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NEW RESULTS ON SEMICLOSEDNESS WITH ILLUSTRATION OF THE SOLUTION OF FEEDBACK CONTROL PROBLEMS

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Abstract

We study in this paper the concept of closable-semiclosed operators in a Hilbert space and their algebraic and topological structures are investigated. We establish a non-trivial correspondence between closable and semiclosed operators. We also provide some necessary and/or sufficient conditions under which semiclosed operaors are closable. Interesting examples are provided and as an illustration, we investigate existence and uniqueness of solutions of feedback control problems where the characteristic operator is semiclosed.

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1 Introduction and motivations

We begin this introduction by establishing some notations and terminologies as well as certain specific details about closed and closable operators. We are usually concerned with a complex Banach space E with norm $\|.\|$, or with an infinite dimensional complex Hilbert space H with inner product $\langle ., . \rangle$ and A a linear operator acting in E or in H. $\perp$ is the orthogonality with respect to the inner product of H. We use $D(A)$, $N(A)$, $R(A)$ and $\sigma(A)$ to denote the domain, the kernel, the range and the spectrum of A, respectively, and write $d(A)$ for the codimension of the range $R(A)$. If A is densely defined linear operator ($D(A)$

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dense in E), the operator A^*, adjoint of A, is defined on the dual space E^* of E by:

$$\langle A^* f, x\rangle = \langle f, Ax\rangle \,, \text{ for all } x \in D(A) \text{ and } f \in D(A^*)$$

with domain $D(A^*) = \{f \in E^* : x \mapsto \langle f, Ax\rangle \text{ is continuous on } D(A)\}$.

Operators that are symmetric, $\langle Af, x\rangle = \langle f, Ax\rangle$, for all $f, x \in D(A)$, play a particularly important role, as they correspond to the observables in the theory of quantum mechanics. A is said to be closed if for any sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} Ax_n = y$ exist in E, one has $x \in D(A)$ and $Ax = y$. Or, since (x_n, Ax_n) belongs to the graph $G(A) = \{(x, Ax) : x \in D(A)\}$ of A, it's clear that A is closed if and only if $G(A)$ is a closed subspace of $E \times E$. If $D(A)$ is closed in E, then the closed graph theorem asserts that A is bounded if and only if A is closed. We denote by $B(E)$ the algebra of bounded linear operators on E. $C(E)$ is the set of closed operators in E. We will be mainly interested in unbounded operators with dense domain, it is these operators who admit an adjoint. $\|x\|_A = \|x\| + \|Ax\|$ defines a norm on $D(A)$ which we call the graph norm. The operator A is closed if and only if $(D(A), \|.\|_A)$ is a Banach space.

An operator A on E is called closable if there exists a closed operator B such that $A \subset B$, i.e. B is an extension of A or $D(A) \subset D(B)$ and $Ax = Bx$ for all $x \in D(A)$, which still implies that $G(A) \subset G(B)$. In that case, there exists a smallest closed extension $\overline{A}$ of A which is called the closure of A. Thus, A is closable if and only if for $(x_n)_{n\in\mathbb{N}}$ in $D(A)$, $y \in E$ such that $\lim\limits_{n\to\infty} x_n = 0$ and $\lim\limits_{n\to\infty} Ax_n = y$ one has $y = 0$. So, if A is closable $G(\overline{A}) = \overline{G(A)}$.

An operator may be not closed for two different reasons. The first reason is that the domain had been chosen too small, but the operator has a closed extension. The second possible reason is that an operator may not have any closed extensions. Let's remember that if E is reflexive and A is densely defined and closable, then A^* is closed and densely defined and $A^{**} = \overline{A}$. Hence, A is continuous if and only if $D(A) = E$, and in this case $D(A^*) = E^*$, $A^* : E^* \longrightarrow E^*$ is continuous and $\|A\| = \|A^*\|$. In addition, if $A \subset B$ then $B^* \subset A^*$ and all densely defined symmetric operator A, i.e. $A \subset A^*$, is closable and $A \subset A^{**} \subset A^*$.

On the other hand, the importance of essential self-adjointness (symmetric, densely defined operators that have a self-adjoint extension) is that it often leads to a non-closed but closable symmetric operator. Therefore, the non-density of the domain of the adjoint makes clear why an operator may not have a closed extension!. Note that there exist non closable operators. Fortunately enough, such operators do not play an essential role in mathematical formulations of quantum mechanics, but their mathematical interest remains proven.

