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A UNIQUE COMMON FIXED POINT FOR CONTRACTIVE MAPPINGS UNDER A NEW CONCEPT

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Abstract

In this paper, we will investigate the existence and uniqueness of common fixed points of certain mappings in the frame of a metric space. The given results cover a number of unique common fixed point theorems especially a result of Phaneendra and Swatmaram [12]. We will also display two examples to illustrate our theorems.

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1 Introduction and preliminaries

We start our work by giving the definition of commuting mappings in a metric space.

Definition 1. *Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be commuting if and only if*

$$fgx = gfx$$

for all x in $\mathcal{X}$.

In 1982, Sessa [14] relaxed the commutativity to the weak commutativity.

Definition 2. *([14]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called weakly commuting if and only if*

$$d(fgx, gfx) \leq d(fx, gx)$$

for all x in $\mathcal{X}$.

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In 1986, Jungck [6] generalized the concept of weak commutativity by introducing the notion of compatible mappings.

Definition 3. *([6]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called compatible if and only if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

In 1995, Jungck and Pathak [9] gave a generalization of the above concept of compatible mappings called biased mappings.

Definition 4. *([9]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is g-biased if and only if whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ and fx_n, $gx_n \to t \in \mathcal{X}$, then*

$$\alpha d(gfx_n, gx_n) \leq \alpha d(fgx_n, fx_n)$$

if $\alpha = \lim inf$ and $\alpha = \lim sup$.

Again, the same authors [9], introduced the concept of weakly biased mappings which represents a convenient generalization of biased mappings.

Definition 5. *([9]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is weakly g-biased if and only if $fp = gp$ implies*

$$d(gfp, gp) \leq d(fgp, fp).$$

In 2012, in [5], we introduced the concept of occasionally weakly biased mappings which is a legitimate generalization of weakly biased mappings given by Jungck and Pathak in [9].

Definition 6. *([5]) Let f and g be self-mappings of a set $\mathcal{X}$. The pair (f, g) is said to be occasionally weakly f-biased and g-biased, respectively, if and only if, there exists a point p in $\mathcal{X}$ such that $fp = gp$ implies*

$$d(fgp, fp) \leq d(gfp, gp),$$
$$d(gfp, gp) \leq d(fgp, fp),$$

respectively.

Let us return back to 1993, Jungck et al. [8] introduced the concept of compatible mappings of type (A) which is equivalent to compatible mappings under the continuity condition.

Definition 7. *([8]) Self-mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be compatible of type (A) if*

$$\lim_{n\to\infty} d(gfx_n, ffx_n) = 0, \lim_{n\to\infty} d(fgx_n, ggx_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that fx_n and $gx_n \to t \in \mathcal{X}$.

After two years, Pathak et al. [11] generalized the above notion by giving the concept of biased mappings of type (A).

Definition 8. *([11]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is said to be g-biased and f-biased of type (A), respectively, if, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ and fx_n, $gx_n \to t \in \mathcal{X}$,*

$$\alpha d(ggx_n, fx_n) \leq \alpha d(fgx_n, gx_n),$$
$$\alpha d(ffx_n, gx_n) \leq \alpha d(gfx_n, fx_n),$$

respectively, where $\alpha = \lim inf$ and if $\alpha = \lim sup$.

Also and in the same paper [11], the authors gave the definition of weakly g-biased of type (A) as follows:

Definition 9. *([11]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is said to be weakly g-biased of type (A) if $fp = gp$ implies*

$$d(ggp, fp) \leq d(fgp, gp).$$

In 1996, the notion of compatible mappings was again generalized in [7] by Jungck himself.

Definition 10. *([7]) Two self-mappings f and g of a metric space $(\mathcal{X}, d)$ are called weakly compatible if and only if f and g commute on the set of coincidence points.*

In 2008, Al-Thagafi and Shahzad [2] introduced the notion of occasionally weakly compatible (owc) mappings as a generalization of weakly compatible mappings. While the paper [2] was under review, Jungck and Rhoades [10] used the concept of owc and proved several results under different contractive conditions (see [1]).

