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REFINED INEQUALITIES FOR THE DISTANCE IN METRIC SPACES

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Abstract

In this note we prove among others that

$$\sum_{1\leq i<j\leq n} p_i p_j d^s(x_i, x_j) \leq \begin{cases} 2^{s-1} \inf_{x\in X}\left[\sum_{k=1}^n p_k(1-p_k)\, d^s(x_k,x)\right], & s\geq 1 \\ \inf_{x\in X}\left[\sum_{k=1}^n p_k(1-p_k)\, d^s(x_k,x)\right], & 0<s<1, \end{cases}$$

where (X,d) is a metric space, $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^n p_i = 1$ and $s > 0$. This generalizes and improves some early upper bounds for the sum $\sum_{1\leq i<j\leq n} p_i p_j d(x_i, x_j)$.

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1 Introduction

Let X be a nonempty set. A function $d : X \times X \to [0,\infty)$ is called a *distance* on X if the following properties are satisfied:

(d) $d(x,y) = 0$ if and only if $x = y$;

(dd) $d(x,y) = d(y,x)$ for any $x, y \in X$ (the *symmetry* of the distance);

(ddd) $d(x,y) \leq d(x,z) + d(z,y)$ for any $x,y,z \in X$ (the *triangle inequality*).

The pair (X,d) is called in the literature a *metric space.*

Important examples of metric spaces are normed linear spaces. We recall that, a linear space E over the real or complex number field $\mathbb{K}$ endowed with a function $\|\cdot\| : E \to [0,\infty)$, is called a *normed space* if $\|\cdot\|$, the *norm*, satisfies the properties:

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(n) $\|x\| = 0$ if and only if $x = 0$;

(nn) $\|\alpha x\| = |\alpha| \|x\|$ for any scalar $\alpha \in \mathbb{K}$ and any vector $x \in E$;

(nnn) $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in E$ (the triangle inequality).

Further, we recall that, the linear space H over the real or complex number field $\mathbb{K}$ endowed with an application $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$ is called an *inner product space*, if the function $\langle \cdot, \cdot \rangle$, called the *inner product*, satisfies the following properties:

(i) $\langle x, x \rangle \geq 0$ for any $x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(ii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for any scalars α, β and any vectors x, y, z;

(iii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for any $x, y \in H$.

It is well know that the function $\|x\| := \sqrt{\langle x, x \rangle}$ defines a norm on H and thus an important example of normed spaces are the inner product spaces.

A fundamental inequality in metric spaces, which obviously follows by the triangle inequality and mathematical induction, is the *generalized triangle inequality*, or the *polygonal inequality* which states that: for any points $x_1, x_2, ..., x_{n-1}, x_n$ $(n \geq 3)$ in a metric space (X, d), we have the inequality

$$d(x_1, x_n) \leq d(x_1, x_2) + ... + d(x_{n-1}, x_n). \tag{1.1}$$

The following result in the general setting of metric spaces holds [2].

Theorem 1. *Let (X, d) be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq \inf_{x \in X} \left[\sum_{i=1}^{n} p_i d(x_i, x) \right]. \tag{1.2}$$

The inequality is sharp in the sense that the multiplicative constant $c = 1$ in front of "inf " cannot be replaced by a smaller quantity.

We have:

Corollary 1. *Let (X, d) be a metric space and $x_i \in X$, $i \in \{1, ..., n\}$. If there exists a closed ball of radius $r > 0$ centered in a point x containing all the points x_i, i.e., $x_i \in \overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}$, then for any $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$ we have the inequality*

$$\sum_{1 \leq i < j \leq n} p_i p_j d(x_i, x_j) \leq r. \tag{1.3}$$

The inequality (1.2) and its consequences were extended to the case of b-metric spaces in [3] and for natural powers of the distance in [1].

In this note we provide some new and improved upper and lower bounds for the sum

$$\sum_{1\leq i<j\leq n} p_i p_j d^s\left(x_i, x_j\right)$$

where (X,d) is a metric space, $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $s > 0$.

