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RICCI SOLITONS ON PSEUDO RIEMANNIAN GENERALIZED SYMMETRIC SPACES

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Abstract

We study the geometry of four-dimensional pseudo Riemannian generalized symmetric spaces of type D; whose metric was explicitly described by Cerny and Kowalski. After describing their curvature properties; we classify the Killing vectors field of these spaces and more particularly, we study the existence of non-trivial (i.e., not Einstein) Ricci solitons; we show that these spaces are shrinking or expanding Ricci solitons but never steady. Moreover this Ricci soliton is not a gradient one.

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1 Introduction

Ricci soliton are self-similar solutions of Hamilton's Ricci flow, they can also be viewed as its fixed points, they are a natural generalization of Einstein metrics. A pseudo-Riemannian manifold (M, g) is said to be a Ricci soliton if there exists a smooth vector field X on M such as the following equation is satisfied

$$L_X g + \varrho = \alpha g, \tag{1}$$

where L_X denotes the Lie derivative with respect to X, ϱ is the Ricci tensor and α is a real number. Moreover, we say that a Ricci soliton (M, g) is a *gradient Ricci soliton* if it admits a vector field X satisfying : $X = \text{grad } h$, for some potential function h.

A Ricci soliton is said to be *shrinking*, *steady* or *expanding* if $\alpha > 0, \alpha = 0$ or $\alpha < 0$ respectively.

Pseudo Riemannian Ricci solitons were severely studied from different perspectives, it has been shown that there are many essential differences between the

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Riemannian case [1], [15] and the Lorentzian case [3], [16]. In [12], the author have proved that there are three-dimensional Riemannian homogeneous Ricci solitons but in [17], it has been shown that there are no left-invariant Riemannian Ricci solitons on three-dimensional Lie groups; although there are soliton metrics for nilpotent Lie groups. Many other authors have investigated the Lorentzian case; as the existence of non-trivial Ricci solitons on conformally flat pr-waves manifolds [2], and on several classes of three-manifolds admitting a parallel degenerate line field [7], as well as the existence of three-dimensional Lorentzian homogeneous left-invariant Ricci solitons [6]. In the dimension 4; Calvaruso and Fino proved, the existence of non-compact homogeneous pseudo-Riemannian Ricci solitons which are not isometric to solvmanifolds [9].

A generalized symmetric space is a pseudo-Riemannian manifold (M, g) which admits at least a regular $s-$structure, those spaces where studied by different authors, from several different points of view. All $2-$dimensional generalized symmetric spaces are symmetric. In dimension 3, they may be identified with $\mathbb{R}^3$ equipped with a metric with all possible signatures. Cerny and Kowalski proved in [13] that four-dimensional proper (that is, non-symmetric) pseudo-Riemannian generalized symmetric spaces may be identified with $\mathbb{R}^4$ equipped with a particular metric and are classified into four classes, named A, B, C and D, and the pseudo-Riemannian metrics can have any signature: the metric of type A is either Riemannian or neutral of signature (2,2), the metric of type C is Lorentzian and the metrics of type B and D are of signature (2,2). In [14]; Kowalski has proved the existence of generalized symmetric Riemannian spaces of arbitrary order, in [8] Calvaruso and De Leo have studied their curvature properties on the algebraic side using the Lie algebras, with respect to suitable pseudo-orthonormal bases and in [11], the authors showed that these spaces can naturally be equipped with some structures defined by their curvature tensors used to characterize symmetric spaces, as the existence of almost Hermitian and almost para-Hermitian structures. A complete classification up to isometry, of non-symmetric simply-connected four-dimensional pseudo-Riemannian generalized symmetric spaces which are algebraic Ricci solitons was been given in [4]; where unlike type B, only types A, C and D are algebraic Ricci solitons. In [10], the authors proved that three dimensional generalized symmetric spaces of any signature and four-dimensional generalized symmetric spaces of type B are Ricci solitons and recently, a complete classification of Ricci solitons on Lorentzian four-dimensional generalized symmetric spaces of type C was proved in [5]. In this paper we find the general solution to the equation (1) for four-dimensional generalized symmetric spaces of type D.

The paper is organized in the following way, in Section 2, we will report the basic description of four-dimensional generalized symmetric spaces. In Section 3, the Levi Civita connection, the curvature tensor and the Ricci tensor of pseudo-Riemannian four-dimensional generalized symmetric spaces of type D will be described in terms of components with respect to coordinate vector fields $\left\{\partial_i = \frac{\partial}{\partial x_i}\right\}$. This provides the needed information for the study, which we make in Section 5. In Section 4, we shall introduce the system of 10 PDEs of Killing vectors and give it's general solution.

In Section 5, Ricci solitons on generalized symmetric spaces of type D are characterized by a system of 10 PDEs. In particular, we show that these spaces admit different vector fields resulting in expanding and shrinking, but never steady Ricci solitons. Finally, it is proved that those Ricci solitons are not gradient.

2 Four-dimensional generalized symmetric spaces

First, we recall the definition of a generalized symmetric space. Let (M, g) be a pseudo-Riemannian manifold. A *regular s-structure* on M is a family of isometries $\{s_x \mid x \in M\}$ of (M, g) such that

- the mapping $\begin{array}{ccc} M \times M & \to & M \\ (x, y) & \mapsto & s_x(y) \end{array}$ is smooth,
- for all x in M; s_x has x as an isolated fixed point.
- for any pair of points x and y in M : $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

The map s_x is called the symmetry centered at x. The *order* of a regular s-structure is the smallest integer $k \geqslant 2$ such that$(s_x)^k = \mathrm{id}_M$ for all $x \in M$. If such an integer does not exist, we say that the regular s-structure has order infinity. A *generalized symmetric space* is a connected, pseudo-Riemannian manifold, carrying at least one regular s-structure. In particular, a generalized symmetric space is a pseudo-Riemannian symmetric space if and only if it admits a regular s-structure of order 2. The *order* of a generalized symmetric space is the minimum of orders of all possible s-structures on it. Moreover, a generalized symmetric space is homogeneous, that is, the full isometry group $I\,(M)$ of M acts transitively on it, such as (M, g) can be identified with $(G/H, g)$, where $G \subset I\,(M)$ is a subgroup of $I\,(M)$ acting transitively on M and H is the isotropy group at a fixed point $o \in M$.

In 1986, Cerny and Kowalski have completely classified generalized symmetric spaces of low dimension. We recall the classification of non-symmetric simply-connected four-dimensional pseudo-Riemannian generalized symmetric spaces by the following theorem.

