ON C3-LIKE FINSLER METRICS UNDER RICCI FLOW

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Abstract

In this paper we have studied the class of Finsler metrics, called C3-like metrics which satisfy the un-normal and normal Ricci flow equation and proved that such metrics are Einstein.

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1 Introduction

The Ricci flow theory, which was introduced by Richard Hamilton, became a very powerful method in understanding the geometry and topology of Riemannian manifolds. Finally in the first decade of twenty first century Perelman able to classify the three dimensional smooth manifolds using Ricci flow equation [5],[6],[7]. The Ricci flow equation is the evolution equation

$$\tfrac{d}{dt}g_{ij} = -2Ric_{ij},$$

for a Riemannian metric g_{ij}, where Ric_{ij} is the Ricci curvature tensor. Hamilton showed that there is a unique solution to this equation for an arbitrary smooth metric on a closed manifold over a sufficiently short time [3],[4].
The concept of Einstein metric is also very important topic in differential geometry as well as in physics. A Riemannian manifold (M,g) is called Einstein if it's Ricci tensor Ric can be given by $Ric = kg$, where k is some constant. It is named after Albert Einstein because this condition is equivalent to saying that the metric is a solution of the vacuum Einstein field equations. A Finsler manifold (M, F) is called Einstein if there exist a scalar function $K(x)$ on M such that $Ric = (n-1)KF^2$.

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Finsler metrics satisfying Ricci flow equation has been an important topic of research. For instance Sadegzadeh and Razavi have studied C-reducible metrics satisfying Ricci flow equation [9], where as Tayebi, Payghan and Najafi have studied semi C-reducible Finsler metrics satisfying Ricci flow equation [10] and proved that such metrics are Einstein when they satisfies the Ricci flow equation. In the present paper we have extended those results to a more general class of Finsler metric, namely C3-like Finsler metric, introduced by Prasad and Singh [8]. More precisely we have prove the following theorems:

Theorem 1. *Let (M, F) be a Finsler manifold of dimension $n \geq 3$. Suppose that F be a C3-like Finsler metric, then every deformation F_t of the metric F satisfying un-normal Ricci flow equation is an Einstein metric.*

Theorem 2. *Let (M, F) be a Finsler manifold of dimension $n \geq 3$. Suppose that F be a C3-like Finsler metric, then every deformation F_t of the metric F satisfying normal Ricci flow equation is an Einstein metric.*

2 Preliminaries

Let M be an n-dimensional smooth manifold and $T_x M$ denotes the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM := \bigcup_{x \in M} T_x M$. We denote the elements of TM by (x, y) where $y \in T_x M$. Let $TM_0 := TM \setminus \{0\}$.

Definition 1. *[2] A Finsler metric on M is a function $F : TM \rightarrow [0, \infty)$ with the following properties:*
(i) F is C^∞ on TM_0,
(ii) F is a positively 1-homogeneous on the fibers of tangent bundle TM.
(iii) The Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.
The pair (M, F) is called a Finsler space, F is called the fundamental function and g_{ij} is called the fundamental tensor.

Let (M, F) be a Finsler space. For a vector $y \in T_x M \setminus \{0\}$, let

$$C_y(u, v, w) := \frac{1}{4} \frac{\partial^3}{\partial s \partial t \partial r} [F^2(y + su + tv + rw)]_{s=t=r=0}$$

where $u, v, w \in T_x M$. Each C_y is a symmetric trilinear form on V. We call the family $C := \{C_y : y \in T_x M \setminus \{0\}\}$ the Cartan torsion. Let $\{b_i\}$ be a basis for $T_x M$ and define $g_{ij} := g_y(b_i, b_j)$, $C_{ijk} := C_y(b_i, b_j, b_k)$.
Then $C_{ijk} = \frac{1}{4} [F^2]_{y^i y^j y^k} = \frac{1}{2} \frac{\partial}{\partial y^k} (g_{ij})$.
Define the mean value of the Cartan torsion by

$$I_y(u) := \Sigma_{i=1}^n g^{ij}(y) C_y(u, b_i, b_j) \, , \, u \in T_x M.$$

We call the family $I := \{I_y | y \in T_x M \setminus \{0\}\}$ the mean Cartan torsion at $x \in M$

$$\text{i.e. } I_i = g^{jk} C_{ijk}.$$

Definition 2. *[8] A Finsler metric F is called C3-like if its Cartan torsion is given by*

$$C_{ijk} = \{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\} + \{b_i I_j I_k + b_j I_i I_k + b_k I_i I_j\}, \tag{1}$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are covectors on TM and of degree -1 and 1, respectively and h_{ij} is angular metric tensor given by $h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}$.