A synthetic example of operator that not have any closed extensions is introduced as follows, let $H = l^2$ the space of square-summable sequences, let A be the operator with domain:

$$D(A) = \left\{x = (x_n)_{n\in\mathbb{N}} \subset l^2 : x_n \neq 0, \text{ for finitely many } n\right\}$$

and

$$Ax = \left(\sum_{j=0}^{\infty} x_j, 0, 0, 0, ... \right).$$

Let's determine A^*. Let e_n be the standard unit vector. Pick $y \in D(A^*)$, then:

$$1.\overline{y_1} = \langle Ae_n, y \rangle = \langle e_n, A^* y \rangle = 1.\overline{(A^* y)_n}, \ n \in \mathbb{N}.$$

This yields that $A^* y = 0$, and we obtain $y_1 = 0$. So, for any $y \in D(A^*)$ we have $y_1 = 0$ and $A^* y = 0$. Now consider the linear operator B given by:

$$D(B) = \left\{ y = (y_n)_{n \in \mathbb{N}} \subset l^2 : y_1 = 0 \right\}, \ By = 0.$$

Let $y \in D(B)$, $\langle Ax, y \rangle = \langle x, By \rangle$, for all $x \in D(A)$. Therefore, $y \in D(A^*)$ and $A^* y = By$. So, $D(A^*) = \left\{ y = (y_n)_{n \in \mathbb{N}} \subset l^2 : y_1 = 0 \right\}$, $A^* y = 0$, for all $y \in D(A^*)$. Since $D(A^*)$ is not dense in l^2, the operator A is not closable.

While unbounded operators cannot be continuous, for some unbounded operators there is some important properties that play a role similar to continuity in a bounded operator such as closedness and closability of operators. Furthermore, every infinite-dimensional normed space admits a non-closable linear operator. The proof requires the axiom of choice and so it is in general non-constructive [15]. In particular, non-closable operators have as spectrum the whole of $\mathbb{C}$. Indeed, if A is non-closable then $\sigma(A) = \mathbb{C}$ and we can check that neither is $A - \lambda$, $\lambda \in \mathbb{C}$. We then have to see that A is not invertible. Suppose that A is boundedly invertible. If $x_n \in D(A)$ for all $n \in \mathbb{N}$, such that $x_n \to 0$ and $Ax_n = y_n \longrightarrow y \neq 0$ as $n \to \infty$, then $A^{-1}Ax_n = A^{-1}y_n \longrightarrow A^{-1}y$ but $A^{-1}Ax_n = x_n \to 0$, as $n \to \infty$, thus $A^{-1}y = 0 \in D(A)$ since $D(A)$ has to be a linear space and $A(0) = 0$, since A is linear, this is a contradiction.

If A is a closed operator on E, then $N(A)$ is a closed subspace of E, this is generally not true for closable operators. Let $A : D(A) \longrightarrow E$ be a densely defined closable operator with closure $\overline{A}$, then $\overline{N(A)} \subsetneq N(\overline{A})$. The inclusion is simple to verify, let's show that it can be strict. Indeed, let $E = l^2(\mathbb{N}^*)$ and $\{e_k : k \in \mathbb{N}^*\}$ the canonical basis of $l^2(\mathbb{N}^*)$. Define:

$$A(x_1, x_2, x_3, ...) = (0, x_2, x_3, ...)$$

with $D(A)$ the subspace of $l^2(\mathbb{N}^*)$ spanned by $\{f_1, e_2, e_3, ...\}$ where $f_1 = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, ...\right)$. Then A is densely defined and $e_1 \notin D(A)$. Hence, $\overline{N(A)} = N(A) = \{0\}$ but $N(\overline{A}) = \mathbb{C}e_1$.

For similar results on the non-closedness of the kernel we can cite the following interesting theorem:

Theorem 1. *Let $f : E \longrightarrow \mathbb{C}$ be a linear functional on E. Then f is a bounded linear functional if and only if $N(f)$ is closed in E.*

Proof. If f is bounded, then $N(f) = f^{-1}(\{0\})$ is closed since $\{0\}$ is closed in E.

Conversely, Suppose that $N(f)$ is closed and f is unbounded, f not identically zero on E. Then, for each $n \in \mathbb{N}$, there exists $x_n \in E$, $\|x_n\| = 1$, such that $|f(x_n)| \geq n$. Let $x \in E$, with $f(x) \neq 0$ and $y_n = x - \frac{f(x)}{f(x_n)} x_n$, $n \in \mathbb{N}^*$. Note that $y_n \in N(f)$, for all $n \in \mathbb{N}^*$, and that the sequence $(y_n)_{n\in\mathbb{N}^*}$ converges to x in E :

$$\|y_n - x\| = \left| \frac{f(x)}{f(x_n)} \right| \leq \frac{|f(x)|}{n} \longrightarrow 0 \text{ as } n \to \infty.$$

So, $x \in N(f)$ which is a contradiction. Therefore, the assumption that f is unbounded is false and then f is bounded on E. □

In fact, there exists an unbounded linear functional $f : E \longrightarrow \mathbb{C}$ iff $\dim E = \infty$. Indeed, choose a normalized countably infinite set of linearly independent vectors $\mathcal{B}' = \{e_k : k \in \mathbb{N}\}$ in E. Extend $\mathcal{B}'$ to a Hamel basis $\mathcal{B}$ for E. Now define, $f : \mathcal{B} \longrightarrow \mathbb{C}$ by setting, $f(\mathcal{B} \backslash \mathcal{B}') = 0$ and $f(e_k) = k$, $k \in \mathbb{N}$. Extend f linearly to whole E, call the extended functional f. Check that f is unbounded and its kernel $N(f)$ is necessarily not closed in E.