Definition 11. *([2]) Two self-mappings f and g of a set $\mathcal{X}$ are occasionally weakly compatible if and only if, there is a point t in $\mathcal{X}$ which is a coincidence point of f and g at which f and g commute.*

Recently, in 2021, in [4] we introduced the concept of weakly f-biased of type (A), and the concepts of occasionally weakly f-biased of type (A) and occasionally weakly g-biased of type (A), and we showed that the two last new definitions coincide with our concepts; occasionally weakly f-biased and occasionally weakly g-biased respectively given in [5].

Definition 12. *([4]) Let f and g be self-mappings of a metric space $(\mathcal{X}, d)$. The pair (f, g) is said to be* ***weakly f-biased of type*** *(A) if $fp = gp$ implies*

$$d(ffp, gp) \leq d(gfp, fp).$$

Definition 13. *([4]) Let f and g be self-mappings of a non-empty set $\mathcal{X}$. The pair (f,g) is said to be **occasionally weakly f-biased of type** (A) **and occasionally weakly g-biased of type** (A), respectively, if and only if, there exists a point p in $\mathcal{X}$ such that $fp = gp$ implies*

$$d(ffp, gp) \leq d(gfp, fp),$$
$$d(ggp, fp) \leq d(fgp, gp),$$

respectively.

In addition that weakly f-biased of type (A) and weakly g-biased of type (A) are occasionally weakly f-biased of type (A) and occasionally weakly g-biased of type (A), respectively, it is also clear from the definitions that if f and g are occasionally weakly compatible or weakly compatible then f, g are both occasionally weakly f-biased and g-biased of type (A). Therefore, occasionally weakly compatible and weakly compatible mappings are subclasses of occasionally weakly biased of type (A) mappings. The next example confirms.

Example 1. *Let $\mathcal{X} = [0,\infty)$ with the usual metric $d(x,y) = |x-y|$. Define f, $g : \mathcal{X} \to \mathcal{X}$ by*

$$fx = \begin{cases} 25x^2 & \text{if } x \in [0,1] \\ \frac{20}{x} & \text{if } x \in (1,\infty), \end{cases} \quad gx = \begin{cases} 1 & \text{if } x \in [0,1] \\ 5x & \text{if } x \in (1,\infty). \end{cases}$$

We have $fx = gx$ if and only if $x = \frac{1}{5}$ or $x = 2$ and

$$0 = d\left(gg\left(\frac{1}{5}\right), f\left(\frac{1}{5}\right)\right) \leq d\left(fg\left(\frac{1}{5}\right), g\left(\frac{1}{5}\right)\right) = 24;$$

that is, the pair (f,g) is occasionally weakly g-biased of type (A). However,

$$40 = d\left(gg(2), f(2)\right) \nleq d\left(fg(2), g(2)\right) = 8,$$

then, f and g are not weakly g-biased of type (A).

On the other hand we have

$$8 = d\left(ff(2), g(2)\right) \leq d\left(gf(2), f(2)\right) = 40;$$

i.e., the pair (f,g) is occasionally weakly f-biased of type (A). But, as

$$24 = d\left(ff\left(\frac{1}{5}\right), g\left(\frac{1}{5}\right)\right) \nleq d\left(gf\left(\frac{1}{5}\right), f\left(\frac{1}{5}\right)\right) = 0;$$

i.e., the pair (f,g) is not weakly f-biased of type (A).

Again, we have

$$fg\left(\frac{1}{5}\right) = 25 \neq 1 = gf\left(\frac{1}{5}\right),$$
$$fg(2) = 2 \neq 50 = gf(2),$$

that is, f and g are neither occasionally weakly compatible nor weakly compatible.

In their paper [12], Phaneendra and Swatmaram obtained a generalization of a result of Banarjee and Thakur [3] by replacing the compatibility with the notion of weakly compatible mappings when any one of the range spaces $f(\mathcal{X})$, $g(\mathcal{X})$, $h(\mathcal{X})$ and $k(\mathcal{X})$ is a complete subspace of $\mathcal{X}$.