In particular :
(i) if $a_i = 0$, then we have $C_{ijk} = b_i I_j I_k + b_j I_i I_k + b_k I_i I_j$, contracting it with g^{ij} implies that $b_i = \frac{1}{3I^2} I^i$, then F is a C2-like Finsler metric,
(ii) if $b_i = 0$, then we have $C_{ijk} = a_i h_{jk} + a_j h_{ki} + a_k h_{ij}$, contracting it with g^{ij} implies that $a_i = \frac{1}{n+1} I_i$, then F is a C-reducible Finsler metric.
(iii) if $a_i = \frac{p}{n+1} I_i$ and $b_i = \frac{q}{3I^2} I_i$, where $p = p(x, y)$ and $q = q(x, y)$ are scalar functions on TM, then F is a semi C-reducible Finsler metric.
Therefore, a C3-like Finsler metric may be consider as a generalization of C-reducible, semi C-reducible and C2-like Finsler metrics.

3 Un-normal Ricci flow equation on C3-like Finsler metrics

The geometric evolution equation

$$\frac{d}{dt} g_{ij} = -2Ric_{ij}, \tag{2}$$

is known as the un-normalized Ricci flow in Riemannian geometry. In principle, the same equation can be used in Finsler setting, because both g_{ij} and Ric_{ij} have been generalised to the broader framework, albeit gaining a y dependence in the process. However, there are two reasons why we shall refrain from doing so.
(i) Not every symmetric covariant 2-tensor $g_{ij}(x, y)$ arises from a Finsler metric $F(x, y)$.
(ii) There is more than one geometrical context in which g_{ij} makes sense.
A deformation of Finsler metrics means a 1-parameter family of metrics $g_{ij}(x, y, t)$, such that $t \in [-\epsilon, \epsilon]$ and $\epsilon > 0$ is sufficiently small. For such a metric $\omega = u_i dx^i$, the volume element as well as the connections attached to it depend on t. The same equation can be used in the Finsler setting. But instead of the above tensor evolution equation, we will use a different form of the above equation. By contracting $\frac{d}{dt} g_{ij} = -2Ric_{ij}$ with y^i and y^j respectively and using Eulers theorem, we get

$$\frac{\partial F^2}{\partial t} = -2F^2 R,$$

where $R = \frac{Ric}{F^2}$. That is,

$$d \log F = -R, F(t = 0) = F_0.$$

This scalar equation directly addresses the evolution of the Finsler metric F, and makes geometrical sense on both the manifold of nonzero tangent vectors TM_0 and the manifold of rays. It is therefore suitable as an un-normalized Ricci flow for Finsler geometry. In this section we will study C3-like Finsler metrics satisfying un-normal Ricci flow equation.

Let us assume that F_t be a deformation of C3-like Finsler metrics which satisfies the Un-Normalised Ricci flow equation given by

$$g'_{ij} = -2Ric_{ij} \quad \text{or} \quad \frac{F'}{F} = -R, \quad \text{where} \quad g'_{ij} = \frac{\partial g_{ij}}{\partial t}$$

Before proving Theorem 1.1 we need the following lemmas:

Lemma 1. *If F_t be a deformation of a C3-like Finsler metrics F on manifold M of dimension $n \geq 3$, then the variation of Cartan torsion is given by*

$$C'_{ijk}I^i I^j I^k = 3(a_i + 3b_i)I^i \|I\|^2 + \frac{1}{2}F^2 R_{,i,j,k}I^i I^j I^k + 3 \|I\|^2 I_m R_{,m}, \quad (3)$$

where $R_{,m} = \frac{\partial R}{\partial y^m}$.