Let A be a linear operator not necessarily closed on E, the property $R(A)$ being closed may be characterized by means of a suitable number associated with A. The reduced minimum modulus of A is defined by:

$$\gamma(A) = \sup \{\gamma : \|Ax\| \geq \gamma d(x, N(A)) \text{ for all } x \in D(A)\}$$

where $d(x, N(A))$ denotes the distance from x to $N(A)$. $\gamma(A)$ satisfies the following properties ([10], [12]):

$$\begin{aligned} R(A) \text{ is closed} &\iff R(A^*) \text{ is closed} \\ &\iff \gamma(A) = \gamma(A^*) > 0 \\ &\iff \exists r > 0,\ \sigma(A^*A) \subset \{0\} \cup [r, +\infty[\,. \end{aligned}$$

In addition, if in particular A is a closed operator and $R(A)$ is complemented in E or $d(A) < \infty$, then $R(A)$ is closed. In general, for a subspace M of E such that $E = M \oplus Y$ this does not imply that M is closed, because considering a non-continuous linear functional f on E, f exists by virtue of Theorem 1.1, and put $M = N(f)$. Then there exists a one-dimensional subspace Y of E such that $E = M \oplus Y$ but M is not closed. Consequently, we dont guarantee that $\dim E/M < \infty$ and then M is closed. However, if M is a range of a closed linear operator then $\dim E/R(A) < \infty$ implies that $R(A)$ is closed.

Also, $\gamma(A) = \infty$ if and only if $N(A)$ is dense in $D(A)$, in particular, there are closable operators that have non-closed infinite dimensional null space (see [12]). Furthermore, the range space of a compact operator is closed, if and only if it is finite-dimensional. So, the important conclusion is that if A is a compact operator acting between infinite-dimensional Hilbert spaces, and $R(A)$ is infinite-dimensional (e.g., an integral operator with a non-degenerate kernel), then $R(A)$ is not closed. The following example shows that a large class of operators of practical interest having non-closed range. Let $H = L^2[a,b]$ and consider the integral operator K,

$$(Kf)(x) = \int_a^b k(x,y)f(y)dy, \ a \leq x \leq b$$

where $k \in L^2([a,b] \times [a,b])$. We know that K is compact and has non-closed range if the kernel is non-degenerate in the sense that k is never written in the form $k(x,y) = \sum_{j=1}^{p} \phi_j(x)\psi_j(x)$ where $\phi_j, \psi_j \in L^2[a,b], j \in \{1,...,p\}$.

Nevertheless, note also that Filmore and Williams have studied in [[5], Theorem 2.9 and its Corollary, p. 266-267] operators with non-closed range. Moreover, linear operator equations of type $Ax = y$; A linear opeartor, are called ill-posed in the sense of Nashed if the range $R(A)$ of A is not closed but contains a closed infinite dimensional subspace.

Cauchy problems on some Banach space E :

$$(\mathcal{P}) \begin{cases} \frac{du}{dt} = Au(t), \ u(0) = x \in E, \\ t \in [0,T], \ 0 < T \leq \infty. \end{cases}$$

where A is a closed or closable operator with sufficiently nice spectral properties, were extensively studied. Under appropriate boundary conditions theses problems are wellposed. However, there are many situations which lead inherently illposed Cauchy problems, for example the heat problem ($A = -\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$) for a rod if heat and flux can be observed at only one point, the solutions will not necessarily depend continuously on the data [4]. Some examples of illposed Cauchy problems are given by Agmon and Nirenberg in [1]. If A is sum, composition, or limit of closed operators, existence results for the abstract Cauchy problem $\mathcal{P}$ depend on the semiclosed character of operator A. Thus, the class of closed operators was extended by Messirdi et al. [13] to a larger set of almost closed linear operators: namely, the operators whose graph, under a convenient norm, is a Banach space continuously embedded in $E \times E$, these latter are called operator ranges by Fillmore and Williams [5]; Julia operators by Dixmier [3] and semiclosed operators by Foias [6] and Kaufman [9]. The class of semiclosed operators contains the set of all closed linear operators and is invariant under addition, composition and limits. From an appropriate description of different concepts of the closedness of linear operators, Messirdi et al. were the first to note in [13], through constructive examples, that there exists semiclosed operators which are not closable and others closable linear operators which are not semiclosed. It then became natural to characterize the intersection of these two classes by studying closable-semiclosed operators.

This paper concern this concept of operators and introduces some new properties of semiclosed subspaces, semiclosed operators and closable-semiclosed operators. Usually these operators act on Hilbert spaces, and much attention is being paid here to their topological actions in operator theory, which include the operator ranges, operators with closed range and quotients of bounded operators. However, this does not prevent us from defining semiclosedness in general on Banach spaces. It is essentially established in this work a non-trivial correspondence between closable and semiclosed operators. We also provide some necessary and/or sufficient conditions under which semiclosed operaors are closable. Interesting examples are provided, they allow readers to understand the semiclosed character, and intuitively understand the comparison results between closable and semiclosed operators. As an illustration, we investigate existence and uniqueness of solutions of certain linear evolution equations with Feedback control, where the characteristic operators might be semiclosed. Laplace transform theory provides analytic tools for a detailed study of these types of problems.

In Section 2, we investigate the class of semiclosed subspaces and semiclosed operators. In Section 3, we characterize the set of closable-semiclosed linear operators what is the main purpose of the paper. In section 4, we study existence and uniqueness of solutions of linear evolution equations with Feedback Control, by using Laplace transform theory for Banach space valued functions.