Theorem 1. *([12]) Let f, g, h and k be self-mappings on a metric space $\mathcal{X}$ satisfying the pair of inclusions $f(\mathcal{X}) \subset k(\mathcal{X})$ and $g(\mathcal{X}) \subset h(\mathcal{X})$, and the inequality*

$$\begin{aligned} d^2(fx, gy) \leq\ & a\max\{d^2(fx, hx), d^2(gy, ky), d^2(hx, ky)\} \\ & + b\max\{d(fx, hx)d(hx, gy), d(fx, ky)d(gy, ky)\} \\ & + cd(hx, gy)d(ky, fx) \end{aligned}$$

for all x, $y \in \mathcal{X}$, where a, b, $c \geq 0$ such that $a + 2b < 1$ and $a + c < 1$. Suppose that one of $f(\mathcal{X})$, $g(\mathcal{X})$, $h(\mathcal{X})$ and $k(\mathcal{X})$ is a complete subspace of $\mathcal{X}$ and the pairs (f, h) and (g, k) are weakly compatible. Then all the four mappings f, g, h and k have a unique common fixed point.

In this contribution we will give two results which improve the above theorem by removing some conditions, extending the constants to functions, and increasing the number of mappings, all this under the new concept of occasionally weakly biased of type (A) mappings. Again, we will present two examples to illustrate our results.

2 Our main results

2.1 A unique common fixed point theorem for four mappings

Theorem 2. *Let (f, h) and (g, k) be occasionally weakly h-biased and k-biased of type (A) self-mappings on a metric space $(\mathcal{X}, d)$ satisfying the inequality*

$$\begin{aligned} d^2(fx, gy) \leq\ & a\max\{d^2(fx, hx), d^2(gy, ky), d^2(hx, ky)\} \\ & + b\max\{d(fx, hx)d(hx, gy), d(fx, ky)d(gy, ky)\} \\ & + cd(hx, gy)d(ky, fx) \end{aligned} \tag{1}$$

for all x, $y \in \mathcal{X}$, where a, b, $c \geq 0$ such that $4a + 2b + c < 1$. Then all the four mappings f, g, h and k have a unique common fixed point.

Proof. By hypothesis, there are two points u and v in $\mathcal{X}$ such that $fu = hu$ implies $d(hhu, fu) \leq d(fhu, hu)$ and $gv = kv$ implies $d(kkv, gv) \leq d(gkv, kv)$.

First, we are going to prove that $fu = gv$. Suppose that $fu \neq gv$, from inequality (1) we have

$$\begin{aligned} d^2(fu, gv) \leq\ & a\max\{d^2(fu, hu), d^2(gv, kv), d^2(hu, kv)\} \\ & + b\max\{d(fu, hu)d(hu, gv), d(fu, kv)d(gv, kv)\} \\ & + cd(hu, gv)d(kv, fu) \\ =\ & (a + c)d^2(fu, gv) \\ <\ & d^2(fu, gv) \end{aligned}$$

which is a contradiction because $a+c \leq 4a+2b+c<1$, thus $fu=gv$.

Now, we assert that $ffu=fu$. If not, then the use of condition (1) gives

$$\begin{aligned} d^2(ffu,gv) &\leq a\max\{d^2(ffu,hfu),d^2(gv,kv),d^2(hfu,kv)\} \\ &\quad +b\max\{d(ffu,hfu)d(hfu,gv),d(ffu,kv)d(gv,kv)\} \\ &\quad +cd(hfu,gv)d(kv,ffu) \\ &= a\max\{d^2(ffu,hfu),0,d^2(hfu,fu)\} \\ &\quad +bd(ffu,hfu)d(hfu,fu) \\ &\quad +cd(hfu,fu)d(fu,ffu). \end{aligned}$$

Since the pair (f,h) is occasionally weakly h-biased of type (A) we have $d(hfu,fu)=d(hhu,fu) \leq d(fhu,hu)=d(ffu,fu)$ and using the triangle inequality, we get

$$\begin{aligned} d^2(ffu,fu) &\leq a\max\{(d(ffu,fu)+d(fu,hfu))^2,0,d^2(hfu,fu)\} \\ &\quad +b(d(ffu,fu)+d(fu,hfu))d(hfu,fu) \\ &\quad +cd(hfu,fu)d(fu,ffu) \\ &\leq a\max\{4d^2(ffu,fu),0,d^2(ffu,fu)\} \\ &\quad +2bd^2(ffu,fu) \\ &\quad +cd^2(fu,ffu) \\ &= (4a+2b+c)d^2(fu,ffu) \\ &< d^2(fu,ffu), \end{aligned}$$

which is a contradiction, therefore $ffu=fu$ and so $hfu=fu$.