2 Main Results

We have the following generalization of the inequality (1.2).

Theorem 2. *Let (X,d) be a metric space and $x_i \in X$, $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$\sum_{1\leq i<j\leq n} p_i p_j d^s\left(x_i, x_j\right) \leq \begin{cases} 2^{s-1}\sum_{k=1}^{n} p_k\left(1-p_k\right) d^s\left(x_k, x\right), & s\geq 1 \\ \\ \sum_{k=1}^{n} p_k\left(1-p_k\right) d^s\left(x_k, x\right), & 0<s<1, \end{cases} \tag{2.1}$$

$$\leq \frac{1}{4}\begin{cases} 2^{s-1}\sum_{k=1}^{n} d^s\left(x_k, x\right), & s\geq 1 \\ \\ \sum_{k=1}^{n} d^s\left(x_k, x\right), & 0<s<1. \end{cases}$$

Proof. We know that, by the convexity property of the power function $f(t) = t^s$, $s \geq 1$ on $[0,\infty)$, we have for a, $b \geq 0$ that

$$(a+b)^s \leq 2^{s-1}\left(a^s + b^s\right).$$

We consider the function $f_s : [0,\infty) \to \mathbb{R}$, $f_s(t) = (t+1)^s - t^s$ we have $f_s'(t) = s\left[(t+1)^{s-1} - t^{s-1}\right]$. Observe that for $0 < s < 1$ and $t > 0$ we have that $f_s'(t) < 0$ showing that f_s is strictly decreasing on the interval $[0,\infty)$. Now for $t_0 = \frac{a}{b}$ $(b > 0, a \geq 0)$ we have $f_s(t_0) < f_s(0)$ giving that $\left(\frac{a}{b}+1\right)^s - \left(\frac{a}{b}\right)^s < 1$, i.e., the inequality

$$(a+b)^s \leq a^s + b^s.$$

Using the triangle inequality, we have for any $x \in X$ and i, $j \in \{1,...,n\}$, that

$$d\left(x_i, x_j\right) \leq d\left(x_i, x\right) + d\left(x, x_j\right). \tag{2.2}$$

If we take the power $s > 0$ in (2.2) we get

$$d^s\left(x_i, x_j\right) \leq \left[d\left(x_i, x\right) + d\left(x, x_j\right)\right]^s \tag{2.3}$$

$$\leq \begin{cases} 2^{s-1}\left(d^s\left(x_i, x\right) + d^s\left(x_j, x\right)\right), & s\geq 1 \\ \\ d^s\left(x_i, x\right) + d^s\left(x_j, x\right), & 0<s<1 \end{cases}$$

for any $x \in X$ and $i, j \in \{1, ..., n\}$.

If we multiply (2.3) by $p_i p_j \geq 0$ and sum over $1 \leq i < j \leq n$ from 1 to n, we get

$$\sum_{1 \leq i < j \leq n} p_i p_j d^s(x_i, x_j) \leq \begin{cases} 2^{s-1} \sum_{1 \leq i < j \leq n} p_i p_j \left(d^s(x_i, x) + d^s(x_j, x)\right), & s \geq 1 \\ \sum_{1 \leq i < j \leq n} p_i p_j \left(d^s(x_i, x) + d^s(x_j, x)\right), & 0 < s < 1. \end{cases} \tag{2.4}$$

Observe that, in general, if $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$ then

$$\sum_{1 \leq i, j \leq n} a_{ij} = \sum_{1 \leq i < j \leq n} a_{ij} + \sum_{1 \leq j < i \leq n} a_{ij} + \sum_{k=1}^{n} a_{kk} = 2 \sum_{1 \leq i < j \leq n} a_{ij} + \sum_{k=1}^{n} a_{kk},$$

which implies that

$$\sum_{1 \leq i < j \leq n} a_{ij} = \frac{1}{2} \left(\sum_{1 \leq i, j \leq n} a_{ij} - \sum_{k=1}^{n} a_{kk} \right).$$