Theorem 1. (Cerny and Kowalski [13]) *Non-symmetric, simply-connected generalized symmetric spaces (M, g) of dimension 4 are of order either 3 or 4, or infinity. All these spaces are indecomposable, and belong, up to isometry, to one of the following four types.*

- Type A. *The underlying homogeneous space is G/H, where*

$$G = \begin{pmatrix} a & b & x_3 \\ c & d & x_4 \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $ad-bc=1$*.* (M,g) *is the space* $\mathbb{R}^4(x_1,x_2,x_3,x_4)$ *with the pseudo-Riemannian metric*

$$\begin{aligned} g &= \lambda\left[\left(1+x_2^2\right)dx_1^2+\left(1+x_1^2\right)dx_2^2-2x_1x_2\,dx_1\,dx_2\right]/\left(1+x_1^2+x_2^2\right) \\ &\quad \pm\left[\left(-x_1+\sqrt{1+x_1^2+x_2^2}\right)dx_3^2 \right. \\ &\quad \left. +\left(x_1+\sqrt{1+x_1^2+x_2^2}\right)dx_4^2-2x_2^2dx_3\,dx_4\right], \end{aligned} \tag{1}$$

where $\lambda\neq 0$ *is a real constant. The order is* $k=3$ *and the possible signatures are* $(4,0)$*,* $(2,2)$ *and* $(0,4)$*.*

- Type B. *The underlying homogeneous space is* G/H*, where*

$$G=\begin{pmatrix} e^{-(x_1+x_2)} & 0 & 0 & a \\ 0 & e^{x_1} & 0 & b \\ 0 & 0 & e^{x_2} & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H=\begin{pmatrix} 1 & 0 & 0 & -w \\ 0 & 1 & 0 & -2w \\ 0 & 0 & 1 & 2w \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(M,g) *is the space* $\mathbb{R}^4(x_1,x_2,x_3,x_4)$ *with the pseudo-Riemannian metric*

$$g=\lambda\left(dx_1^2+dx_2^2+dx_1dx_2\right)+e^{-x_2}\left(2dx_1+dx_2\right)dx_4+e^{-x_1}\left(dx_1+2dx_2\right)dx_3, \tag{2}$$

where λ *is a real constant. The order is* $k=3$ *and the signature is always* $(2,2)$*.*

- Type C. *The underlying homogeneous space* G/H *is the matrix group*

$$G=\begin{pmatrix} e^{-x_4} & 0 & 0 & x_1 \\ 0 & e^{x_4} & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(M,g) *is the space* $\mathbb{R}^4(x_1,x_2,x_3,x_4)$ *with the Lorentzian metric*

$$g=\varepsilon\left(e^{-2x_4}dx_1^2+e^{2x_4}dx_2^2\right)+dx_3dx_4 \quad \textit{with } \varepsilon=\pm 1. \tag{3}$$

The order is $k=4$ *and the possible signatures are* $(1,3),(3,1)$*.*

- Type D. *The underlying homogeneous space is* G/H *where*

$$G=\begin{pmatrix} a & b & x_1 \\ c & d & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad H=\begin{pmatrix} e^{x_4} & 0 & 0 \\ 0 & e^{-x_4} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $ad - bc = 1$. (M, g) is the space $\mathbb{R}^4(x_1, x_2, x_3, x_4)$ with the pseudo-Riemannian metric

$$\begin{aligned} g \quad = \quad & -2\cosh(2x_3)\cos(2x_4)\,dx_1\,dx_2 + \lambda\left(dx_3^2 - \cosh^2(2x_3)\,dx_4^2\right) \\ & + (\sinh(2x_3) - \cosh(2x_3)\sin(2x_4))\,dx_1^2 \\ & + (\sinh(2x_3) + \cosh(2x_3)\sin(2x_4))\,dx_2^2, \end{aligned} \tag{4}$$

where $\lambda \neq 0$ is a real constant. The order is infinite and the signature is $(2, 2)$.

3 Curvature of four-dimensional generalized symmetric space of type D

Let (M, g) be a four-dimensional generalized symmetric space of type D and denote by ∇ and R the Levi-Civita connection and the Riemann curvature tensor of M, respectively. Throughout this paper, we will always use the sign convention

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

The Ricci tensor of (M, g) is defined by $\varrho(X, Y) = \mathrm{tr}\{Z \to R(X, Z)Y\}$.

We shall report the nonvanishing Levi-Civita connection, the Riemann curvature tensor, and the corresponding Ricci tensor with respect to the coordinates vector fields $\left\{\partial_1 = \frac{\partial}{\partial x_1}, \partial_2 = \frac{\partial}{\partial x_2}, \partial_3 = \frac{\partial}{\partial x_3}, \partial_4 = \frac{\partial}{\partial x_4}\right\}$.

Lemma 1. *Let M be a four-dimensional generalized symmetric space of type D. Then the non-vanishing components of the Levi-Civita connection ∇ of M are given by*

$$\begin{aligned}
&\nabla_{\partial_1}\partial_1 = \tfrac{1}{\lambda}(-\cosh(2x_3) + \sinh(2x_3)\sin(2x_4))\partial_3 - \tfrac{\cos(2x_4)}{\lambda\cosh(2x_3)}\partial_4, \\
&\nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = \tfrac{1}{\lambda}\sinh(2x_3)\cos(2x_4)\partial_3 + \tfrac{\sin(2x_4)}{\lambda\cosh(2x_3)}\partial_4, \\
&\nabla_{\partial_1}\partial_3 = \nabla_{\partial_3}\partial_1 = -\sin(2x_4)\partial_1 - \cos(2x_4)\partial_2, \\
&\nabla_{\partial_1}\partial_4 = \nabla_{\partial_4}\partial_1 = \sinh(2x_3)\cosh(2x_3)\cos(2x_4)\partial_1 \\
&\qquad\qquad\qquad - \cosh(2x_3)(-\cosh(2x_3) + \sinh(2x_3)\sin(2x_4))\partial_2, \\
&\nabla_{\partial_2}\partial_2 = -\tfrac{1}{\lambda}(\cosh(2x_3) + \sinh(2x_3)\sin(2x_4))\partial_3 + \tfrac{\cos(2x_4)}{\lambda\cosh(2x_3)}\partial_4, \\
&\nabla_{\partial_2}\partial_3 = \nabla_{\partial_3}\partial_2 = -\cos(2x_4)\partial_1 + \sin(2x_4)\partial_2, \\
&\nabla_{\partial_2}\partial_4 = \nabla_{\partial_4}\partial_2 = -\cosh(2x_3)(\cosh(2x_3) + \sinh(2x_3)\sin(2x_4))\partial_1 \\
&\qquad\qquad\qquad - \sinh(2x_3)\cosh(2x_3)\cos(2x_4)\partial_2, \\
&\nabla_{\partial_3}\partial_4 = \nabla_{\partial_4}\partial_3 = 2\tanh(2x_3)\partial_4, \\
&\nabla_{\partial_4}\partial_4 = 2\cosh(2x_3)\sinh(2x_3)\partial_3.
\end{aligned}$$

