, 1)$. Using (24), the critical point of the corresponding problem has the form

$$\begin{cases} \sigma_{ii}^\gamma = \frac{t}{n(n-1)} v^\gamma, i \in \{1, \ldots, n-1\} \\ \sigma_{nn}^\gamma = v^\gamma \end{cases} \tag{25}$$

By use of (25) and $\sum_{i=1}^\gamma \sigma_{ii}^\gamma = c^\gamma$, we arrive at

$$\begin{cases} \sigma_{ii}^\gamma = \frac{t}{(n+t)(n-1)} c^\gamma, i \in \{1, \ldots, n-1\} \\ \sigma_{nn}^\gamma = \frac{n}{(n+t)} c^\gamma. \end{cases} \tag{26}$$

For an arbitrary fixed point $p \in G$, the 2-form $\mathcal{B} : T_pG \times T_pG \to \mathbf{R}$ has the following form

$$\mathcal{B}(X, Y) = Hess(\mathcal{F}_\gamma(X, Y)) + \langle h(X, Y), (grad(\mathcal{F}))(x_\circ) \rangle \tag{27}$$

where h and $\langle , \rangle$ are the second fundamental form of G in $\mathbf{R}^n$ and standard inner product on $\mathbf{R}^n$ respectively. The Hessian matrix of $\mathcal{F}_\gamma$ is of the form

$$Hess(\mathcal{F}_\gamma) = \begin{pmatrix} 2\frac{(n+t)(n-1)}{t} - 2 & -2 & \ldots & -2 & -2 \\ -2 & 2\frac{(n+t)(n-1)}{t} - 2 & \ldots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \ldots & 2\frac{(n+t)(n-1)}{t} - 2 & -2 \\ -2 & -2 & \ldots & -2 & \frac{2t}{n} \end{pmatrix}.$$

Though G is totally geodesic in $\mathbf{R}^n$, take a tangent vector $X = (X_1, \ldots, X_n)$ at any arbitrary point p on G, verifying the relation $\sum_{i=1}^n X_i = 0$, we have the following

$$\begin{aligned} \mathcal{B}(X, X) &= \frac{2(n^2 - n + tn - 2t)}{t} \sum_{i=1}^{n-1} X_i^2 + \frac{2t}{n} X_n^2 - 2\Big(\sum_{i=1}^n X_i\Big)^2 \\ &= \frac{2(n^2 - n + tn - 2t)}{t} \sum_{i=1}^{n-1} X_i^2 + \frac{2t}{n} X_n^2 \\ &\geq 0 \end{aligned} \tag{28}$$

Hence the point $(\sigma_{11}^{\gamma}, \ldots, \sigma_{nn}^{\gamma})$ is the global minimum point by Lemma 3 and $\mathcal{F}_{\gamma}(\sigma_{11}^{\gamma}, \ldots, \sigma_{nn}^{\gamma}) = 0$. Thus, we have $\mathcal{Q} \geq 0$ and hence

$$\begin{aligned} 2\tau \;\; \leq \;\; & t\mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) + \Big[n(n-1)p + 2(n-1)qtr(B)\Big] \\ & -(\Lambda_1+\Lambda_2)(n-1)a - \Lambda_2(\Lambda_1-\Lambda_2)(n-1)b - (\Lambda_1-\Lambda_2)(n-1)n\pi(H), \end{aligned}$$

whereby, we obtain

$$\begin{aligned} \rho \;\; \leq \;\; & \frac{t}{n(n-1)}\mathcal{C} + \frac{(n+t)(n^2-n-t)}{n^2 t}\mathcal{C}(L) + \left[p + \frac{2}{n}qtr(B)\right] \\ & -\frac{(\Lambda_1+\Lambda_2)}{n}a - \frac{\Lambda_2(\Lambda_1-\Lambda_2)}{n}b - (\Lambda_1-\Lambda_2)\pi(H), \end{aligned}$$

for every tangent hyperplane L of M. If we take the infimum over all tangent hyperplanes L, the result trivially follows. Moreover the equality sign holds iff

$$\sigma_{ij}^{\gamma} = 0, \; \forall \; i,j \in \{1,\ldots,n\}, \; i \neq j \text{ and } \gamma \in \{n+1,\ldots,m\} \tag{29}$$

and

$$\sigma_{nn}^{\gamma} = \frac{n(n-1)}{t}\sigma_{11}^{\gamma} = \cdots = \frac{n(n-1)}{t}\sigma_{n-1n-1}^{\gamma}, \quad \forall \gamma \in \{n+1,\ldots,m\}. \tag{30}$$

From (29) and (30), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\widetilde{M}$, such that the shape operator takes the form (14) with respect to the orthonormal tangent and orthonormal normal frames.

In the same way, we can prove (ii).

□

5 Conflicts of interest

The authors declare no conflict of interest.

References

[1] Agashe, N.S. and Chafle, M.R., *A semi-symmetric non-metric connection on a Riemannian manifold*, Indian J. Pure Appl. Math. **23** (1992), 399-409.

[2] Agashe, N.S. and Chafle, M.R., *On submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection*, Tensor, **55** (1994), 120-130.

[3] Casorati, F., *Mesure de la courbure des surface suivant 1'idee commune. Ses rapports avec les mesures de coubure gaussienne et moyenne*, Acta Math. **14**, (1999), 95-110.

