Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 2(64), No. 2 - 2022, 75-84
https://doi.org/10.31926/but.mif.2022.2.64.2.6

EQUIVALENCE OF K-FUNCTIONAL AND MODULUS OF SMOOTHNESS CONSTRUCTED BY FOURIER-BESSEL TRANSFORM IN THE SPACE $L_{2,\gamma}(\mathbb{R}^n_+)$

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Abstract

Using a generalized shift operator, we define generalized modulus of smothness in the space $L_{2,\gamma}(\mathbb{R}^n_+)$. Based on the Laplace-Bessel differential operator we define Sobolev-type space and K-functionals. In this paper paper we prove the equivalence theorem for a K-functional and a modulus of smoothness for the Fourier-Bessel transformation on $\mathbb{R}^n_+$.

2000 *Mathematics Subject Classification:* 42B10, 42A38.
Key words: Laplace-Bessel differential operator; generalized shift operator; K-functional; modulus of smoothness.

1 Introduction and preliminaries

In [4], El Hamma and Daher proved the equivalence theorem for a K-functional and a modulus of smoothness for the Dunkl transform in the Hilbert space $L^2_k(\mathbb{R}^n)$, Using a generalized spherical mean operator.

In this work, we prove the analog of this result (see[4]) for the Fourier-Bessel transform in the space $L_{2,\gamma}(\mathbb{R}^n_+)$. For this purpose, we use a generalized shift operator in the place of the generalized spherical mean operator.

Suppose that $\mathbb{R}^n$ is the n-dimensional euclidien space, $x = (x_1, ..., x_n)$, $\xi = (\xi_1, ..., \xi_n)$ are vectors in $\mathbb{R}^n$, $(x, \xi) = x_1\xi_1 + ... + x_n\xi_n$, $|x| = \sqrt{(x, x)}$ and we write $x = (x', x_n)$ with $x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}$.

Let $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) = (x', x_n) : x_n > 0\}$. S^n_+ denote the unit sphere on $\mathbb{R}^n_+$ defined by $S^n_+ = \{x \in \mathbb{R}^n_+ : |x| = 1\}$. $\mathbb{S}_+ = \mathbb{S}(\mathbb{R}^n_+)$ be the space of functions

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which are the restrictions to $\mathbb{R}^n_+$ of the test functions of the Schwartz that are even with respect to x_n. decreasing sufficiently rapidly at infinity, together with all derivatives of the form

$$\Lambda^\alpha_\gamma = \Lambda^{\alpha'}_{x'} B^{\alpha_n}_n = \Lambda^{\alpha_1}_1 .....\Lambda^{\alpha_{n-1}}_{n-1} B^{\alpha_n}_n = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} ..... \frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}} B^{\alpha_n}_n,$$

i.e., for all $\psi \in \mathbb{S}_+$, $\sup_{x\in\mathbb{R}^n_+} |x^\beta \Lambda^\alpha_\gamma \psi| < \infty$, where $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ are multi-indexes, and $x^\beta = x_1^{\beta_1} ....x_n^{\beta_n}$ and $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}$ is the Bessel differential expansion.

Let $L_{2,\gamma} = L_{2,\gamma}(\mathbb{R}^n_+)$, where $\gamma > 0$, denote the space of mesurable functions with the following finite norm

$$\|f\|_{L_{2,\gamma}} = \left(\int_{\mathbb{R}^n_+} |f(x)|^2 x_n^\gamma dx \right)^{\frac{1}{2}}.$$

For $\gamma \geq 0$, let j_γ denote the normalized Bessel function of the first kind of order γ given by

$$j_\gamma(z) = \Gamma(\gamma+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\gamma+1)} \left(\frac{z}{2}\right)^{2n},$$

where Γ is the gamma-function (see[11]).

The function $u = j_{\frac{\gamma-1}{2}}(z)$ satisfies the differential equation

$$B_{x_n} u(x, y) = B_{y_n} u(x, y),$$

with the initial conditions $u(x, 0) = f(x)$ and $u_y(x, 0) = 0$ is function infinitely differentiable, even, and, moreover entire analytic.