2 New results on semiclosedness

Definition 1. *1) For a subspace $M \subset H$, we say that it is semiclosed if it admits an inner product $\langle .,. \rangle_M$ with respect to which it becomes a Hilbert space, and the embedding:*

$$\begin{aligned} J \quad : \quad & (M, \langle .,. \rangle_M) \longrightarrow (H, \langle .,. \rangle) \\ & x \mapsto Jx = x \end{aligned}$$

is a continuous operator.

2) A linear operator $A : D(A) \longrightarrow H$ is semiclosed if its graph $G(A)$ is a semiclosed subspace in $H \times H$. Equivalently, if there exist an inner product $\langle .,. \rangle_*$ on $G(A)$ such that $(G(A), \langle .,. \rangle_*)$ is Hilbert and $C > 0$ with $\|X\|^2 \leq C \langle X, X \rangle_*$ for all $X \in G(A)$. The set of all semiclosed operators on H is denoted by $SC(H)$.

Remark 1. *This definition is naturally generalizable to Banach spaces by replacing the inner product with the norm of space.*

Example 1. *1) Classical Sobolev spaces $H^s(\mathbb{R}^n)$, $n \in \mathbb{N}^*$, $s \geq 0$, are semiclosed subspaces in $L^2(\mathbb{R}^n)$ with the corresponding inner product:*

$$\langle \varphi, \psi \rangle_s = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}d\xi,$$

where $\mathcal{F}$ is the Fourier transform on $L^2(\mathbb{R}^n)$.

2) The intersection, sum and product of closed (semiclosed) subspaces of H are semiclosed. Indeed, if M and N are two semiclosed subspaces of H, then the *standard inner product defined on* $M \times N$ *is given by:*

$$\langle (x_M, x_N), (y_M, y_N) \rangle_{M\times N} = \langle x_M, y_M \rangle_M + \langle x_N, y_N \rangle_N$$

for all $x_M, y_M \in M$ *and* $x_N, y_N \in N$, *where* $\langle ., . \rangle_M$ *and* $\langle ., . \rangle_N$ *are the corresponding inner products on* M *and* N, *repectively, with:*

$$\|x_M\| \leq \alpha\sqrt{\langle x_M, x_M \rangle_M}, \;\; \|x_N\| \leq \beta\sqrt{\langle x_N, x_N \rangle_N}; \;\; \alpha, \beta > 0.$$

It is immediate to show that $\left(M \times N, \langle ., . \rangle_{M\times N}\right)$ *is complete, on the other hand we have:*

$$\begin{aligned} \|(x_M, x_N)\|_{H\times H} &= \sqrt{\|x_M\|^2 + \|x_N\|^2} \leq \alpha \langle x_M, x_M \rangle_M + \beta \langle x_N, x_N \rangle_N \\ &\leq \max(\alpha, \beta) \langle (x_M, x_N), (x_M, x_N) \rangle_{M\times N}. \end{aligned}$$

So, $M \times N$ *is a semiclosed subspace of* $H \times H$ *with respect to* $\langle ., . \rangle_{M\times N}$. *We check in the same way that* $M + N$ *and* $M \cap N$ *are semiclosed in* H.

3) Let M *be a proper dense semiclosed subspace of* H. *Then the identity operator* $I_M : M \longrightarrow H$ *is not closed but it is semiclosed operator on* H, *since:*

$$\begin{aligned} G(I_M) &= \{(x, x) : x \in M\} = (M \times M) \cap \Delta, \\ \Delta &= \{(x, y) \in H \times H : x = y\} \end{aligned}$$

is semiclosed in $H \times H$ *as intersection of the semiclosed subspace* $M \times M$ *and the closed set* Δ *in* $H \times H$.

4) In particular, the continuous injection of $H^s(\mathbb{R}^n)$, $s \geq 0$, *into* $L^2(\mathbb{R}^n)$, *is a semiclosed operator on* $L^2(\mathbb{R}^n)$.

Remark 2. *1) The set of semiclosed subspaces of a Hilbert space forms a complete lattice with respect to intersection and sum see [9].*

2) Let $A \in SC(H)$, *then* $D(A)$ *and* $R(A)$ *are semiclosed subspaces of* H. *The restriction of* A *to a semiclosed subspace is a semiclosed operator ([13]).*

3) Semiclosed operators are also used for introducing the quotient of bounded operators [7], [14].

There are, among others, some examples showing that all semiclosed subspaces or semiclosed operators are not necessarily closed and that every subspace or every operator is not necessarily semiclosed. In fact, it is indicated in [15] that considering proper dense subspaces, one can construct non semiclosed operators via the axiom of choice and Theorem 1.1.

We start with a number of new characterizations of semiclosed operators. First, semiclosed operators are induced by positive bounded operators.

Proposition 1. *Let* $A \in SC(H)$, *then there is a unique operator* $T \in B(H \times H)$, T *positive and self-adjoint, such that* $\langle X, Y \rangle_{H\times H} = \langle X, TY \rangle_*$ *for all* $X \in G(A)$ *and* $Y \in H \times H$, *where* $\langle ., \rangle_{H\times H}$ *is the natural inner product on* $H \times H$.