Therefore

$$\begin{aligned} &\sum_{1 \leq i < j \leq n} p_i p_j \left(d^s(x_i, x) + d^s(x_j, x)\right) \\ &= \frac{1}{2} \left(\sum_{1 \leq i, j \leq n} p_i p_j \left(d^s(x_i, x) + d^s(x_j, x)\right) - 2 \sum_{k=1}^{n} p_k^2 d^s(x_k, x) \right) \\ &= \sum_{k=1}^{n} p_k d^s(x_k, x) - \sum_{k=1}^{n} p_k^2 d^s(x_k, x) = \sum_{k=1}^{n} p_k (1 - p_k) d^s(x_k, x) \end{aligned}$$

and by using (2.4) we deduce the first inequality in (2.1).

The second part follows by the fact that

$$p_k (1 - p_k) \leq \frac{1}{4} (p_k + 1 - p_k)^2 = \frac{1}{4}$$

for all $k \in \{1, ..., n\}$. □

Remark 1. *By taking the infimum over $x \in X$ in (2.1), we get*

$$\sum_{1 \leq i < j \leq n} p_i p_j d^s(x_i, x_j) \leq \begin{cases} 2^{s-1} \inf_{x \in X} \left[\sum_{k=1}^{n} p_k (1 - p_k) d^s(x_k, x)\right], & s \geq 1 \\ \inf_{x \in X} \left[\sum_{k=1}^{n} p_k (1 - p_k) d^s(x_k, x)\right], & 0 < s < 1, \end{cases} \tag{2.5}$$

$$\leq \frac{1}{4} \begin{cases} 2^{s-1} \inf_{x \in X} \left[\sum_{k=1}^{n} d^s(x_k, x)\right], & s \geq 1 \\ \inf_{x \in X} \left[\sum_{k=1}^{n} d^s(x_k, x)\right], & 0 < s < 1. \end{cases}$$

For $s = 1$ we derive

$$\sum_{1 \le i < j \le n} p_i p_j d(x_i, x_j) \le \inf_{x \in X} \left[\sum_{k=1}^{n} p_k (1 - p_k) d(x_k, x) \right],$$

which is a better inequality than (1.2) since

$$\sum_{k=1}^{n} p_k (1 - p_k) d(x_k, x) \le \sum_{i=1}^{n} p_i d(x_i, x).$$

Corollary 2. *Let (X, d) be a metric space and $x_i \in X$, $i \in \{1, ..., n\}$. Then we have the inequality*

$$\sum_{1 \le i < j \le n} d^s(x_i, x_j) \le (n-1) \begin{cases} 2^{s-1} \sum_{k=1}^{n} d^s(x_k, x), & s \ge 1 \\ \\ \sum_{k=1}^{n} d^s(x_k, x), & 0 < s < 1. \end{cases} \tag{2.6}$$

Follows by the first inequality in (2.1) for $p_k = \frac{1}{n}$, $k \in \{1, ..., n\}$.

Corollary 3. *Let (X, d) be a metric space and $x_i \in X$, $i \in \{1, ..., n\}$. If there exists a closed ball of radius $r > 0$ centered in a point x containing all the points x_i, i.e., $x_i \in \overline{B}(x, r) := \{y \in X : d(x, y) \le r\}$, then for any $p_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$ we have the inequalities*

$$\sum_{1 \le i < j \le n} p_i p_j d^s(x_i, x_j) \le \sum_{k=1}^{n} p_k (1 - p_k) \begin{cases} 2^{s-1} r^s, & s \ge 1 \\ \\ r^s, & 0 < s < 1. \end{cases} \tag{2.7}$$

We also have the following lower bound:

Theorem 3. *Let (X, d) be a metric space and $x_i \in X$, $p_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$2^{s-1} \left(\sum_{1 \le i < j \le n} p_i p_j d(x_i, x_j) \right)^s \le \sum_{1 \le i < j \le n} p_i p_j d^s(x_i, x_j), \; s > 1. \tag{2.8}$$