Proof. Let us assume that F_t be a deformation of a Finsler metric F on a 2-dimensional manifold M satisfies Ricci flow equation given by (2).

By definition of Ricci tensor, we have

$$\begin{aligned} Ric_{ij} &= \frac{1}{2}[RF^2]_{y^i y^j} \\ &= Rg_{ij} + \frac{1}{2}F^2 R_{,i,j} + R_{,i}y_j + R_{,j}y_i \end{aligned} \quad (4)$$

where $R_{,i} = \frac{\partial R}{\partial y^i}$ and $R_{,i,j} = \frac{\partial^2 R}{\partial y^i \partial y^j}$.

Taking vertical derivative of (4) and from the fact $y_{i,j} = g_{ij}$ and $FF_k = y_k$ we have

$$\begin{aligned} Ric_{ij,k} = 2RC_{ijk} + \frac{1}{2}F^2 R_{,i,j,k} + (g_{jk}R_{,i} + g_{ij}R_{,k} + g_{ki}R_{,j}) \\ + \{R_{,j,k}y_i + R_{,i,j}y_k + R_{,k,i}y_j\}. \end{aligned} \quad (5)$$

Multiplying (5) with $I^i I^j I^k$ and using $y_i I^i = y^i I_i = 0$ we obtain

$$Ric_{ij,k}I^i I^j I^k = 2RC_{ijk}I^i I^j I^k + \frac{1}{2}F^2 R_{,i,j,k}I^i I^j I^k + 3 \|I\|^2 I_m R_{,m} \quad (6)$$

Now multiplying (1) by $I^i I^j I^k$ we have

$$C_{ijk}I^i I^j I^k = 3(a_i + 3b_i)I^i \|I\|^2. \quad (7)$$

Since F_t satisfies the Ricci flow equation so using (7) we have

$$C'_{ijk}I^i I^j I^k = 3(a_i + 3b_i)I^i \|I\|^2 + \frac{1}{2}F^2 R_{,i,j,k}I^i I^j I^k + 3 \|I\|^2 I_m R_{,m}. \quad (8)$$

$\square$

Lemma 2. *If F_t be a deformation of C3-like Finsler metrics F on a manifold M of dimension $n \geq 3$, then $C'_{ijk} I^i I^j I^k$ is divisible by $\|I\|^2$.*

Proof. Since $g^{ij} g_{jk} = \delta^i_k$, then differentiating with respect to t and using $g'_{ij} = -2Ric_{ij}$ we have $g'^{il} = 2Ric^{il}$.

At first we will calculate the value of I'_i

$$
\begin{aligned}
I'_i &= (g^{jk} C_{ijk})' \\
&= (g^{jk})' C_{ijk} + g^{jk} C'_{ijk} \\
&= 2Ric^{jk} C_{ijk} - g^{jk} Ric_{jk,i} \\
&= Ric^{jk} g_{jk,i} - (g^{jk} Ric_{jk})_{,i} + g^{jk}_{,i} Ric_{jk} \\
&= -(g^{jk} Ric_{jk}), i \\
&= -\rho_i
\end{aligned}
\tag{9}
$$

where $\rho := g^{jk} Ric_{jk}$ and $\rho_i = \frac{\partial \rho}{\partial y^i}$.

Since $y_i := F F_{y^i}$, differentiating with respect to t we have

$$
y'_i = -2Ric_{im} y^m.
$$

Now we compute h'_{ij}.

$$
\begin{aligned}
h'_{ij} &= (g_{ij} - F^{-2} y_i y_j)' \\
&= g'_{ij} - \left[-2F^{-3} F' y_i y_j + F^{-2} y'_i y_j + F^{-2} y_i y'_j \right] \\
&= -2Ric_{ij} + 2F^{-2} \frac{F'}{F} y_i y_j - F^{-2} (y'_i y_j + y_i y'_j) \\
&= -2Ric_{ij} + 2F^{-2} \frac{F'}{F} y_i y_j + 2F^{-2} (Ric_{im} y_j + Ric_{jm} y_i) y^m \\
&= -2Ric_{ij} - 2Rl_i l_j + 2(Ric_{im} l_j + Ric_{jm} l_i) l^m \\
&= -2Ric_{ij} + 2R(h_{ij} - g_{ij}) + 2(Ric_{im} l_j + Ric_{jm} l_i) l^m
\end{aligned}
\tag{10}
$$

where $l_i = \frac{y_i}{F}$.