The only non-zero components of the Riemann curvature tensor R, are

$$
\begin{aligned}
&R_{\partial_1,\partial_2}\partial_1 = [\tfrac{2\cosh(2x_3)\cos(2x_4)}{\lambda}]\partial_1 + [\tfrac{2\sinh(2x_3)-2\cosh(2x_3)\sin(2x_4)}{\lambda}]\partial_2,\\
&R_{\partial_1,\partial_2}\partial_2 = [\tfrac{-2\sinh(2x_3)-2\cosh(2x_3)\sin(2x_4)}{\lambda}]\partial_1 - [\tfrac{2\cosh(2x_3)\cos(2x_4)}{\lambda}]\partial_2,\\
&R_{\partial_1,\partial_2}\partial_3 = \tfrac{-2}{\lambda\cosh(2x_3)}\partial_4,\\
&R_{\partial_1,\partial_2}\partial_4 = \tfrac{-2\cosh(2x_3)}{\lambda}\partial_3,\\
&R_{\partial_1,\partial_3}\partial_1 = \tfrac{-\sinh(2x_3)+\cosh(2x_3)\sin(2x_4)}{\lambda}\partial_3,\\
&R_{\partial_1,\partial_3}\partial_2 = \tfrac{\cosh(2x_3)\cos(2x_4)}{\lambda}\partial_3 - \tfrac{1}{\lambda\cosh(2x_3)}\partial_4,\\
&R_{\partial_1,\partial_3}\partial_3 = \partial_1,\\
&R_{\partial_1,\partial_3}\partial_4 = \cosh^2(2x_3)\cos(2x_4)\partial_1 + \cosh(2x_3)\,[\sinh(2x_3) - \cosh(2x_3)\sin(2x_4)]\,\partial_2,\\
&R_{\partial_1,\partial_4}\partial_1 = \tfrac{[\cosh(2x_3)\sin(2x_4)-\sinh(2x_3)]}{\lambda}\partial_4,\\
&R_{\partial_1,\partial_4}\partial_2 = \tfrac{-\cosh(2x_3)}{\lambda}\partial_3 + \tfrac{[\cosh(2x_3)\cos(2x_4)]}{\lambda}\partial_4,\\
&R_{\partial_1,\partial_4}\partial_3 = -\cosh^2(2x_3)\cos(2x_4)\partial_1 - \cosh(2x_3)(\sinh(2x_3)\\
&\qquad\qquad - \cosh(2x_3)\sin(2x_4))\partial_2,\\
&R_{\partial_1,\partial_4}\partial_4 = -\cosh^2(2x_3)\partial_1,\\
&R_{\partial_2,\partial_3}\partial_1 = \tfrac{\cosh(2x_3)\cos(2x_4)}{\lambda}\partial_3 + \tfrac{1}{\lambda\cosh(2x_3)}\partial_4,\\
&R_{\partial_2,\partial_3}\partial_2 = \tfrac{-\sinh(2x_3)-\cosh(2x_3)\sin(2x_4)}{\lambda}\partial_3,\\
&R_{\partial_2,\partial_3}\partial_3 = \partial_2,\\
&R_{\partial_2,\partial_3}\partial_4 = -\cosh(2x_3)(\sinh(2x_3) + \cosh(2x3)\sin(2x_4))\\
&\qquad\qquad -[\cosh^2(2x_3)\cos(2x_4)]\partial_2,\\
&R_{\partial_2,\partial_4}\partial_1 = \tfrac{\cosh(2x_3)}{\lambda}\partial_3 + \tfrac{[\cosh(2x_3)\cos(2x_4)]}{\lambda}\partial_4,\\
&R_{\partial_2,\partial_4}\partial_2 = \left[\tfrac{-\sinh(2x_3)-\cosh(2x_3)\sin(2x_4)}{\lambda}\right]\partial_4,\\
&R_{\partial_2,\partial_4}\partial_3 = \cosh(2x_3)(\sinh(2x_3)+\cosh(2x_3)\sin(2x_4))\partial_1 + \cosh^2(2x_3)\cos(2x_4)\partial_2,\\
&R_{\partial_2,\partial_4}\partial_4 = -\cosh^2(2x_3)\,\partial_2,\\
&R_{\partial_3,\partial_4}\partial_1 = -2\cosh^2(2x_3)\cos(2x_4)\partial_1 - 2\cosh(2x_3)(\sinh(2x_3)\\
&\qquad\qquad - \cosh(2x_3)\sin(2x_4))\partial_2,\\
&R_{\partial_3,\partial_4}\partial_2 = 2\cosh(2x_3)(\sinh(2x_3)\\
&\qquad\qquad + \cosh(2x_3)\sin(2x_4))\partial_1 + 2\cosh^2(2x_3)\cos(2x_4)\partial_2,\\
&R_{\partial_3,\partial_4}\partial_3 = -4\partial_4,\\
&R_{\partial_3,\partial_4}\partial_4 = -4\cosh^2(2x_3)\,\partial_3,
\end{aligned}
$$

and the ones obtained by them using the symmetries of the curvature tensor.

The non-zero components of the Ricci curvature tensor are given by

$$\varrho_{33} = -6, \qquad \varrho_{44} = 6\cosh^2(2x_3).$$

4 Killing vectors on four-dimensional generalized symmetric space of type D

In this section we give the Killing vectors on the four-dimensional generalized symmetric spaces (M,g) of type D. Let $X=\sum_{i=1}^{4} f_i\partial_i$ be an arbitrary vector field on (M,g), where $f_1,..,f_4$ are smooth functions on M of the variables x_1,x_2,x_3,x_4.

The Lie derivative of the metric (4) with respect to X is given by:

$$\left\{\begin{array}{rl}
(L_Xg)_{\partial_1,\partial_1} = & 2(\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\partial_1 f_1 \\
& -2\cosh(2x_3)\cos(2x_4)\partial_1 f_2+2(\cosh(2x_3)-\sinh(2x_3)\sin(2x_4))f_3 \\
& -2\cosh(2x_3)\cos(2x_4)f_4, \\
(L_Xg)_{\partial_1,\partial_2} = & -\cosh(2x_3)\cos(2x_4)\partial_1 f_1+(\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\partial_2 f_1 \\
& +(\sinh(2x_3)+\cosh(2x_3)\sin(2x_4))\partial_1 f_2-2\sinh(2x_3)\cos(2x_4)f_3 \\
& -\cosh(2x_3)\cos(2x_4)\partial_2 f_2+2\cosh(2x_3)\sin(2x_4)f_4, \\
(L_Xg)_{\partial_1,\partial_3} = & \lambda\partial_1 f_3+(\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\partial_3 f_1 \\
& -\cosh(2x_3)\cos(2x_4)\partial_3 f_2, \\
(L_Xg)_{\partial_1,\partial_4} = & -\lambda\cosh^2(2x_3)\partial_1 f_4+(\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\partial_4 f_1 \\
& -\cosh(2x_3)\cos(2x_4)\partial_4 f_2, \\
(L_Xg)_{\partial_2,\partial_2} = & -2\cosh(2x_3)\cos(2x_4)\partial_2 f_1+2(\sinh(2x_3) \\
& +\cosh(2x_3)\sin(2x_4))\partial_2 f_2+2(\sinh(2x_3)\sin(2x_4) \\
& +\cosh(2x_3))f_3+2\cosh(2x_3)\cos(2x_4)f_4, \\
(L_Xg)_{\partial_2,\partial_3} = & \lambda\partial_2 f_3-\cosh(2x_3)\cos(2x_4)\partial_3 f_1 \\
& +(\sinh(2x_3)+\cosh(2x_3)\sin(2x_4))\partial_3 f_2, \\
(L_Xg)_{\partial_2,\partial_4} = & -\lambda\cosh^2(2x_3)\partial_2 f_4-\cosh(2x_3)\cos(2x_4)\partial_4 f_1 \\
& +(\sinh(2x_3)+\cosh(2x_3)\sin(2x_4))\partial_4 f_2, \\
(L_Xg)_{\partial_3,\partial_3} = & 2\lambda\partial_3 f_3, \\
(L_Xg)_{\partial_3,\partial_4} = & \lambda\partial_4 f_3-\lambda\cosh^2(2x_3)\partial_3 f_4, \\
(L_Xg)_{\partial_4,\partial_4} = & -2\lambda\cosh^2(2x_3)\partial_4 f_4-4\lambda\cosh(2x_3)\sinh(2x_3)f_3.
\end{array}\right. \tag{1}$$