Proof. We use Jensen's discrete inequality for the power function $f(t) = t^s$, $s > 1$ to write

$$\frac{\sum_{1 \le i, j \le n} p_i p_j d^s(x_i, x_j)}{\sum_{1 \le i, j \le n} p_i p_j} \ge \left(\frac{\sum_{1 \le i, j \le n} p_i p_j d(x_i, x_j)}{\sum_{1 \le i, j \le n} p_i p_j} \right)^s. \tag{2.9}$$

Observe that

$$\sum_{1 \le i, j \le n} p_i p_j = \left(\sum_{i=1}^{n} p_i \right)^2 = 1,$$

$$\sum_{1 \le i, j \le n} p_i p_j d^s(x_i, x_j) = 2 \sum_{1 \le i < j \le n} p_i p_j d^s(x_i, x_j)$$

and

$$\sum_{1\le i,j\le n} p_i p_j d\left(x_i,x_j\right) = 2\sum_{1\le i<j\le n} p_i p_j d\left(x_i,x_j\right).$$

By (2.9) we get

$$2\sum_{1\le i<j\le n} p_i p_j d^s\left(x_i,x_j\right) \ge \left(2\sum_{1\le i<j\le n} p_i p_j d\left(x_i,x_j\right)\right)^s = 2^s\left(\sum_{1\le i<j\le n} p_i p_j d\left(x_i,x_j\right)\right)^s, \tag{2.10}$$

and the inequality (2.8) is proved. □

Corollary 4. *Let (X,d) be a metric space and $x_i \in X$, $i \in \{1,...,n\}$. Then we have the inequality*

$$\left(\frac{2}{n^2}\right)^{s-1}\left(\sum_{1\le i<j\le n} d\left(x_i,x_j\right)\right)^s \le \sum_{1\le i<j\le n} d^s\left(x_i,x_j\right),\ s>1. \tag{2.11}$$

3 Applications

If $(E,\|\cdot\|)$ is a normed linear space and $x_i \in E$, $i \in \{1,...,n\}$, $p_i \ge 0$ $(i \in \{1,...,n\})$ with $\sum_{i=1}^n p_i = 1$, then by (2.1) we have the inequality

$$\begin{aligned}&\sum_{1\le i<j\le n} p_i p_j \left\|x_i - x_j\right\|^s \\ &\le \begin{cases} 2^{s-1}\inf_{x\in X}\left[\sum_{k=1}^n p_k\left(1-p_k\right)\left\|x_k - x\right\|^s\right],\ s\ge 1 \\ \\ \inf_{x\in X}\left[\sum_{k=1}^n p_k\left(1-p_k\right)\left\|x_k - x\right\|^s\right],\ 0<s<1, \end{cases} \\ &\le \frac{1}{4}\begin{cases} 2^{s-1}\inf_{x\in X}\left[\sum_{k=1}^n \left\|x_k - x\right\|^s\right],\ s\ge 1 \\ \\ \inf_{x\in X}\left[\sum_{k=1}^n \left\|x_k - x\right\|^s\right],\ 0<s<1. \end{cases}\end{aligned} \tag{3.1}$$

In particular, for the uniform distribution $p_i = \frac{1}{n}$, we have

$$\sum_{1\le i<j\le n} \left\|x_i - x_j\right\|^s \le \begin{cases} 2^{s-1}(n-1)\inf_{x\in X}\left[\sum_{i=1}^n \left\|x_i - x\right\|^s\right],\ s\ge 1, \\ \\ (n-1)\inf_{x\in X}\left[\sum_{i=1}^n \left\|x_i - x\right\|^s\right],\ 0<s<1. \end{cases} \tag{3.2}$$