Multiplying (1) by g^{ij} we have

$$
a_i = \frac{1}{n+1} \left\{ (1 - 2I^m b_m) I_i - \|I^2\| b_i \right\}.
\tag{11}
$$

After differentiating (1) with respect to t we have

$$
\begin{aligned}
C'_{ijk} = (a_i h'_{jk} + a_j h'_{ki} + a_k h'_{ij}) + (a'_i h_{jk} + a'_j h_{ki} + a'_k h_{ij}) \\
- (b_i(\rho_j I_k + I_j \rho_k) + b_j(\rho_i I_k + I_i \rho_k) + b_k(\rho_i I_j + I_i \rho_j)).
\end{aligned}
\tag{12}
$$

Now contracting (12) with $I^i I^j I^k$ we have

$$
C'_{ijk} I^i I^j I^k = \left(a_i h'_{jk} + a_j h'_{ki} + a_k h'_{ij} \right) I^i I^j I^k + 3a'_i I^i \|I\|^2 - 6b_i \rho_j \|I\|^2 I^i I^j.
\tag{13}
$$

Using (10) and (11) in (13) we obtain

$$C'_{ijk}I^iI^jI^k = \frac{-6}{n+1}Ric_{ij}I^iI^j\left(1-3b_mI^m\right)\|I^2\| + 3a'_iI^i\|I^2\| - 6b_i\rho_j\|I^2\|I^iI^j. \quad (14)$$

Therfore the result follows. $\square$

Proof of Theorem 1.1

In the view of (8) and (14) it follows that $R_{,i,j,k}I^iI^jI^k$ is divisible by $\|I^2\|$. Then we can write

$$R_{,i,j,k} = A_{ij}I_k + B_ig_{jk}. \quad (15)$$

By contracting $R_{,i,j,k}$ with y^j or y^k we have $R_{,i} = 0$. So F_t is Einstein.

4   Normal Ricci flow equation on C3-like Finsler metrics

If M is compact, then so is SM, and we can normalize the above equation by requiring that the flow keeps the volume of SM constant. Recalling the Hilbert form $\omega := F_{y^i}dx^i$, that volume is

$$Vol_{SM} := \int_{SM} \frac{(-1)^{\frac{(n-1)(n-2)}{2}}}{(n-1)!}\omega \wedge (d\omega)^{(n-1)} := \int_{SM} dV_{SM}.$$

During the evolution, F, ω and consequently the volume form dV_{SM} and the volume Vol_{SM}, all depend on t. On the other hand, the domain of integration SM, being the quotient space of TM_0 under the equivalence relation $z \equiv y$, $z = \lambda y$ for some $\lambda > 0$, is totally independent of any Finsler metric and hence does not depend on t. We have from insights in [1]

$$\tfrac{d}{dt}(dV_{SM}) = \left[g_{ij} - g'_{ij} - n\tfrac{d}{dt}\log F\right]dV_{SM}.$$

A normalized Ricci flow for Finsler metrics is proposed by Bao and it is given by

$$\frac{d}{dt}\log F = -R + \frac{1}{Vol(SM)}\int_{SM} RdV, \quad F(t=0) = F_0, \quad (16)$$

where the given manifold M is compact. Now, we let $Vol(SM) = 1$. Then all of Ricci-constant metrics are exactly the fixed points of the above flow. Let

$$Ric_{ij} = \frac{1}{2}(F^2R)_{.y^i.y^j}. \quad (17)$$

Differentiating (16) with respect to y^i and y^j we have,

$$\frac{d}{dt}(g_{ij}) = -2Ric_{ij} + \frac{2}{Vol(SM)}\int_{SM} RdV\, g_{ij}, \quad g(t=0) = g_0. \quad (18)$$

Starting with any familiar metric on M as the initial data F_0, we may deform it using the proposed normalized Ricci flow, in the hope of arriving at a Ricci constant metric.