Now, suppose that $ggv \neq gv$. Using inequality (1) we obtain

$$\begin{aligned} d^2(fu,ggv) &\leq a\max\{d^2(fu,hu),d^2(ggv,kgv),d^2(hu,kgv)\} \\ &\quad +b\max\{d(fu,hu)d(hu,ggv),d(fu,kgv)d(ggv,kgv)\} \\ &\quad +cd(hu,ggv)d(kgv,fu) \\ &= a\max\{0,d^2(ggv,kgv),d^2(gv,kgv)\} \\ &\quad +bd(gv,kgv)d(ggv,kgv) \\ &\quad +cd(gv,ggv)d(kgv,gv). \end{aligned}$$

As the pair (g,k) is occasionally weakly k-biased of type (A), we have $d(kgv,gv)=d(kkv,gv) \leq d(gkv,kv)=d(ggv,gv)$. Again, using the triangle inequality we get

$$\begin{aligned} d^2(gv,ggv) &\le a\max\{0,(d(ggv,gv)+d(gv,kgv))^2,d^2(gv,kgv)\} \\ &\quad +bd(gv,kgv)(d(ggv,gv)+d(gv,kgv)) \\ &\quad +cd(gv,ggv)d(kgv,gv) \\ &\le a\max\{0,4d^2(ggv,gv),d^2(gv,ggv)\} \\ &\quad +2bd^2(gv,ggv) \\ &\quad +cd^2(gv,ggv) \\ &= (4a+2b+c)d^2(gv,ggv) \\ &< d^2(gv,ggv). \end{aligned}$$

This contradiction implies that $ggv = gv$ and so $kgv = gv$; i.e., $gfu = fu$ and $kfu = fu$. Put $fu = hu = gv = kv = w$, therefore w is a common fixed point of mappings f, g, h and k.

Finally, let w and t be two distinct common fixed points of mappings f, g, h and k. Then, $w = fw = gw = hw = kw$ and $t = ft = gt = ht = kt$. From (1) we have

$$\begin{aligned} d^2(ft,gw) &\le a\max\{d^2(ft,ht),d^2(gw,kw),d^2(ht,kw)\} \\ &\quad +b\max\{d(ft,ht)d(ht,gw),d(ft,kw)d(gw,kw)\} \\ &\quad +cd(ht,gw)d(kw,ft); \end{aligned}$$

i.e.,

$$\begin{aligned} d^2(t,w) &\le (a+c)d^2(t,w) \\ &< d^2(t,w), \end{aligned}$$

which is a contradiction, thus $t = w$. □

Now, we give an illustrative example which highlights our result.

Example 2. *Let $\mathcal{X} = [0,\infty)$ with the metric $d(x,y) = |x-y|$. Define*

$$fx = \begin{cases} \frac{9}{10} \text{ if } x \in [0,1) \\ 1 \text{ if } x \in [1,\infty), \end{cases} \quad gx = \begin{cases} \frac{4}{5} \text{ if } x \in [0,1) \\ 1 \text{ if } x \in [1,\infty), \end{cases}$$

$$hx = \begin{cases} 10 \text{ if } x \in [0,1) \\ \frac{1}{x^3} \text{ if } x \in [1,\infty), \end{cases} \quad kx = \begin{cases} 20 \text{ if } x \in [0,1) \\ \frac{1}{x^4} \text{ if } x \in [1,\infty). \end{cases}$$

First, it is clear to see that f and h are occasionally weakly h-biased of type (A) and g and k are occasionally weakly k-biased of type (A). Take $a = \frac{1}{7}$, $b = \frac{1}{9}$ and $c = \frac{1}{6}$, we get

1. *for x, $y \in [0,1)$, we have $fx = \frac{9}{10}$, $gy = \frac{4}{5}$, $hx = 10$, $ky = 20$ and*

$$\begin{aligned} \frac{1}{100} &\le \frac{1}{7}\max\left\{\frac{8281}{100},\frac{9216}{25},100\right\}+\frac{1}{9}\max\left\{\frac{2093}{25},\frac{9168}{25}\right\}+\frac{4393}{150} \\ &= \frac{128831}{1050}, \end{aligned}$$