Denote $\overline{x_p} := \sum_{i=1}^{n} p_i x_i$, then we have the inequalities

$$\sum_{1\le i<j\le n} p_i p_j \|x_i - x_j\|^s \tag{3.3}$$
$$\le \begin{cases} 2^{s-1} \sum_{k=1}^{n} p_k (1-p_k) \|x_k - \overline{x_p}\|^s, & s \ge 1 \\ \\ \sum_{k=1}^{n} p_k (1-p_k) \|x_k - \overline{x_p}\|^s, & 0 < s < 1, \end{cases}$$
$$\le \frac{1}{4} \begin{cases} 2^{s-1} \sum_{k=1}^{n} \|x_k - \overline{x_p}\|^s, & s \ge 1 \\ \\ \sum_{k=1}^{n} \|x_k - \overline{x_p}\|^s, & 0 < s < 1. \end{cases}$$

By triangle inequality we have that

$$\sum_{j=1}^{n} p_j \|x_i - x_j\| \ge \left\| \sum_{j=1}^{n} p_j (x_i - x_j) \right\| = \left\| x_i - \sum_{j=1}^{n} p_j x_j \right\| = \|x_i - \overline{x_p}\|.$$

Therefore

$$\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \|x_i - x_j\| \ge \sum_{i=1}^{n} p_i \|x_i - \overline{x_p}\| \tag{3.4}$$

and since

$$\begin{aligned} 2^{s-1} \left(\sum_{1\le i<j\le n} p_i p_j \|x_i - x_j\| \right)^s &= 2^{s-1} \left(\frac{1}{2} \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \|x_i - x_j\| \right)^s \\ &= \frac{1}{2} \left(\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \|x_i - x_j\| \right)^s \\ &\ge \frac{1}{2} \left(\sum_{i=1}^{n} p_i \|x_i - \overline{x_p}\| \right)^s \end{aligned}$$

and by (2.8) we derive

$$\frac{1}{2} \left(\sum_{i=1}^{n} p_i \|x_i - \overline{x_p}\| \right)^s \le \sum_{1\le i<j\le n} p_i p_j \|x_i - x_j\|^s.$$

If

$$\overline{x} := \frac{x_1 + ... + x_n}{n}$$

denotes the gravity center of the vectors x_i, $i \in \{1, ..., n\}$, then we have the inequality

$$\sum_{1\le i<j\le n} \|x_i - x_j\|^s \le \begin{cases} 2^{s-1} (n-1) \sum_{i=1}^{n} \|x_i - \overline{x}\|^s, & \text{if } s \ge 1, \\ \\ (n-1) \sum_{i=1}^{n} \|x_i - \overline{x}\|^s, & \text{if } 0 < s < 1. \end{cases} \tag{3.5}$$

Proposition 1. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space, $x_i \in H, (i \in \{1,...,n\})$ and assume that there exists the vectors a, $A \in H$ so that either*

$$Re\langle A - x_i, x_i - a\rangle \geq 0, \text{ for } i \in \{1,...,n\},$$

or, equivalently,

$$\left\| x_i - \frac{a+A}{2} \right\| \leq \frac{1}{2}\|A - a\|, \text{ for } i \in \{1,...,n\}.$$

Then for any $p_i \geq 0$, $i \in \{1,...,n\}$ with $\sum_{i=1}^{n} p_i = 1$ one has the inequality

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\|^s \leq \begin{cases} \frac{1}{2}\|A-a\|^s \sum_{k=1}^{n} p_k(1-p_k), & s \geq 1 \\ \\ \frac{1}{2^s}\|A-a\|^s \sum_{k=1}^{n} p_k(1-p_k), & 0 < s < 1. \end{cases} \tag{3.6}$$

In particular, if $p_i = \frac{1}{n}$, $i \in \{1,...,n\}$ then by (3.6) we get

$$\sum_{1\leq i<j\leq n} \|x_i - x_j\|^s \leq \begin{cases} \frac{1}{2}(n-1)\|A-a\|^s, & s \geq 1 \\ \\ \frac{1}{2^s}(n-1)\|A-a\|^s, & 0 < s < 1. \end{cases} \tag{3.7}$$

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