So X is a Killing vector field on M if and only if the following system of 10 PDEs holds

$$
\left\{
\begin{array}{ll}
2(\sinh(2x_3) - \cosh(2x_3)\sin(2x_4))\partial_1 f_1 - 2\cosh(2x_3)\cos(2x_4)\partial_1 f_2 & \\
\quad +2(\cosh(2x_3) - \sinh(2x_3)\sin(2x_4)) f_3 - 2\cosh(2x_3)\cos(2x_4) f_4 & = 0, \\
 & \\
-\cosh(2x_3)\cos(2x_4)\partial_1 f_1 + (\sinh(2x_3) - \cosh(2x_3)\sin(2x_4))\partial_2 f_1 & \\
-\cosh(2x_3)\cos(2x_4)\partial_2 f_2 + (\sinh(2x_3) + \cosh(2x_3)\sin(2x_4))\partial_1 f_2 & \\
-2\sinh(2x_3)\cos(2x_4) f_3 + 2\cosh(2x_3)\sin(2x_4) f_4 & = 0, \\
 & \\
\lambda\partial_1 f_3 + (\sinh(2x_3) - \cosh(2x_3)\sin(2x_4))\partial_3 f_1 & \\
\quad -\cosh(2x_3)\cos(2x_4)\partial_3 f_2 & = 0, \\
 & \\
-\lambda\cosh^2(2x_3)\partial_1 f_4 + (\sinh(2x_3) - \cosh(2x_3)\sin(2x_4))\partial_4 f_1 & \\
\quad -\cosh(2x_3)\cos(2x_4)\partial_4 f_2 & = 0, \\
 & \\
-2\cosh(2x_3)\cos(2x_4)\partial_2 f_1 + 2(\sinh(2x_3) & \\
\quad +\cosh(2x_3)\sin(2x_4))\partial_2 f_2 + (\sinh(2x_3)\sin(2x_4) & \\
\quad +\cosh(2x_3)) f_3 + \cosh(2x_3)\cos(2x_4) f_4 & = 0, \\
 & \\
\lambda\partial_2 f_3 - \cosh(2x_3)\cos(2x_4)\partial_3 f_1 + (\sinh(2x_3) & \\
\quad +\cosh(2x_3)\sin(2x_4))\partial_3 f_2 & = 0, \\
 & \\
-\lambda\cosh^2(2x_3)\partial_2 f_4 - \cosh(2x_3)\cos(2x_4)\partial_4 f_1 + (\sinh(2x_3) & \\
\quad +\cosh(2x_3)\sin(2x_4))\partial_4 f_2 & = 0, \\
 & \\
\lambda\partial_3 f_3 & = 0, \\
 & \\
\lambda\partial_4 f_3 - \lambda\cosh^2(2x_3)\partial_3 f_4 & = 0, \\
 & \\
-\lambda\cosh^2(2x_3)\partial_4 f_4 - 2\lambda\cosh(2x_3)\sinh(2x_3) f_3 & = 0.
\end{array}
\right.
\tag{2}
$$

Therefore, we give the solution of this system by:

$$
\left\{
\begin{array}{l}
f_1 = c_1 x_1 + (c_2 + c_3)\, x_2 + c_4, \\
f_2 = -c_1 x_2 + (c_2 - c_3)\, x_1 + c_5, \\
f_3 = c_1 \sin(2x_4) + c_2 \cos(2x_4), \\
f_4 = [c_1 \cos(2x_4) - c_2 \sin(2x_4)] \tanh(2x_3) + c_3.
\end{array}
\right.
$$

with $c_1, ..., c_5$ real constants.

Then the general solution of Killing vectors field system (2), holds for the vector field

$$\begin{aligned} X &= [c_1x_1 + (c_2 + c_3)\,x_2 + c_4]\,\frac{\partial}{\partial x_1} + [-c_1x_2 + (c_2 - c_3)\,x_1 + c_5]\,\frac{\partial}{\partial x_2} \\ &\quad + [c_1 \sin(2x_4) + c_2 \cos(2x_4)]\,\frac{\partial}{\partial x_3} \\ &\quad +[[c_1 \cos(2x_4) - c_2 \sin(2x_4)] \tanh(2x_3) + c_3]\frac{\partial}{\partial x_4}, \end{aligned} \tag{3}$$

with $c_1, ..., c_5$ real constants.

Hence, the following vectors form a basis of the Lie Algebra of Killing vector fields $i\,(M)$ on generalized symmetric spaces of type D, whose dimension is 5.

$$\begin{cases} V_1 = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + \sin(2x_4)\frac{\partial}{\partial x_3} + \cos(2x_4)\tanh(2x_3)\,\frac{\partial}{\partial x_4}, \\ V_2 = x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + \cos(2x_4)\frac{\partial}{\partial x_3} - \sin(2x_4)\tanh(2x_3)\,\frac{\partial}{\partial x_4}, \\ V_3 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}, \\ V_4 = \frac{\partial}{\partial x_1}, \\ V_5 = \frac{\partial}{\partial x_2}. \end{cases}$$

5 Ricci solitons on four-dimensional generalized symmetric space of type D

In this section we study the existence of Ricci solitons on the four-dimensional generalized symmetric spaces (M, g) of type D. Let $X = f_1\partial_1 + f_2\partial_2 + f_3\partial_3 + f_4\partial_4$ be an arbitrary vector field on (M, g), where $f_1, .., f_4$ are smooth functions of the variables x_1, x_2, x_3, x_4.