Proof. Let $Y \in H\times H$ and define $f_Y : H\times H \longrightarrow \mathbb{C}$ by $f_Y(X) = \langle X, Y\rangle_{H\times H}$. The restriction of f_Y to $G(A)$ is bounded on $(G(A), \langle .,.\rangle_*)$, because for $X \in G(A)$,

$$|f_Y(X)| \leq \sqrt{C}\,\|X\|_*\,\|Y\|_{H\times H}$$

where $\|.\|_{H\times H} = \sqrt{\langle .,.\rangle_{H\times H}}$ and $\|.\|_* = \sqrt{\langle .,.\rangle_*}$. By Riesz representation theorem, there exists a unique $Z \in G(A)$ such that $f_Y(X) = \langle X, Z\rangle_*$.

Define $T : \left(H \times H, \langle .,.\rangle_{H\times H}\right) \longrightarrow (G(A), \langle .,.\rangle_*)$ by $TY = Z$. Then $\langle X, Y\rangle_{H\times H} = \langle X, TY\rangle_*$ for all $X \in G(A)$, $Y \in H \times H$. Clearly, $T(H \times H) \subset G(A)$ and for each $Y \in H \times H$,

$$\|TY\|_{H\times H} \leq \sqrt{C}\,\|TY\|_* \leq \sqrt{C'}\,\|Y\|_{H\times H}\,;\ C' > 0.$$

Furthermore, for all $X, Y \in H \times H$,

$$\begin{aligned}\langle TY, X\rangle_{H\times H} &= \langle TY, TX\rangle_* = \overline{\langle TX, TY\rangle_*} = \overline{\langle TX, Y\rangle_{H\times H}} = \langle Y, TX\rangle_{H\times H}\,,\\ \langle TY, Y\rangle_{H\times H} &= \langle TY, TY\rangle_* = \|TY\|_*^2 \geq 0.\end{aligned}$$

So, T is bounded positive self-adjoint operator on $H \times H$. □

Remark 3. *1) It can be deduced from the proof of the previous proposition that $G(A)$ is invariant under T, and for all semiclosed subspace M of H, and for each Hilbert inner product $\langle .,.\rangle_M$ on M such that $(M, \langle .,.\rangle_M)$ is continuously embedded in $(H, \langle .,.\rangle)$, there is a unique bounded positive selfadjoint operator S on H such that $\langle x, y\rangle = \langle x, Sy\rangle_M$, for all $x \in M$ and $y \in H$. M is invariant under S. In addition, by virtue of uniqueness of the Riesz vector, $T(M \times \{0\}) = S(M) \times \{0\}$, $T(\{0\}\times M) = \{0\}\times S(M)$ and $T(M\times M) = S(M)\times S(M)$. So, $M\times\{0\}$, $\{0\}\times M$ and $M \times M$ are invariant under T.*

2) Unlike closed operators, we have already mentioned that if $A \in SC(H)$, $N(A)$ is not necessarely closed in H, nevertheless it is in $D(A)_*$, where $D(A)_*$ is the domain $D(A)$ equipped with Hilbert structure induced by the inner product $\langle .,.\rangle_*$.

3) Proposition 2.5 also shows that each semiclosed subspace of H possesses a unique topology in the sense that if $\langle .,.\rangle_{1,M}$ and $\langle .,.\rangle_{2,M}$ are inner products on M semiclosed subspace of H such that $\left(M, \langle .,.\rangle_{j,M}\right)$ is continuously embedded in $(H, \langle .,.\rangle)$, $j \in \{1, 2\}$, then the topology induced on M by $\langle .,.\rangle_{1,M}$ coincides with that induced by $\langle .,.\rangle_{2,M}$.

We know that sum and product of two semiclosed operators is also a semiclosed operator which is not the case for closed operators (see [13]).

Proposition 2. *Let $A, B \in SC(H)$ such that $D(A)\cap D(B)$ and $B^{-1}(D(A))$ are not trivial, then $A + B \in SC(H)$ and $AB \in SC(H)$.*

Hence, the set $SC(H)$ is closed under addition and multiplication, but it is not a vector space (see [9], [13]). In fact, $SC(H)$ is the smallest family containing all closed operators and itself closed under sum and product, precisely if $Cl(H))$ denotes the set of all closable operators in H, then $C(H) \subset Cl(H)$ and $C(H) \subset Cl(H) \cap SC(H)$. However, only Messirdi et al. have shown in [13] and [15] that the classes $Cl(H)$ and $SC(H)$ are not comparable, since there is semiclosed operators that are not closable and others closable that are not semiclosed. Constructive examples are given in [13].

Proposition 3. *Let $A \in SC(H)$.*
1) If $D(A) = H$ then A is bounded.
2) A still has a densely defined semiclosed extension.
3) $R(A)$ is closed if and only if $R(A) \oplus N$ is closed for some semiclosed subspace N of H.

For assertions (1) and (2), see [13]. On the other hand, it is clear that the sum of two closed subspaces of an infinite dimensional Banach space E is not necessarily closed. The following lemma due to Neubauer, see [11], gives sufficient conditions under which semiclosed subspaces are closed, it also allows to show the assertion (3) of the previous proposition.