2. *for* x, $y \in [1, \infty)$, *we have* $fx = 1 = gy$, $hx = \frac{1}{x^3}$, $ky = \frac{1}{y^4}$ *and*

$$\begin{aligned} 0 \quad \leq \quad & \frac{1}{7}\max\left\{\left(1-\frac{1}{x^3}\right)^2, \left(1-\frac{1}{y^4}\right)^2, \left(\frac{1}{x^3}-\frac{1}{y^4}\right)^2\right\} \\ & +\frac{1}{9}\max\left\{\left(1-\frac{1}{x^3}\right)^2, \left(1-\frac{1}{y^4}\right)^2\right\} \\ & +\frac{1}{6}\left(1-\frac{1}{x^3}\right)\left(1-\frac{1}{y^4}\right), \end{aligned}$$

3. *for* $x \in [0, 1)$, $y \in [1, \infty)$, *we have* $fx = \frac{9}{10}$, $gy = 1$, $hx = 10$, $ky = \frac{1}{y^4}$ *and*

$$\begin{aligned} \frac{1}{100} \quad \leq \quad & \frac{1}{7}\max\left\{\frac{8281}{100}, \left(1-\frac{1}{y^4}\right)^2, \left(10-\frac{1}{y^4}\right)^2\right\} \\ & +\frac{1}{9}\max\left\{\frac{819}{10}, \left|\left(\frac{9}{10}-\frac{1}{y^4}\right)\left(1-\frac{1}{y^4}\right)\right|\right\} \\ & +\frac{3}{2}\left|\frac{9}{10}-\frac{1}{y^4}\right|, \end{aligned}$$

4. *finally, for* $x \in [1, \infty)$, $y \in [0, 1)$, *we have* $fx = 1$, $gy = \frac{4}{5}$, $hx = \frac{1}{x^3}$, $ky = 20$ *and*

$$\begin{aligned} \frac{1}{25} \quad \leq \quad & \frac{1}{7}\max\left\{\left(1-\frac{1}{x^3}\right)^2, \frac{9216}{25}, \left(20-\frac{1}{x^3}\right)^2\right\} \\ & +\frac{1}{9}\max\left\{\left|\left(1-\frac{1}{x^3}\right)\left(\frac{4}{5}-\frac{1}{x^3}\right)\right|, \frac{1824}{5}\right\} \\ & +\frac{19}{6}\left|\frac{4}{5}-\frac{1}{x^3}\right|, \end{aligned}$$

so, all hypotheses of the above theorem are satisfied and 1 *is the unique common fixed point of mappings* f, g, h *and* k.

Now, we will extend constants a, b and c of the above theorem to functions.

Theorem 3. *Let* (f, h) *and* (g, k) *be occasionally weakly* h*-biased and* k*-biased of type* (A) *self-mappings on a metric space* $(\mathcal{X}, d)$ *satisfying the inequality*

$$\begin{aligned} d^2(fx, gy) \quad \leq \quad & a(d(hx, ky))\max\{d^2(fx, hx), d^2(gy, ky), d^2(hx, ky)\} \\ & +b(d(hx, ky))\max\{d(fx, hx)d(hx, gy), d(fx, ky)d(gy, ky)\} \\ & +c(d(hx, ky))d(hx, gy)d(ky, fx) \end{aligned} \qquad (2)$$

for all x, $y \in \mathcal{X}$, *where* a, b, $c : [0, \infty) \to [0, 1)$ *are non-decreasing functions which satisfying the following condition*

$$4a(t) + 2b(t) + c(t) < 1 \ \forall t > 0.$$

Then all the four mappings f, g, h *and* k *have a unique common fixed point.*

Proof. By hypothesis, there are two points u and v in $\mathcal{X}$ such that $fu = hu$ implies $d(hhu, fu) \leq d(fhu, hu)$ and $gv = kv$ implies $d(kkv, gv) \leq d(gkv, kv)$.

First, we are going to prove that $fu = gv$. Suppose that $fu \neq gv$, from inequality (2) we have

$$\begin{aligned} d^2(fu, gv) &\leq a(d(hu, kv)) \max\{d^2(fu, hu), d^2(gv, kv), d^2(hu, kv)\} \\ &\quad + b(d(hu, kv)) \max\{d(fu, hu)d(hu, gv), d(fu, kv)d(gv, kv)\} \\ &\quad + c(d(hu, kv))d(hu, gv)d(kv, fu) \\ &= (a(d(fu, gv)) + c(d(fu, gv)))d^2(fu, gv) \\ &< d^2(fu, gv) \end{aligned}$$

which is a contradiction, thus $fu = gv$.