Lemma 1. *The following inequalities are valid for the Bessel function* $j_{\frac{\gamma-1}{2}}$

1. $\left|1 - j_{\frac{\gamma-1}{2}}(x)\right| \leq 2|x|$
2. $\left|1 - j_{\frac{\gamma-1}{2}}(x)\right| \geq c$ *with* $|x| \geq 1$, *where* $c > 0$ *is a certain constant.*

Proof. Analog of Lemma 2.9 in [3]. □

The Fourier-Bessel transform is defined on $\mathbb{S}_+$ by

$$F_\gamma f(x) = \int_{\mathbb{R}^n_+} f(y) e^{-i(x',y')} j_{\frac{\gamma-1}{2}}(x_n y_n) y_n^\gamma dy.$$

The inverse Fourier-Bessel transform is given by the formula

$$F_\gamma^{-1} f(x) = C_{n,\gamma} F_\gamma f(-x', x_n),$$

where

$$C_{n,\gamma} = (2\pi)^{n-1} 2^{\gamma-1} \Gamma^2((\gamma+1)/2),$$

(see [6, 11, 12]).

We define the Laplace-Bessel differential operator

$$D_\gamma = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\gamma}{x_n} \frac{\partial}{\partial x_n} \quad , \qquad \gamma > 0,$$

which is associated to the Fourier-Bessel transform by the formula

$$F_\gamma(D_\gamma f)(x) = -|x|^2 F_\gamma f(x). \tag{1}$$

In [2], for $f \in L_{2,\gamma}$, the Parseval's identity is given by

$$\|F_\gamma f\|_{L_{2,\gamma}} = C_{n,\gamma} \|f\|_{L_{2,\gamma}}.$$

The generalized shift operator (D_γ-shift) T^y defined by

$$T^y f(x) = C_\gamma \int_0^\pi f(x' - y', \sqrt{x_n^2 - 2x_n y_n cos\theta + y_n^2}) sin^{\gamma-1}\theta d\theta,$$

where $C_\gamma = \pi^{-\frac{1}{2}} \Gamma\left(\frac{\gamma+1}{2}\right) \left[\Gamma(\frac{\gamma}{2})\right]^{-1}$(see[7, 8, 9, 10]).
From [5], we have the following formula

$$F_\gamma[T^y f(x)] = j_{\frac{\gamma-1}{2}}(x_n y_n) F_\gamma[f(x)]. \tag{2}$$

In [1], we have

$$\|T^y f(x)\|_{L_{2,\gamma}} \leq \|f\|_{L_{2,\gamma}}. \tag{3}$$

With the help of the generalized shift operator for any $f \in L_{2,\gamma}$ we define differences of the order m ($m \in \mathbb{N} = \{1, 2, ...\}$) with a step $y_n > 0$.

$$\Delta_{y_n}^m f(x) = (I - T^y)^m f(x),$$

with $y = (y', y_n)$ and I is the unit operator.

Now, we define the generalized modulus of smoothness of the mth order by the formula

$$w_m(f, \delta)_{2,\gamma} = \sup_{0<y_n\leq\delta} \left\|\Delta_{y_n}^m f\right\|_{L_{2,\gamma}} \quad , \qquad \delta > 0.$$

Let $W^m_{2,\gamma}$ be the Sobolev space of $f \in L_{2,\gamma}$ such that $D^j_\gamma f \in L_{2,\gamma}$, $j = 1, 2, ..., m$ (the action of differential operator D^j_γ is understood in the sense of distributions), i.e.,

$$W^m_{2,\gamma} = \left\{ f \in L_{2,\gamma} \ : \ D^j_\gamma f \in L_{2,\gamma} \quad ; \quad j = 1, 2, ..., m \right\} 0$$

The generalized K-functional is defined by

$$\begin{aligned} K_m(f,t)_{2,\gamma} &= K(f,t\ ;\ L_{2,\gamma}\ ;\ W^m_{2,\gamma}) \\ &= inf\left\{ \|f-g\|_{L_{2,\gamma}} + t\|D^m_\gamma g\|_{L_{2,\gamma}}\ ;\ g \in W^m_{2,\gamma} \right\}, \end{aligned}$$

where $f \in L_{2,\gamma}$, $t > 0$.

2 Main results

In this section we give the main result of this paper, and in what follows, f is an arbitrary function of the space $L_{2,\gamma}$; $c, c_1, c_2, c_3, ...$ are positive constants.