Lemma 1. *(Neubauer's lemma) Let M and N be semiclosed subspaces of E. If $M + N$ and $M \cap N$ are closed in E, then both M and N are closed.*

Lemma 2. *Let M_1 and M_2 be two subspaces of E such that M_1 is semiclosed, M_2 and $M_1 + M_2$ are closed. Then,*
1) $\overline{M_1} = M_1 + \overline{M_1 \cap M_2}$,
2) $\overline{M_1 \cap M_2} = \overline{M_1} \cap M_2$.

Proof. 1) Let's put $M_0 = M_1 + \overline{M_1 \cap M_2}$. Thus, M_0 is a semiclosed subspace of E and since $\overline{M_1 \cap M_2} \subset M_2$ and $M_1 \subset M_0$, we have:

$$M_0 + M_2 \subset M_1 + M_2 \subset M_0 + M_2.$$

So, $M_0 + M_2 = M_1 + M_2$ is closed in E. In addition,

$$M_0 \cap M_2 = \left(M_1 + \overline{M_1 \cap M_2}\right) \cap M_2 \subset \overline{M_1 \cap M_2} \subset M_0 \cap M_2.$$

Thus, $M_0 \cap M_2 = \overline{M_1 \cap M_2}$ is also closed in E. Now, using the Neubauer's lemma we deduce that M_0 is closed in E. Precisely, $M_1 \subset M_0 \subset \overline{M_1}$ and then $M_0 = \overline{M_1}$.
2) Therefore,

$$\overline{M_1} \cap M_2 = M_0 \cap M_2 = \overline{M_1 \cap M_2}.$$

□

Furthermore, if A a linear operator on E, it is simple to establish that:

Lemma 3.

$$(E \times \{0\}) + G(A) = E \times \{0\} + \{0\} \times R(A),$$

and

$$(E \times \{0\}) \cap G(A) = N(A) \times \{0\}.$$

The following result brings interesting informations about closed extensions of a semiclosed operator.

Lemma 4. *Let $A \in SC(E)$ with closed range $R(A)$ in E, then:*

$$\overline{G(A)} = G(A) + \overline{N(A)} \times \{0\}.$$

Proof. As $R(A)$ is closed in E and $G(A)$ is semiclosed in $E \times E$, it follows from Lemma 2.11, that $(E \times \{0\}) + G(A) = E \times \{0\} + \{0\} \times R(A)$ is closed in $E \times E$. So, using Lemma 2.10, with $M_1 = G(A)$ and $M_2 = E \times \{0\}$, we obtain:

$$\overline{G(A)} = G(A) + \overline{G(A) \cap (E \times \{0\})}.$$

Thus, by virtue of Lemma 2.11, we have $\overline{G(A) \cap (E \times \{0\})} = \overline{N(A) \times \{0\}} = \overline{N(A)} \times \{0\}$, and then the requested result. □

We are now able to prove our first fundamental result:

Theorem 2. *Let $A \in SC(E)$.*
1) If $N(A)$ and $R(A)$ are closed in H, then $A \in C(E)$.
2) If $R(A)$ is closed, then A is closable if and only if $\overline{N(A)} \cap D(A) = N(A)$.

Proof. 1) If $N(A)$ and $R(A)$ are closed in E, we deduce from Lemma 2.12, that:

$$\overline{G(A)} = G(A) + N(A) \times \{0\} = G(A)$$

so $G(A)$ is closed in $E \times E$, which implies that A is closed.

2) Suppose that $R(A)$ is closed in E. If $\overline{N(A)} \cap D(A) = N(A)$, show that A is closable on E. Let $(0, v) \in \overline{G(A)} = G(A) + \overline{N(A)} \times \{0\}$, then there are $s \in D(A)$ and $t \in \overline{N(A)}$ such that $(0, v) = (s, As) + (t, 0)$. So,

$$\begin{cases} s + t = 0 \\ As = v \end{cases},$$

from where $s = -t \in D(A) \cap \overline{N(A)} = N(A)$ and $v = As = 0$. $\overline{A} = A^{**}$.

Conversely, if A is closable just check that $\overline{N(A)} \cap D(A) \subset N(A)$. Let $u \in \overline{N(A)} \cap D(A)$, then there is a sequence $(u_n)_{n \in \mathbb{N}}$ such that u_n belongs to $N(A)$ for all $n \in \mathbb{N}$ and $u_n \longrightarrow u$ in E as $n \to \infty$. Hence, $A(u_n - u) = Au \longrightarrow Au$ in E as $n \to \infty$ and thus $Au = 0$. □

3 Closable-Semiclosed operators

We deduce from Theorem 2.13, two fundamental characterizations of the class, $SC(E) \cap Cl(E)$, of closable-semiclosed operators.

Let

$$\widehat{SC}(E) = \{A \in SC(E) : N(A) \text{ and } R(A) \text{ are closed}\},$$

and

$$\widetilde{SC}(E) = \left\{A \in SC(E) : R(A) \text{ is closed and } \overline{N(A)} \cap D(A) = N(A)\right\}.$$

Corollary 1.

$$\widehat{SC}(E) = \{A \in C(E) : \gamma(A) > 0\}\,,$$

$$\widetilde{SC}(E) = \{A \in Cl(E) : N(A) \textit{ is closed and } \gamma(A) > 0\}\,.$$

In Hilbert spaces, it is possible to obtain finer results than those established essentially in Lemmas 2.10, 2.11 and 2.12.