By using (4) and (1) in (1), a standard calculation gives that a four-dimensional generalized symmetric space of type D, is a Ricci soliton if and only if the following system of 10 PDEs holds,

$$\left\{\begin{array}{ll}
2(\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\partial_1 f_1 - 2\cosh(2x_3)\cos(2x_4)\partial_1 f_2 & \\
\quad -2\cosh(2x_3)\cos(2x_4)f_4 + 2(\cosh(2x_3)-\sinh(2x_3)\sin(2x_4))f_3 & \\
\quad +\alpha(\cosh(2x_3)\sin(2x_4)-\sinh(2x_3)) & =0, \\
-\cosh(2x_3)\cos(2x_4)\partial_1 f_1 + (\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\partial_2 f_1 & \\
\quad +(\sinh(2x_3)+\cosh(2x_3)\sin(2x_4))\partial_1 f_2 - \cosh(2x_3)\cos(2x_4)\partial_2 f_2 & \\
\quad -2\sinh(2x_3)\cos(2x_4)f_3 + 2\cosh(2x_3)\sin(2x_4)f_4 & \\
\quad +\alpha\cosh(2x_3)\cos(2x_4) & =0, \\
\lambda\partial_1 f_3 + (\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\partial_3 f_1 - \cosh(2x_3)\cos(2x_4)\partial_3 f_2 & =0, \\
-\lambda\cosh^2(2x_3)\partial_1 f_4 + (\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\partial_4 f_1 & \\
\quad -\cosh(2x_3)\cos(2x_4)\partial_4 f_2 & =0, \\
-2\cosh(2x_3)\cos(2x_4)\partial_2 f_1 + 2(\sinh(2x_3)+\cosh(2x_3)\sin(2x_4))\partial_2 f_2 & \\
\quad +2\cosh(2x_3)\cos(2x_4)f_4 \; + 2(\sinh(2x_3)\sin(2x_4) & \\
\quad +\cosh(2x_3))f_3 - \alpha(\cosh(2x_3)\sin(2x_4)+\sinh(2x_3)) & =0, \\
\lambda\partial_2 f_3 - \cosh(2x_3)\cos(2x_4)\partial_3 f_1 + (\sinh(2x_3)+\cosh(2x_3)\sin(2x_4))\partial_3 f_2 & =0, \\
-\lambda\cosh^2(2x_3)\partial_2 f_4 - \cosh(2x_3)\cos(2x_4)\partial_4 f_1 + (\sinh(2x_3) & \\
\quad +\cosh(2x_3)\sin(2x_4))\partial_4 f_2 & =0, \\
2\lambda\partial_3 f_3 - \alpha\lambda - 6 & =0, \\
\partial_4 f_3 - \cosh^2(2x_3)\partial_3 f_4 & =0, \\
2\lambda\cosh^2(2x_3)\partial_4 f_4 + 4\lambda\cosh(2x_3)\sinh(2x_3)f_3 - \cosh^2(2x_3)(\alpha\lambda+6) & =0.
\end{array}\right. \tag{1}$$

The eight equation in (1) yields

$$f_3\left(x_1, x_2, x_3, x_4\right) = \mu x_3 + F_3^1\left(x_1, x_2, x_4\right), \tag{2}$$

where F_3^1 is a smooth function depending on x_1, x_2, x_4 and $\mu = \frac{\alpha\lambda+6}{2\lambda}$. Deriving the nineth equation in (1) with respect to x_3, we obtain the following differential equation of second order

$$\partial_3^2 f_4 + 4\tanh\left(2x_3\right)\partial_3 f_4 = 0,$$

hence, integrating we get

$$f_4\left(x_1, x_2, x_3, x_4\right) = \frac{1}{2}F_4^1\left(x_1, x_2, x_4\right)\tanh\left(2x_3\right) + F_4^2\left(x_1, x_2, x_4\right), \tag{3}$$

where F_4^1 and F_4^2 are smooth functions depending on x_1, x_2, x_4.

Next, from the tenth equation in (1), we get

$$\partial_4 f_4 = \mu - 2\tanh(2x_3) f_3, \tag{4}$$

and deriving the nineth equation in (1) with respect to x_4, we have

$$\partial_4^2 f_3 - \cosh^2(2x_3)\partial_3\partial_4 f_4 = 0, \tag{5}$$

so replacing (2) and (4) into equation (5) ; we find the following differential equation of second order

$$\partial_4^2 F_3^1 + 4F_3^1 = \mu\left[-4x_3 - 2\sinh(2x_3)\cosh(2x_3)\right], \tag{6}$$

but F_3^1 is depending on x_1, x_2, x_4 only, then this is only possible if

$$\mu = 0, \tag{7}$$

then we deduce that

$$\alpha = \frac{-6}{\lambda}. \tag{8}$$

Therefore, a four-dimensional pseudo-Riemannian generalized symmetric space of type D, can be shrinking or expanding Ricci soliton but never steady one, because $\alpha \neq 0$.

Replacing μ into (6) then integrating; we find

$$F_3^1(x_1, x_2, x_4) = F_3^2(x_1, x_2)\cos(2x_4) + F_3^3(x_1, x_2)\sin(2x_4), \tag{9}$$

where F_3^2 and F_3^3 are smooth functions depending on x_1, x_2 and since $\mu = 0$ then f_3 is independant of x_3 and we have

$$f_3(x_1, x_2, x_4) = F_3^2(x_1, x_2)\cos(2x_4) + F_3^3(x_1, x_2)\sin(2x_4), \tag{10}$$

and (4) becomes

$$\partial_4 f_4 = -2\tanh(2x_3) f_3, \tag{11}$$

which by derivation with respect to x_3 gives

$$\partial_3\partial_4 f_4 = -\frac{4}{\cosh^2(2x_3)} F_3^1. \tag{12}$$

On the other hand; deriving f_4 twice from (3) with respect to x_3, then to x_4 we obtain

$$\partial_4\partial_3 f_4 = \frac{\partial_4 F_4^1}{\cosh^2(2x_3)}, \tag{13}$$

so, substituting (9) and (13) into (12) , hence integrating we get

$$F_4^1(x_1, x_2, x_4) = 2\left[F_3^3(x_1, x_2)\cos(2x_4) - F_3^2(x_1, x_2)\sin(2x_4)\right] + H(x_1, x_2), \tag{14}$$

where H is a smooth function depending on x_1, x_2.