Lemma 2. *Let $f \in L_{2,\gamma}$. Then*

$$\|\Delta^m_{y_n} f\|_{L_{2,\gamma}} \leq 2^m \|f\|_{L_{2,\gamma}}$$

Proof. we use the proof of recurrence for m and the formula (3). □

Lemma 3. *Let $f \in W^m_{2,\gamma}$, $t > 0$. The following inequality is true*

$$w_m(f,t)_{2,\gamma} \leq c_1 t^{2m} \|D^m_\gamma f\|_{L_{2,\gamma}}$$

Proof. Assume that $y_n \in (0,t]$, $\Delta^m_{y_n} f = (I - T^y)^m f$

Using formulas (1) and (2) and the Parseval equality we obtain

$$\|\Delta^m_{y_n} f\|_{L_{2,\gamma}} = \frac{1}{C_{n,\gamma}} \left\| \left(1 - j_{\frac{\gamma-1}{2}}(\xi_n y_n)\right)^m F_\gamma[f](\xi) \right\|_{L_{2,\gamma}} \tag{4}$$

and

$$\|D^m_\gamma f\|_{L_{2,\gamma}} = \frac{1}{C_{n,\gamma}} |\xi|^{2m} \|F_\gamma[f](\xi)\|_{L_{2,\gamma}}$$

Formula (4) implies that

$$\begin{aligned} \|\Delta^m_{y_n} f\|_{L_{2,\gamma}} &= \frac{1}{C_{n,\gamma}} y_n^{2m} \left\| \frac{\left(1 - j_{\frac{\gamma-1}{2}}(\xi_n y_n)\right)^m}{y_n^{2m}\xi_n^{2m}} \xi_n^{2m} F_\gamma[f](\xi) \right\|_{L_{2,\gamma}} \\ &= \frac{1}{C_{n,\gamma}} y_n^{2m} \left\| \frac{\left(1 - j_{\frac{\gamma-1}{2}}(\xi_n y_n)\right)^m}{(y_n\xi_n)^{2m}} \xi_n^{2m} F_\gamma[f](\xi) \right\|_{L_{2,\gamma}} \end{aligned}$$

According to lemma 1, for all $s \in \mathbb{R}$ we have the inequality

$$\left|\left(1 - j_{\frac{\gamma-1}{2}}(s)\right)^{2m} s^{-2m}\right| \leq c_1,$$

where $c_1 = 2^{2m}$.

Then

$$\|\Delta^m_{y_n} f\|_{L_{2,\gamma}} \leq \frac{c_1}{C_{n,\gamma}} y_n^{2m} \|\xi_n^{2m} F_\gamma[f](\xi)\|_{L_{2,\gamma}}.$$

Since

$$\|\xi_n^{2m} F_\gamma[f](\xi)\|_{L_{2,\gamma}} = \xi_n^{2m} \|F_\gamma[f](\xi)\|_{L_{2,\gamma}} \leq |\xi|^{2m} \|F_\gamma[f](\xi)\|_{L_{2,\gamma}}.$$

We obtain

$$\|\xi_n^{2m} F_\gamma[f](\xi)\|_{L_{2,\gamma}} \leq C_{n,\gamma} \|D^m_\gamma f\|_{L_{2,\gamma}}.$$

Then

$$\|\Delta^m_{y_n} f\|_{L_{2,\gamma}} \leq c_1 y_n^{2m} \|D^m_\gamma f\|_{L_{2,\gamma}} \leq c_1 t^{2m} \|D^m_\gamma f\|_{L_{2,\gamma}}$$

Calculating the supremum with respect to all $y_n \in (0, t]$, we obtain

$$w_m(f, t)_{2,\gamma} \leq c_1 t^{2m} \|D^m_\gamma f\|_{L_{2,\gamma}}$$

□

For any $f \in L_{2,\gamma}$ and any number $\nu > 0$, let us define the function

$$P_\nu(f)(x) = F_\gamma^{-1}(F_\gamma[f](\xi)\chi_\nu(\xi_n)),$$

where $\chi_\nu(\xi_n)$ is the function defined by $\chi_\nu(\xi_n) = 1$, for $\xi_n \leq \nu$ and $\chi_\nu(\xi_n) = 0$ for $\xi_n > \nu$, F_γ^{-1} is the inverse Fourier-Bessel transform.