Lemma 5. *Let M and N be semiclosed subspaces of H. If $M + N$ is closed in H and $M \cap N$ is closed in M, then:*

1) N is closed in H,

2) There is a closed subspace H_0 in H such that $H_0 \subset M$ and $H_0 + N = M + N$,

3) $(M \cap N)^{\perp} = M^{\perp} + N^{\perp}$.

Proof. Let H_0 be the orthogonal complement of $M \cap N$ in M. Then:

$$M = H_0 + M \cap N,\ \ H_0 \cap N \subset H_0 \cap M \cap N = \{0\}\,,$$

and $H_0 + N = M + N$ since $H_0 + N \subset M + N \subset H_0 + N$. Thus, $H_0 \cap N = \{0\}$ and $H_0 + N$ is closed in H. Using Neubauer's lemma and Lemma 2.10, we obtain that N and H_0 are closed in H and $\overline{M} \cap N = \overline{M \cap N}$. In addition, it is simple to show that $M + N$ closed is equivalent to $M^{\perp} + N^{\perp} = \overline{M}^{\perp} + N^{\perp} = \left(\overline{M} \cap N\right)^{\perp} = (M \cap N)^{\perp}$ (see [11]). □

Corollary 2. *Let M and N be semiclosed subspaces of H.*

1) If $M + N$ is closed in H, then:

$$\begin{aligned}
&\overline{M} = M + \overline{(M \cap N)},\ \ \overline{N} = N + \overline{(M \cap N)},\\
&\overline{M} \cap N = \overline{(M \cap N)} \cap N,\ \ M \cap \overline{N} = \overline{(M \cap N)} \cap M,\\
&\overline{M} \cap \overline{N} = \overline{M} \cap N = M \cap \overline{N} = \overline{(M \cap N)},\\
&(M \cap N)^{\perp} = M^{\perp} + N^{\perp}.
\end{aligned}$$

2) If $M + N$ is closed in H and M and N are dense in H, then $M \cap N$ is dense in H.

Proof. 1) is an immediate consequence of Lemma 2.10 and Lemma 3.2.

2)

$$(M \cap N)^{\perp} = M^{\perp} + N^{\perp} = \{0\} + \{0\} = \{0\}\,.$$

□

Proposition 4. *Let $A \in SC(H)$ be densely defined. If $R(A)$ is closed then $R(A^*) = N(A)^{\perp}$ is closed in H.*

Proof. A^* exists since $D(A)$ is assumed to be dense in H and $G(A^*) = V(G(A))^{\perp} = G(-A)^{\perp}$, where V is the isometry $V(x, y) = (-y, x)$ on $H \times H$. Thus,

$$H \times \{0\} + G(-A) = H \times \{0\} + \{0\} \times R(A)$$

is closed in $H \times H$. Then Lemma 2.11 and Lemma 3.2 give:

$$\begin{aligned} (N(A) \times \{0\})^{\perp} &= ((H \times \{0\}) \cap G(-A))^{\perp} = (\{0\} \times H) + G(A^*) \\ &= (\{0\} \times H) + (R(A^*) \times \{0\}), \end{aligned}$$

which implies $R(A^*) = N(A)^{\perp}$. □

By using successively Neubauer's lemma, Lemma 2.10 and Theorem 2.13, we obtain our second main result:

Theorem 3. *Let $A \in SC(H)$ be densely defined. If $N(A) \cap R(A)$ and $N(A) + R(A)$ are closed in H, then $N(A)$ and $R(A)$ are closed in H. Thus, $A \in C(H)$ is with closed range.*

$$\begin{aligned} &\{A \in SC_d(H) : N(A) \cap R(A) \text{ and } N(A) + R(A) \text{ are closed}\} \\ &\subset \{A \in C_d(H) : \gamma(A) > 0\}. \end{aligned}$$

where the index "d" means that operators are all densely defined in the corresponding class.

Remark 4. *Let $L(H)$ denotes the class of densely defined semiclosed operators A such that $\overline{N(A)} \cap D(A) = N(A)$. It is clear from the above that:*

$$B(H) \subset C_d(H) \subset L(H),$$

$$\{A \in SC_d(H) \cap Cl_d(H) : N(A) \text{ is closed in } H\} \subset L(H),$$

$$\begin{aligned} &\widetilde{SC}_d(H) = \{A \in L(H) : \gamma(A) > 0\} \\ &= \{A \in Cl_d(H) : N(A) \text{ closed and } \gamma(A) > 0\}, \end{aligned}$$

$$\widehat{SC}_d(H) \subset L(H).$$

The new class $L(H)$ is caracterized by the kernel and the domain of operators. But the kernel defines only a part of the image ($Ax = 0$, $A \in L(H)$), so the behavior of the rest of the points is completely undefined. Take for example the direct sum of an unbounded operator and a null operator, so we will have an operator of $L(H)$ whose only useful information it emits is that concerning its kernel. The unbounded part of such operators is completely arbitrary and as it does not contain any particular information. So the additional property concerning the range had to intervene to formulate the class $L(H)$ via $SC_d(H)$ and $Cl_d(H)$. The class $L(H)$ undoubtedly opens other horizons of research for interested readers...