Proof of Theorem 1.2

Here we consider Finsler manifolds that satisfies the normal Ricci flow equation. Then ,

$$\frac{dg_{ij}}{dt} = -2Ric_{ij} + 2\int_{SM} RdV\, g_{ij}, \quad d(\log F) = \frac{F'}{F} = -R + \int_{SM} RdV. \quad (19)$$

In this case,

$$y_i' = \left(-2Ric_{im} + 2\int_{SM} RdV\, g_{im}\right) y^m. \quad (20)$$

Therefore,

$$
\begin{aligned}
I_i' &= (g^{jk}C_{ijk})' \\
&= (g^{jk})'C_{ijk} + g^{jk}C_{ijk}' \\
&= 2\left[Ric^{jk} - \int_{SM} RdV\, g^{jk}\right]C_{ijk} + g^{jk}[Ric_{jk,i} + 2\int_{SM} RdV\, C_{ijk}] \\
&= 2Ric^{jk}g_{jk,i} - (g^{jk}Ric_{jk})_{,i} + g_{,}^{jk}Ric_{jk} \\
&= -(g^{jk}Ric_{jk}),i \\
&= -\rho_i.
\end{aligned}
\quad (21)
$$

And

$$
\begin{aligned}
h_{ij}' &= (g_{ij} - F^{-2}y_iy_j)' \\
&= g_{ij}' - \left[-2F^{-3}F'y_iy_j + F^{-2}y_i'y_j + F^{-2}y_iy_j'\right] \\
&= g_{ij}' + 2F^{-2}\frac{F'}{F}y_iy_j - F^{-2}\left(y_i'y_j + y_iy_j'\right) \\
&= -2Ric_{ij} + 2\int_{SM} RdV\, g_{ij} + 2\left(-R - \int_{SM} RdV\right)l_il_j \\
&\quad +2(Ric_{im}l_j + Ric_{jm}l_i)l^m \\
&= -2Ric_{ij} + 2\int_{SM} RdV\, g_{ij} + 2(-R - \int_{SM} RdV)(g_{ij} - h_{ij}) \\
&\quad +2(Ric_{im}l_j + Ric_{jm}l_i)l^m \\
&= -2Ric_{ij} - 2R(h_{ij} - g_{ij}) + 2\int_{SM} RdV\, h_{ij} \\
&\quad +2(Ric_{im}l_j + Ric_{jm}l_i)l^m.
\end{aligned}
\quad (22)
$$

Therefore,

$$h_{ij} = -2Ric_{ij} - 2R(h_{ij} - g_{ij}) + 2\int_{SM} RdV\, h_{ij} + 2(Ric_{im}l_j + Ric_{jm}l_i)l^m. \quad (23)$$

Now from the un-normalised case,

$$C'_{ijk}I^iI^jI^k = (a_ih'_{jk} + a_jh'_{ki} + a_kh'_{ij})I^iI^jI^k + (a'_ih_{jk} + a'_jh_{ki} + a'_kh_{ij})I^iI^jI^k$$
$$- \{b_i(\rho_jI_k + I_j\rho_k) + b_j(\rho_iI_k + I_i\rho_k) + b_k(\rho_iI_j + I_i\rho_j)\}I^iI^jI^k. \quad (24)$$

Now using (22) and (11) we get

$$C'_{ijk}I^iI^jI^k = \frac{-6}{n+1}Ric_{ij}I^iI^j(1 - 3b_mI^m)\|I^2\| + 3a'_iI_i\|I^2\|$$
$$- 6b_i\rho_jI^iI^j\|I^2\| + 6a_k\int_{SM}RdVI^k\|I^2\|. \quad (25)$$

So by the same arguement of un-normalised case the result follows.

Acknowledgment

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