One can easily prove that the function $P_\nu(f)$ is infinitely differentiable and belong to all classes $W^m_{2,\gamma}$.

Lemma 4. *For any function $f \in L_{2,\gamma}$. The following inequality is true*

$$\|f - P_\nu(f)\|_{L_{2,\gamma}} \leq c_2 \left\|\Delta^m_{\frac{1}{\nu}} f\right\|_{L_{2,\gamma}}, \nu > 0$$

Proof. It follows from $\left|1 - j_{\frac{\gamma-1}{2}}(t)\right| \geq c$ with $|t| \geq 1$ (see lemma 1) and Parseval equality that

$$\begin{aligned}\|f-P_\nu(f)\|_{L_{2,\gamma}} &= \frac{1}{C_{n,\gamma}}\|(1-\chi_\nu(\xi_n))F_\gamma[f](\xi)\|_{L_{2,\gamma}}\\ &= \frac{1}{C_{n,\gamma}}\left\|\frac{1-\chi_\nu(\xi_n)}{\left(1-j_{\frac{\gamma-1}{2}}(\frac{\xi_n}{\nu})\right)^m}\left(1-j_{\frac{\gamma-1}{2}}\left(\frac{\xi_n}{\nu}\right)\right)^m F_\gamma[f](\xi)\right\|_{L_{2,\gamma}}\end{aligned}$$

Note that

$$\sup_{|\xi|\in\mathbb{R}}\frac{1-\chi_\nu(\xi_n)}{\left|1-j_{\frac{\gamma-1}{2}}(\frac{\xi_n}{\nu})\right|^m}\leq\frac{1}{c^m}$$

Then

$$\begin{aligned}\|f-P_\nu(f)\|_{L_{2,\gamma}} &\leq \frac{c^{-m}}{C_{n,\gamma}}\left\|\left(1-j_{\frac{\gamma-1}{2}}\left(\frac{\xi_n}{\nu}\right)\right)^m F_\gamma[f](\xi)\right\|_{L_{2,\gamma}}\\ &\leq c_2\left\|\Delta^m_{\frac{1}{\nu}}f\right\|_{L_{2,\gamma}}\end{aligned}$$

□

Corollary 1. *For $f\in L_{2,\gamma}$*

$$\|f-P_\nu(f)\|_{L_{2,\gamma}}\leq c_2 w_m(f,\frac{1}{\nu})_{2,\gamma}$$

Lemma 5. *The following inequality is true:*

$$\left\|D^m_\gamma(P_\nu(f))\right\|_{L_{2,\gamma}}\leq c_4 v^{2m}\left\|\Delta^m_{\frac{1}{\nu}}f\right\|_{L_{2,\gamma}}$$

Proof. Using the Parseval equality, we have

$$\begin{aligned}\left\|D^m_\gamma(P_\nu(f))\right\|_{L_{2,\gamma}} &= \frac{1}{C_{n,\gamma}}\left\|F_\gamma[D^m_\gamma(P_\nu(f))]\right\|_{L_{2,\gamma}}\\ &= \frac{1}{C_{n,\gamma}}\left\||\xi|^{2m}\chi_\nu(\xi_n)F_\gamma[f](\xi)\right\|_{L_{2,\gamma}}\\ &= \frac{1}{C_{n,\gamma}}\left\|\frac{|\xi|^{2m}\chi_\nu(\xi_n)}{\left(1-j_{\frac{\gamma-1}{2}}(\frac{\xi_n}{\nu})\right)^m}\left(1-j_{\frac{\gamma-1}{2}}\left(\frac{\xi_n}{\nu}\right)\right)^m F_\gamma[f](\xi)\right\|_{L_{2,\gamma}}\end{aligned}$$