4 Feedback Control Problems governed by semiclosed operators

The class of semiclosed operators is quite large. It contains basically all linear operators appearing in applications by imposing certain conditions, such that operators commute with the Bochner and Stieltjes integral for sufficiently regular functions (see [8], [15]). One of the basic results of one parameter semigroup theory is that the abstract Cauchy problem ($\mathcal{P}$) is well-posed if and only if the linear operator A generates a strongly continuous semigroup, this is not usually the case when the operator A is assumed semiclosed. In this section, we will study the abstract Cauchy problem ($\mathcal{P}$), with $A = B + CS$, B and S are closed and C is bounded in E, S is called the observation operator and C the feedback control. If $u \in C^1([0,T];E) \cap C([0,T]) ; (D(A), \langle .,. \rangle_*)$ satisfies ($\mathcal{P}$), then u is called a classical solution of ($\mathcal{P}$) where $(D(A), \langle .,. \rangle_*)$ is a Hilbert space continuously embedded in H. Given an observation operator S, the control C has to be chosen such that ($\mathcal{P}$) has unique solutions for all $x \in E$. The operator A is semiclosed, then under certain additional conditions on $N(A)$ and $R(A)$ the operator can be closable and this will solve the problem. Even if the operator A is closable, we might not want to switch to its closure, because much of the basic information on the underlying evolution problem is contained in the domain of A and would be lost by considering the larger, and often difficult to describe domain of the closure of A.

In many situations A will not be closable, so none of the usual semigroup techniques can be used to study problem ($\mathcal{P}$). As a special example of ($\mathcal{P}$), we consider $A = A_1 + A_2$ on

$$E = C_0([0,+\infty[) = \left\{ u : [0,+\infty[\longrightarrow \mathbb{C} : \text{ continuous and } \lim_{t\to+\infty} u(t) = 0 \right\},$$

such that:

$$A_1 u(t) = t\frac{du}{dt}(t) = tu'(t)$$

with domain

$$D(A_1) = \left\{ x \in E \cap C^1(]0,+\infty[) : \lim_{t\to+\infty} tu'(t) = 0 \text{ and } \lim_{t\to 0} tu'(t) \text{ exists} \right\},$$

and

$$A_2 u(t) = u'(0)c(t)$$

where $c(t)$ is any fixed function of E and

$$D(A_2) = \{ u \in E : u \text{ differentiable at } 0 \}.$$

Hence, $Au(t) = tu'(t) + u'(0)c(t)$ is defined from $D(A) = D(A_1) \cap D(A_2)$ to E. A_1 generates the strongly continuous semigroup $T(s)u(t) = u(te^s)$, $s,t \geq 0$, $T(s)u(0) = u(0)$, $\|T(s)u\|_E = \|u\|_E$ and $T(s)u(t) \to 0$ as $s \to +\infty$ for all $t > 0$. On the other hand, let $u_n(t) = -\frac{e^{-nt}}{n}$ be a sequence of elements of $D(A)$, then

$u_n \longrightarrow 0$ and $Au_n \longrightarrow c$ in E as $n \to \infty$, so for any control function $c \neq 0$, the operator A will not be closable in E. However, for any $c \in E$, the operator A is semiclosed in E, with domain the completion of $D(A)$ with respect to the graph norm $\|u\| + \|A_1 u\| + \|A_2 u\|$. We will show next that for control functions c and initial states x which are differentiable at 0, the problem $(\mathcal{P})$ has unique solution.

The link between the generator A_1 and the semigroup $T(s)$ is given via the Laplace transform:

$$(\lambda - A_1)^{-1}x = \int_0^{+\infty} e^{-\lambda t}T(t)x dt,\ x \in E.$$

If u and c are differentiable on $[0, +\infty[$, then an easy computation shows that the resolvent equation $(zI - A)y(z) = x$ has a unique solution given by:

$$y(z) = (z - A_1)^{-1}x + \frac{x'(0)}{z - 1 - c'(0)}(z - A_1)^{-1}c$$

where $1 + c'(0) \neq z$, $Re\, z > 1$ and $y(z) = \widehat{u}(z)$ for all $z > \max\{1, Re(1 + c'(0))\}$, $\widehat{u}(z) = \int_0^{+\infty} e^{-zt}u(t)dt$ is the Laplace transform of $u(t)$. Thus, from elementary Laplace transform theory, we obtain:

$$u(t) = T(t)x + x'(0)\int_0^t e^{(1+c'(0))(t-s)}T(s)cds.$$

For all $x, c \in E$ differentiable at 0, it is clear that $u \in L^1_{loc}([0, +\infty[; D(A))$, $\int_0^t u(s)ds \in L^1_{loc}([0, +\infty[; E)$ and $u(t)$ and $\int_0^t u(s)ds$ are differentiable functions. So, by virtue of Hille-Yosida Theorem and Laplace transform method, the function u defined above is the unique solution of the problem $(\mathcal{P})$ with initial data $x \in E$. For further details, the reader may consult. [[2], [1]].

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