Deriving f_4 from (3) with respect to x_4, then replacing into (11) and by taking (10) and (14) into account; we have

$$\partial_4 F_4^2 = 0,$$

so F_4^2 is depending only on x_1, x_2, then because of (14) we get

$$\begin{aligned} & f_4(x_1, x_2, x_3, x_4) \\ = & \left(F_3^3(x_1,x_2)\cos(2x_4) - F_3^2(x_1,x_2)\sin(2x_4) + \frac{1}{2}H(x_1,x_2)\right)\tanh(2x_3) \\ & +F_4^2(x_1,x_2). \end{aligned}$$

We replace in the ninth equation in (1), by using (10), we have

$$H = 0,$$

so

$$\begin{aligned} f_4(x_1, x_2, x_3, x_4) = & \left[F_3^3(x_1,x_2)\cos(2x_4) - F_3^2(x_1,x_2)\sin(2x_4)\right]\tanh(2x_3) \\ & +F_4^2(x_1,x_2). \end{aligned} \tag{15}$$

We combine the first, the second and the fifth equation of the system (1), hence we get

$$\partial_1 f_1 + \partial_2 f_2 = \alpha. \tag{16}$$

By using the third and the sixth equation of the system (1), we have

$$\partial_3 f_1 = \lambda\left[\sinh(2x_3) + \cosh(2x_3)\sin(2x_4)\right]\partial_1 f_3 + \lambda\cosh(2x_3)\cos(2x_4)\,\partial_2 f_3, \tag{17}$$

$$\partial_3 f_2 = \lambda\cosh(2x_3)\cos(2x_4)\,\partial_1 f_3 + \lambda\left[\sinh(2x_3) - \cosh(2x_3)\sin(2x_4)\right]\partial_2 f_3, \tag{18}$$

and by using the fourth and the seventh equation of the system (1), we get

$$\begin{aligned} \partial_4 f_1 = & -\lambda\cosh^2(2x_3)\left[(\sinh(2x_3) + \cosh(2x_3)\sin(2x_4))\,\partial_1 f_4 \right. \\ & \left. + \cosh(2x_3)\cos(2x_4)\,\partial_2 f_4\right], \end{aligned} \tag{19}$$

$$\begin{aligned} \partial_4 f_2 = & -\lambda\cosh^2(2x_3)\left[\cosh(2x_3)\cos(2x_4)\,\partial_1 f_4 \right. \\ & \left. +(\sinh(2x_3) - \cosh(2x_3)\sin(2x_4)\,\partial_2 f_4)\right]. \end{aligned} \tag{20}$$

Then, deriving (17) with respect to x_1 and (18) with respect to x_2; next replacing f_3 and taking into account (16); we obtain

$$\begin{aligned} & \partial_1\partial_2 F_3^2(x_1,x_2)\left(2\cosh(2x_3)\cos^2(2x_4)\right) \\ & +\partial_1\partial_2 F_3^3(x_1,x_2)\left(2\cosh(2x_3)\cos(2x_4)\sin(2x_4)\right) \\ & +\partial_1^2 F_3^2(x_1,x_2)\cos(2x_4)\left(\sinh(2x_3) + \cosh(2x_3)\sin(2x_4)\right) \\ & +\partial_1^2 F_3^3(x_1,x_2)\sin(2x_4)\left(\sinh(2x_3) + \cosh(2x_3)\sin(2x_4)\right) \\ & +\partial_2^2 F_3^2(x_1,x_2)\cos(2x_4)\left(\sinh(2x_3) - \cosh(2x_3)\sin(2x_4)\right) \\ & +\partial_2^2 F_3^3(x_1,x_2)\sin(2x_4)\left(\sinh(2x_3) - \cosh(2x_3)\sin(2x_4)\right) = 0, \end{aligned}$$

for any values of x_3, x_4. Since F_3^2, F_3^3 are independent of x_3 and x_4 and since the family

$\{\cosh(2x_3)\cos^2(2x_4), \cosh(2x_3)\cos(2x_4)\sin(2x_4),$
$\cos(2x_4)(\sinh(2x_3)+\cosh(2x_3)\sin(2x_4)),$
$\sin(2x_4)(\sinh(2x_3)+\cosh(2x_3)\sin(2x_4)),$
$\cos(2x_4)(\sinh(2x_3)-\cosh(2x_3)\sin(2x_4)),$
$\sin(2x_4)(\sinh(2x_3)-\cosh(2x_3)\sin(2x_4))\}$ is linearly independant, we have then

$$\partial_1\partial_2 F_3^2(x_1,x_2) = \partial_1^2 F_3^2(x_1,x_2) = \partial_2^2 F_3^2(x_1,x_2) = 0,$$
$$\partial_1\partial_2 F_3^3(x_1,x_2) = \partial_1^2 F_3^3(x_1,x_2) = \partial_2^2 F_3^3(x_1,x_2) = 0,$$

therefore

$$f_3(x_1,x_2,x_4) = \left(a_3^1 x_1 + a_3^2 x_2 + a\right)\cos(2x_4) + \left(a_3^3 x_1 + a_3^4 x_2 + b\right)\sin(2x_4), \quad (21)$$

with $a, b, a_3^1, a_3^2, a_3^3, a_3^4$ real constants.

Next, replacing f_4 and deriving the fourth and the seventh equation of the system (1), with respect to x_1 and with respect to x_2; we get

$$\partial_1\partial_2 F_4^2(x_1,x_2) = \partial_1^2 F_4^2(x_1,x_2) = \partial_2^2 F_4^2(x_1,x_2) = 0,$$

therefore

$$\begin{aligned} & f_4(x_1,x_2,x_3,x_4) \\ = & \left[\left(a_3^3 x_1 + a_3^4 x_2 + b\right)\cos(2x_4) - \left(a_3^1 x_1 + a_3^2 x_2 + a\right)\sin(2x_4)\right]\tanh(2x_3) \\ & + \left(b_4^1 x_1 + b_4^2 x_2 + c\right), \end{aligned} \quad (22)$$

with b_4^1, b_4^2 and c real constants.

Now, by using (21), (22) and deriving (17), (18), (19) and (20) with respect to x_1 and with respect to x_2, a standard calculation yields

$$\partial_1\partial_3 f_1 = \partial_2\partial_3 f_1 = 0,$$
$$\partial_1\partial_3 f_2 = \partial_2\partial_3 f_2 = 0,$$
$$\partial_1\partial_4 f_1 = \partial_2\partial_4 f_1 = 0,$$
$$\partial_1\partial_4 f_2 = \partial_2\partial_4 f_2 = 0,$$

and deriving twice (17) and (18) with respect to x_3, yields

$$\begin{aligned} \partial_3^3 f_1 &= 4\partial_3 f_1 \\ \partial_3^3 f_2 &= 4\partial_3 f_2 \end{aligned}$$

so, integrating gives

$$\partial_3 f_1 = A_1(x_4)\, e^{2x_3} + A_2(x_4)\, e^{-2x_3}$$

and

$$\partial_3 f_2 = B_1(x_4)\, e^{2x_3} + B_2(x_4)\, e^{-2x_3}$$

where A_1, A_2, B_1 and B_2 are smooth functions depending on x_4, hence integrating we obtain

$$\begin{aligned} f_1(x_1,x_2,x_3,x_4) &= \frac{1}{2}\left[A_1(x_4)e^{2x_3} + A_2(x_4)e^{-2x_3}\right] + G_1(x_1,x_2), \\ f_2(x_1,x_2,x_3,x_4) &= \frac{1}{2}\left[B_1(x_4)e^{2x_3} - B_2(x_4)e^{-2x_3}\right] + G_2(x_1,x_2), \end{aligned} \tag{23}$$

with G_1 and G_2 smooth functions depending on x_1, x_2, which give by (16) that

$$\partial_1 G_1 + \partial_2 G_2 = \alpha, \tag{24}$$

Adding the first and the fifth equation of the system (1) and using (23), we obtain the following system

$$\begin{cases} \partial_1 G_2 + \partial_2 G_1 = 2\left(a_3^1 x_1 + a_3^2 x_2 + a\right), \\ \partial_1 G_1 - \frac{\alpha}{2} = a_3^3 x_1 + a_3^4 x_2 + b. \end{cases} \tag{25}$$