Now, we assert that $ffu = fu$. If not, then the use of condition (2) gives

$$\begin{aligned} d^2(ffu, gv) &\leq a(d(hfu, kv)) \max\{d^2(ffu, hfu), d^2(gv, kv), d^2(hfu, kv)\} \\ &\quad + b(d(hfu, kv)) \max\{d(ffu, hfu)d(hfu, gv), \\ &\quad d(ffu, kv)d(gv, kv)\} \\ &\quad + c(d(hfu, kv))d(hfu, gv)d(kv, ffu) \\ &= a(d(hfu, fu)) \max\{d^2(ffu, hfu), 0, d^2(hfu, fu)\} \\ &\quad + b(d(hfu, fu))d(ffu, hfu)d(hfu, fu) \\ &\quad + c(d(hfu, fu))d(hfu, fu)d(fu, ffu). \end{aligned}$$

Since the pair (f, h) is occasionally weakly h-biased of type (A) we have $d(hfu, fu) = d(hhu, fu) \leq d(fhu, hu) = d(ffu, fu)$, and as a, b and c are non-decreasing, using the triangle inequality, we get

$$\begin{aligned} d^2(ffu, fu) &\leq a(d(hfu, fu)) \max\{(d(ffu, fu) + d(fu, hfu))^2, 0, d^2(hfu, fu)\} \\ &\quad + b(d(hfu, fu))(d(ffu, fu) + d(fu, hfu))d(hfu, fu) \\ &\quad + c(d(hfu, fu))d(hfu, fu)d(fu, ffu) \\ &\leq a(d(ffu, fu)) \max\{4d^2(ffu, fu), 0, d^2(ffu, fu)\} \\ &\quad + 2b(d(ffu, fu))d^2(ffu, fu) \\ &\quad + c(d(ffu, fu))d^2(fu, ffu) \\ &= (4a(d(ffu, fu)) + 2b(d(ffu, fu)) + c(d(ffu, fu)))d^2(fu, ffu) \\ &< d^2(fu, ffu), \end{aligned}$$

which is a contradiction, therefore $ffu = fu$ and so $hfu = fu$.

Now, suppose that $ggv \neq gv$. Using inequality (2) we obtain

$$\begin{aligned} d^2(fu, ggv) &\leq a(d(hu, kgv)) \max\{d^2(fu, hu), d^2(ggv, kgv), d^2(hu, kgv)\} \\ &\quad + b(d(hu, kgv)) \max\{d(fu, hu)d(hu, ggv), \\ &\quad d(fu, kgv)d(ggv, kgv)\} \\ &\quad + c(d(hu, kgv))d(hu, ggv)d(kgv, fu) \\ &= a(d(gv, kgv)) \max\{0, d^2(ggv, kgv), d^2(gv, kgv)\} \\ &\quad + b(d(gv, kgv))d(gv, kgv)d(ggv, kgv) \\ &\quad + c(d(gv, kgv))d(gv, ggv)d(kgv, gv). \end{aligned}$$

As the pair (g, k) is occasionally weakly k-biased of type (A), we have $d(kgv, gv) = d(kkv, gv) \leq d(gkv, kv) = d(ggv, gv)$, and since functions a, b and c are non-decreasing, again using the triangle inequality we get

$$\begin{aligned} d^2(gv, ggv) &\leq a(d(gv, kgv)) \max\{0, (d(ggv, gv) + d(gv, kgv))^2, d^2(gv, kgv)\} \\ &\quad + b(d(gv, kgv))d(gv, kgv)(d(ggv, gv) + d(gv, kgv)) \\ &\quad + c(d(gv, kgv))d(gv, ggv)d(kgv, gv) \\ &\leq a(d(gv, ggv)) \max\{0, 4d^2(ggv, gv), d^2(gv, ggv)\} \\ &\quad + 2b(d(gv, ggv))d^2(gv, ggv) \\ &\quad + c(d(gv, ggv))d^2(gv, ggv) \\ &= (4a(d(gv, ggv)) + 2b(d(gv, ggv)) + c(d(gv, ggv)))d^2(gv, ggv) \\ &< d^2(gv, ggv). \end{aligned}$$

This contradiction implies that $ggv = gv$ and so $kgv = gv$; i.e., $gfu = fu$ and $kfu = fu$. Put $fu = hu = gv = kv = w$, therefore w is a common fixed point of mappings f, g, h and k.