Since there exist $c_3>0$ such that $|\xi|\leq c_3\,\xi_n$. So that

$$\begin{aligned}\sup_{\xi_n\le v}\frac{|\xi|^{2m}\chi_\nu(\xi_n)}{\left|1-j_{\frac{\gamma-1}{2}}(\frac{\xi_n}{\nu})\right|^m} &= v^{2m}\sup_{\xi_n\le v}\frac{\left(\frac{|\xi|}{\nu}\right)^{2m}}{\left|1-j_{\frac{\gamma-1}{2}}(\frac{\xi_n}{\nu})\right|^m}\\ &\le c_3^{2m}v^{2m}\sup_{\xi_n\le v}\frac{\left(\frac{\xi_n}{\nu}\right)^{2m}}{\left|1-j_{\frac{\gamma-1}{2}}(\frac{\xi_n}{\nu})\right|^m}\\ &\le c_3^{2m}v^{2m}\sup_{|t|\le 1}\frac{t^{2m}}{\left|1-j_{\frac{\gamma-1}{2}}(t)\right|^m}\end{aligned}$$

Let

$$c_4=c_3^{2m}\sup_{|t|\le 1}\frac{t^{2m}}{\left|1-j_{\frac{\gamma-1}{2}}(t)\right|^m}$$

Then we have:

$$\left\|D_\gamma^m(P_\nu(f))\right\|_{L_{2,\gamma}}\le c_4v^{2m}\left\|\Delta_{\frac{1}{\nu}}^m(f)\right\|_{L_{2,\gamma}}$$

□

Corollary 2. *For* $f\in L_{2,\gamma}$

$$\left\|D_\gamma^m(P_\nu(f))\right\|_{L_{2,\gamma}}\le c_4v^{2m}w_m(f,\frac{1}{\nu})_{2,\gamma}$$

Theorem 1. *There exists positive constants* c_5 *and* c_6 *which satisfying the inequality*

$$c_5w_m(f,\delta)_{2,\gamma}\le K_m(f,\delta^{2m})_{2,\gamma}\le c_6w_m(f,\delta)_{2,\gamma}$$

Proof. Firstly prove of the inequality

$$c_5w_m(f,\delta)_{2,\gamma}\le K_m(f,\delta^{2m})_{2,\gamma}$$

Let $y_n\in(0,\delta]$, $g\in W_{2,\gamma}^m$. By lemmas 2 and 3, we have

$$\begin{aligned}\|\Delta_{y_n}^m(f)\|_{L_{2,\gamma}} &\le \|\Delta_{y_n}^m(f-g)\|_{L_{2,\gamma}}+\|\Delta_{y_n}^m g\|_{L_{2,\gamma}}\\ &\le 2^m\|f-g\|_{L_{2,\gamma}}+c_1y_n^{2m}\|D_\gamma^m g\|_{L_{2,\gamma}}\\ &\le c_7\left(\|f-g\|_{L_{2,\gamma}}+\delta^{2m}\|D_\gamma^m g\|_{L_{2,\gamma}}\right),\end{aligned}$$

where $c_7=max(2^m,c_1)$.

Calculating the supremum with respect to $y_n\in(0,\delta]$ and the infimum with respect to all possible functions $g\in W_{2,\gamma}^m$, we obtain

$$w_m(f,\delta)_{2,\gamma} \leq c_7 K_m(f,\delta^{2m})_{2,\gamma}$$

whence we get the inequality.

Now, we prove the inequality

$$K_m(f,\delta^{2m})_{2,\gamma} \leq c_6 w_m(f,\delta)_{2,\gamma}.$$

Let $\nu > 0$ be a positive real number. As $P_\nu(f) \in W^m_{2,\gamma}$, it follows from the definition of the K-functional and corollaries 1 and 2, that

$$\begin{aligned} K_m(f,\delta^{2m})_{2,\gamma} &\leq \|f - P_\nu(f)\|_{L_{2,\gamma}} + \delta^{2m}\|D^m_\gamma P_\nu(f)\|_{L_{2,\gamma}} \\ &\leq c_2 w_m(f,1/\nu)_{2,\gamma} + c_4\nu^{2m}\delta^{2m} w_m(f,1/\nu)_{2,\gamma} \\ &\leq c_2 w_m(f,1/\nu)_{2,\gamma} + c_4(\nu\delta)^{2m} w_m(f,1/\nu)_{2,\gamma} \end{aligned}$$

Since ν is an arbitrary positive value, choossing $\nu = \dfrac{1}{\delta}$, we obtain

$$K_m(f,\delta^{2m})_{2,\gamma} \leq c_6 w_m(f,\delta)_{2,\gamma},$$

with $c_6 = c_2 + c_4$ and this ends the proof.

□

Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions.

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