The second equation in (25) yields

$$G_1(x_1,x_2) = \frac{a_3^3}{2}x_1^2 + a_3^4 x_1 x_2 + \left(b + \frac{\alpha}{2}\right)x_1 + H_1(x_2),$$

where H_1 is a smooth function. Next, in the system (25) deriving the first equation with respect to x_1 and the second equation with respect to x_2, we get

$$G_2(x_1,x_2) = \left(2a_3^1 - a_3^4\right)\frac{x_1^2}{2} + x_1 H_2(x_2) + H_3(x_2),$$

where H_2 and H_3 are smooth functions and by using (24), it becomes

$$G_2(x_1,x_2) = \left(2a_3^1 - a_3^4\right)\frac{x_1^2}{2} + \theta_1 x_1 - a_3^3 x_1 x_2 + \left(\frac{\alpha}{2} - b\right)x_2 - \frac{a_3^4}{2}x_2^2 + \gamma, \tag{26}$$

with γ a real constant.

Replacing in the first equation of the system (25) gives

$$G_1(x_1,x_2) = \frac{a_3^3}{2}x_1^2 + a_3^4 x_1 x_2 + \left(b + \frac{\alpha}{2}\right)x_1 + \left(2a_3^2 + a_3^3\right)\frac{x_2^2}{2} + (2a - \theta_1)x_2 + \beta, \tag{27}$$

with β a real constant.

Replacing f_1 and f_2, in the second equation of the system (1), by taking into account of (26) and (27), we obtain

$$\begin{aligned} G_1(x_1,x_2) &= \tfrac{a_3^3}{2}x_1^2 + a_3^4 x_1 x_2 + \left(b + \tfrac{\alpha}{2}\right)x_1 + \left(2a_3^2 + a_3^3\right)\tfrac{x_2^2}{2} + (a + c)x_2 + \beta, \\ G_2(x_1,x_2) &= \left(2a_3^1 - a_3^4\right)\tfrac{x_1^2}{2} + (a - c)x_1 - a_3^3 x_1 x_2 + \left(\tfrac{\alpha}{2} - b\right)x_2 - \tfrac{a_3^4}{2}x_2^2 + \gamma. \end{aligned} \tag{28}$$

Hence, replacing f_3 in the equations (17) and (18), by taking into account (28) , we get

$$\begin{aligned} A_1(x_4) &= \frac{\lambda}{2}\left[a_3^1\cos(2x_4) + a_3^3\sin(2x_4)(1+\sin(2x_4)) + a_3^2\cos^2(2x_4)\right. \\ &\quad \left. + \left(a_3^1 + a_3^4\right)\cos(2x_4)\sin(2x_4)\right] \\ A_2(x_4) &= \frac{\lambda}{2}\left[a_3^1\cos(2x_4) + a_3^3\sin(2x_4)(1-\sin(2x_4)) - a_3^2\cos^2(2x_4)\right. \\ &\quad \left. - \left(a_3^1 + a_3^4\right)\cos(2x_4)\sin(2x_4)\right] \\ B_1(x_4) &= \frac{\lambda}{2}\left[\cos(2x_4)\sin(2x_4)\left(a_3^3 - a_3^2\right) + \cos(2x_4)\left(a_3^1\cos(2x_4) + a_3^2\right)\right. \\ &\quad \left. + a_3^4\sin(2x_4)(1-\sin(2x_4))\right] \\ B_2(x_4) &= \frac{\lambda}{2}\left[\cos(2x_4)\sin(2x_4)\left(a_3^3 - a_3^2\right) + \cos(2x_4)\left(a_3^1\cos(2x_4) - a_3^2\right)\right. \\ &\quad \left. - a_3^4\sin(2x_4)(1+\sin(2x_4))\right] \end{aligned}$$

Therefore, we have

$$\begin{aligned} f_1(x_1,x_2,x_3,x_4) &= \frac{\lambda}{4}\{[a_3^1\cos(2x_4) + a_3^3\sin(2x_4)(1+\sin(2x_4)) + a_3^2\cos^2(2x_4) \\ &\quad + \left(a_3^1 + a_3^4\right)\cos(2x_4)\sin(2x_4)]e^{2x_3} + [a_3^1\cos(2x_4) \\ &\quad + a_3^3\sin(2x_4)(1-\sin(2x_4)) - a_3^2\cos^2(2x_4) \\ &\quad - \left(a_3^1 + a_3^4\right)\cos(2x_4)\sin(2x_4)]e^{-2x_3}\} + \frac{a_3^3}{2}x_1^2 + a_3^4 x_1 x_2 \frac{x_2^2}{2} \\ &\quad + \left(b + \frac{\alpha}{2}\right)x_1 + \left(2a_3^2 + a_3^3\right) + (a+c)\,x_2 + \beta, \\ f_2(x_1,x_2,x_3,x_4) &= \frac{\lambda}{4}\{[\cos(2x_4)\sin(2x_4)\left(a_3^3 - a_3^2\right) + \cos(2x_4)\left(a_3^1\cos(2x_4) + a_3^2\right) \\ &\quad + a_3^4\sin(2x_4)(1-\sin(2x_4))]e^{2x_3} - [\cos(2x_4)\sin(2x_4)\left(a_3^3 - a_3^2\right) \\ &\quad + \cos(2x_4)\left(a_3^1\cos(2x_4) - a_3^2\right) \\ &\quad - a_3^4\sin(2x_4)(1+\sin(2x_4))]e^{-2x_3}\} - \frac{a_3^4}{2}x_2^2 \\ &\quad + \left(2a_3^1 - a_3^4\right)\frac{x_1^2}{2} + (a-c)\,x_1 - a_3^3 x_1 x_2 + \left(\frac{\alpha}{2} - b\right)x_2 + \gamma, \\ f_3(x_1,x_2,x_4) &= \left(a_3^1 x_1 + a_3^2 x_2 + a\right)\cos(2x_4) + \left(a_3^3 x_1 + a_3^4 x_2 + b\right)\sin(2x_4), \\ f_4(x_1,x_2,x_3,x_4) &= [\left(a_3^3 x_1 + a_3^4 x_2 + b\right)\cos(2x_4) \\ &\quad - \left(a_3^1 x_1 + a_3^2 x_2 + a\right)\sin(2x_4)]\tanh(2x_3) \\ &\quad + \left[\left(a_3^4 - a_3^1\right)x_1 + \left(a_3^2 + a_3^3\right)x_2 + c\right]. \end{aligned}$$