[4] Chen, B.Y. , *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. (Basel) **60** (1993), 568-578.

[5] Chen, B.Y., *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*, Glasg. Math. J. **41** (1999), 33-41.

[6] Chen, B.Y., *Mean curvature and shape operator of isometric immersions in real space forms*, Glasg. Math. J. **38** (1996), 87-97.

[7] Chen, B.Y., *Hypersurfaces of a conformally flat space*, Tensor (N. S.) **26** (1972), 318-322.

[8] Chen, B.Y., *Minimal Submanifolds in a Riemannian Manifold*, University of Kansas Press, Lawrence 1968.

[9] De, U. C. and Gazi, A. K., *on the existance of nearly quasi-Einstien manifolds*, Novi. Sad. J. Math., **39**, (2009), 111-117.

[10] Decu, S., Haesen, S. and Verstralelen, L., *Optimal inequalities involving Casorati curvatures*, Bull. Transylv. Univ. Brasov, Ser B, **49**, (2007), 85-93.

[11] Decu, S., Haesen, S. and Verstralelen, L., *Optimal inequalities characterizing quasi-umbilical submanifolds*, J. Inequalities Pure. Appl. Math, **9**, (2008), Article ID 79, 7pp.

[12] Golab, S., *On semi-symmetric and quarter-symmetric linear connections.*, Tensor N.S., **29** (1975), 249-254.

[13] Haesen, S., Kowalczyk, D. and Verstralelen, L., *On the extrinsic principal directions of Riemannnian submanifolds*, Note Math. **29** (2009), 41-53.

[14] Gülbahar, M., Kihc, E., Keles, S. and Triphati, M. M., *Some basic inequalities for submanifolds of nearly quasi-constant curvature manifolds*, Dynamical Systems **16** (2014), 156-167.

[15] Hayden, H.A., *Subspaces of a space with torsion*, Proc. London Math. Soc. **34** (1932), 27-50.

[16] Imai, T., *Notes on semi-symmetric metric connections*, Tensor **24** (1972), 293-296.

[17] Kowalczyk, D., *Casorati curvatures*, Bull. Transilvania Univ. Brasov Ser. III **50** (2008), no. 1, 2009-2013.

[18] Lee, C. W., Lee, J.W., Vilcu, G. E. and Yoon, D. Y.,*Optimal inequalities for the Casorati curvatures of the submanifolds of Generalized space form endowed with seni-symmetric metric connections*, Bull. Korean Math. Soc. **52** (2015), 1631-1647.

[19] Lee, J. W. and Vilcu, G. E., *Inequalities for generalized normalized δ-Casorati curvatures of slant submanifolds in quaternion space forms*, Taiwanese J. Math. **19** (2015), 691-702.

[20] Mihai, A. and Özgür, C., *Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection*, Taiwanese J. Math. **14** (2010), 1465-1477.

[21] Nakao, Z., *Submanifolds of a Riemannian manifold with semisymmetric metric connections*. Proc. Amer. Math. Soc., **54** (1976), 261-266.

[22] Oprea, T., *Optimization methods on Riemannian manifolds*. An. Univ. Bucur. Mat. **54** (2005), 127-136.

[23] Özgür, C. and Chen, B. Y., *Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature*, Turk. J. Math. **35** (2011), 501-509.

[24] Özgür, C. and De, A., *Chen inequalities for submanifolds of a Riemannian manifold of nearly quasi-constant curvature*, Publ. Math. Debrecen **82** (2013), 439-450.

[25] Özgür, C. and Mihai, A., *Chen inequalities for submanifolds of real space forms with a semi-symmetric non-metric connection*, Canad. Math. Bull. **55** (2012), 611-622.

[26] Qu, Q. and Wang, Y., *Multiply warped products with a quarter-symmetric connection*, J. Math. Anal. Appl. **431** (2015), 955-987.

[27] Su, M., Zhang, L. and Zhang, P., *Some inequalities for submanifolds in a Riemannian manifold of nearly quasi-constant curvature*, Filomat, **31** (2017), 2467- 2475.

[28] Verstralelen, L., *Geometry of submanifolds I, The first Casorati curvature indicatrices*, Kragujevac J. Math. **37** (2013), 5-23.

[29] Verstralelen, L., *The geometry of eye and brain*, Soochow J. Math. **30** (2004), 367-376.

[30] Wang, Y., *Chen inequalities for submanifolds of complex space forms and Sasakian space forms with quarter-symmetric connections*, Int. J. Geom. Meth. Mod. Phy. **16** (2019), 1950118.

[31] Yano, K., *On semi-symmetric metric connection*. Rev. Roumaine Math. Pures Appl., **15** (1970), 1579-1586.

[32] P. Zhang, X. Pan, L. Zhang, *Inequalities for submanifolds of a Riemannian manifold of nearly quasi-constant curvature with a semi-symmetric non-metric connection*, Rev. Un. Mat. Argentina, **56** (2015), 1-19.

[33] P. Zhang, L. Zhang, W. Song, *Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection*, Taiwanese J. Math., **18** (2014), 1841-1862.

[34] P. Zhang, *Remarks on Chen's inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature*, Vietnam J. Math., **43** (2015), 557-569.

[35] P. Zhang, L. Zhang, *Casorati inequalities for submanifolds in a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection*, Symmetry, **8** (2016), 19.