Finally, by replacing in the fourth and the seventh equation of the system (1), we prove that

$$\begin{aligned} a_3^4 + \left(a_3^3 - a_3^2\right)\cos(2x_4)\sin(2x_4) - \left(a_3^1 + a_3^4\right)(\sin(2x_4))^2 = 0 \\ a_3^3 - \left(a_3^1 + a_3^4\right)\cos(2x_4)\sin(2x_4) - \left(a_3^3 - a_3^2\right)(\sin(2x_4))^2 = 0 \end{aligned}$$

which gives since the family $\left\{1, \cos(2x_4)\sin(2x_4), (\sin(2x_4))^2\right\}$ is linearly independant that

$$a_3^1 = a_3^2 = a_3^3 = a_3^4 = 0.$$

The calculations above proved that the general solution of the Ricci soliton system (1) is explicitly given by $X = f_1\partial_1 + f_2\partial_2 + f_3\partial_3 + f_4\partial_4$ where

$$\begin{aligned} f_1 &= \left(b + \frac{\alpha}{2}\right) x_1 + (a+c)\, x_2 + \beta, \\ f_2 &= (a-c)\, x_1 + \left(\frac{\alpha}{2} - b\right) x_2 + \gamma\ , \\ f_3 &= a\cos(2x_4) + b\sin(2x_4)\,, \\ f_4 &= [b\cos(2x_4) - a\sin(2x_4)]\tanh(2x_3) + c, \end{aligned} \tag{29}$$

for arbitrary real constants a, b, c, β, γ.

Then, the Ricci soliton equation (1) holds for the metric g described in (4) with the vector field

$$\begin{aligned} X = {} & \left[\tfrac{1}{2}(\alpha + 2b)\, x_1 + (a+c)\, x_2 + \beta\right]\partial_1 + \left[(a-c)\, x_1 + \tfrac{1}{2}(\alpha - 2b)\, x_2 + \gamma\ \right]\partial_2 \\ & + [a\cos(2x_4) + b\sin(2x_4)]\,\partial_3 + [(b\cos(2x_4) - a\sin(2x_4))\tanh(2x_3) + c]\,\partial_4, \end{aligned} \tag{30}$$

for arbitrary real constants a, b, c, β, γ.

Therefore, the four-dimensional pseudo-Riemannian generalized symmetric spaces of type D, admit appropriate vector fields for which (1) holds.

For any value of α not zero, we have the following result :

Theorem 2. *A four-dimensional pseudo-Riemannian generalized symmetric space of type D is expanding and shrinking Ricci solitons but never steady.*

Comparing between the general solution (3) of the Killing vectors field system and the general solution (30) of the Ricci soliton system; we notice that this last one can not be a Killing vector field, otherwise $\alpha = 0$; while four-dimensional pseudo-Riemannian generalized symmetric space of type D can not be a steady Ricci soliton.

Now, let $X = \text{grad}\ h$ be an arbitrary gradient vector field on four-dimensional pseudo-Riemannian generalized symmetric space of type D with potential function h.

X is then given by

$$\begin{aligned} \text{grad}\ h = {} & [-(\cosh(2x_3)\sin(2x_4) + \sinh(2x_3))(\partial_1 h) - \cosh(2x_3)\cos(2x_4)(\partial_2 h)]\,\partial_1 \\ & + [-\cosh(2x_3)\cos(2x_4)\,\partial_1 h + (\cosh(2x_3)\sin(2x_4) - \sinh(2x_3))\,\partial_2 h]\,\partial_2 \\ & + \tfrac{1}{\lambda}(\partial_3 h)\,\partial_3 - \tfrac{1}{\lambda\cosh^2(2x_3)}(\partial_4 h)\,\partial_4. \end{aligned} \tag{31}$$

By a standard calculation we prove, using (29) that the four-dimensional pseudo-Riemannian generalized symmetric space of type D is gradient Ricci soliton, if and only if the following system yields

$$\left\{\begin{array}{ll} -\left(\cosh\left(2x_3\right)\sin\left(2x_4\right)+\sinh\left(2x_3\right)\right)\partial_1 h & \\ \quad -\cosh\left(2x_3\right)\cos\left(2x_4\right)\partial_2 h-\left(b+\frac{\alpha}{2}\right)x_1-\left(a+c\right)x_2-\beta & =0, \\ \left(\cosh\left(2x_3\right)\sin\left(2x_4\right)-\sinh\left(2x_3\right)\right)\partial_2 h & \\ \quad -\cosh\left(2x_3\right)\cos\left(2x_4\right)\partial_1 h-\left(a-c\right)x_1-\left(\frac{\alpha}{2}-b\right)x_2-\gamma & =0, \\ \partial_3 h-\lambda\left[a\cos\left(2x_4\right)+b\sin\left(2x_4\right)\right] & =0, \\ \partial_4 h+\lambda\cosh\left(2x_3\right)\sinh\left(2x_3\right)\left[b\cos\left(2x_4\right)-a\sin\left(2x_4\right)\right] & \\ \quad +c\lambda\cosh^2\left(2x_3\right) & =0. \end{array}\right. \tag{32}$$

Hence, by using the derivative of the last equation in (32) with respect to x_3 and the derivative of the third one with respect to x_4, we obtain

$$a\sin\left(2x_4\right)-b\cos\left(2x_4\right)-c\tanh\left(2x_3\right)=0,$$

for every x_3 and x_4, which gives since the family $\{\sin\left(2x_4\right),\cos\left(2x_4\right),\tanh\left(2x_3\right)\}$ is linearly independant, that

$$a=b=c=0,$$

consequently, we obtain that

$$\partial_3 h=\partial_4 h=0,$$

thus, h is independant of x_3 and x_4, so the first and the second equation in (32) , become

$$\left\{\begin{array}{ll} \left[\cosh\left(2x_3\right)\sin\left(2x_4\right)+\sinh\left(2x_3\right)\right]\partial_1 h+\cosh\left(2x_3\right)\cos\left(2x_4\right)\partial_2 h+\frac{1}{2}\alpha x_1+\beta & =0, \\ \cosh\left(2x_3\right)\cos\left(2x_4\right)\partial_1 h-\left[\cosh\left(2x_3\right)\sin\left(2x_4\right)-\sinh\left(2x_3\right)\right]\partial_2 h+\frac{1}{2}\alpha x_2+\gamma & =0. \end{array}\right.$$

Finally, by using the derivative of the two equations above with respect to x_3, we prove that

$$\partial_1 h=\partial_2 h=0.$$

We deduce that h is a real constant, but in this case; h does not verify the first and the second equation in (32). Thus, we have shown the following corollary.

Corollary 1. *A four-dimensional pseudo-Riemannian generalized symmetric space of type D is not a gradient Ricci soliton.*

Conclusion

Generalized symmetric spaces has been studied by different authors, and in this paper we were interested by the four-dimensionnal ones of type D, equipped with a pseudo-Riemannian metric g. We described their curvature properties as the Levi-Civita connection, the Riemann curvature tensor and the Ricci curvature tensor, next we classify the Killing vectors field of these spaces then we studied the existence of non-trivial, i.e., not Einstein Ricci solitons; we showed that these spaces are shrinking or expanding Ricci solitons but never steady. Moreover we proved that this Ricci soliton is not a gradient one.

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