Finally, let w and t be two distinct common fixed points of mappings f, g, h and k. Then, $w = fw = gw = hw = kw$ and $t = ft = gt = ht = kt$. From (2) we have

$$\begin{aligned} d^2(ft, gw) &\leq a(d(ht, kw)) \max\{d^2(ft, ht), d^2(gw, kw), d^2(ht, kw)\} \\ &\quad + b(d(ht, kw)) \max\{d(ft, ht)d(ht, gw), d(ft, kw)d(gw, kw)\} \\ &\quad + c(d(ht, kw))d(ht, gw)d(kw, ft); \end{aligned}$$

i.e.,

$$\begin{aligned} d^2(t, w) &\leq (a(d(t, w)) + c(d(t, w)))d^2(t, w) \\ &< d^2(t, w), \end{aligned}$$

which is a contradiction, thus $t = w$. □

Again, we give an example which illustrates our above theorem.

Example 3. *Let* $\mathcal{X} = [0, \frac{\pi}{2}]$ *with the metric* $d(x, y) = |x - y|$. *Define*

$$fx = \begin{cases} \frac{\pi}{4} \text{ if } x \in [0, \frac{\pi}{4}] \\ \frac{2\pi}{9} \text{ if } x \in (\frac{\pi}{4}, \frac{\pi}{2}], \end{cases} \quad gx = \begin{cases} \frac{\pi}{4} \text{ if } x \in [0, \frac{\pi}{4}] \\ \frac{\pi}{5} \text{ if } x \in (\frac{\pi}{4}, \frac{\pi}{2}], \end{cases}$$

$$hx = \begin{cases} x \text{ if } x \in [0, \frac{\pi}{4}] \\ \frac{\pi}{2} \text{ if } x \in (\frac{\pi}{4}, \frac{\pi}{2}], \end{cases} \quad kx = \begin{cases} x \text{ if } x \in [0, \frac{\pi}{4}] \\ 0 \text{ if } x \in (\frac{\pi}{4}, \frac{\pi}{2}]. \end{cases}$$

First, it is clear to see that f *and* h *are occasionally weakly* h*-biased of type* (A) *and* g *and* k *are occasionally weakly* k*-biased of type* (A). *Take* $a(t) = \frac{\sin t}{7}$, $b(t) = \frac{1}{9}$ *and* $c(t) = \frac{\sin t}{6}$, *we get*

1. *for* $x, y \in [0, \frac{\pi}{4}]$, *we have* $fx = \frac{\pi}{4}$, $gy = \frac{\pi}{4}$, $hx = x$, $ky = y$ *and*

$$\begin{aligned} 0 \quad \leq \quad & \frac{\sin|x-y|}{7} \max\left\{\left(\frac{\pi}{4} - x\right)^2, \left(\frac{\pi}{4} - y\right)^2, (x-y)^2\right\} \\ & + \frac{1}{9} \max\left\{\left(\frac{\pi}{4} - x\right)^2, \left(\frac{\pi}{4} - y\right)^2\right\} \\ & + \frac{\sin|x-y|}{6} \left(\frac{\pi}{4} - x\right)\left(\frac{\pi}{4} - y\right), \end{aligned}$$

2. *for* $x, y \in (\frac{\pi}{4}, \frac{\pi}{2}]$, *we have* $fx = \frac{2\pi}{9}$, $gy = \frac{\pi}{5}$, $hx = \frac{\pi}{2}$, $ky = 0$ *and*

$$\frac{\pi^2}{2025} \leq \frac{1}{7} \max\left\{\frac{25\pi^2}{324}, \frac{\pi^2}{25}, \frac{\pi^2}{4}\right\} + \frac{1}{9} \max\left\{\frac{\pi^2}{12}, \frac{2\pi^2}{45}\right\} + \frac{\pi^2}{90},$$

3. *for* $x \in [0, \frac{\pi}{4}]$, $y \in (\frac{\pi}{4}, \frac{\pi}{2}]$, *we have* $fx = \frac{\pi}{4}$, $gy = \frac{\pi}{5}$, $hx = x$, $ky = 0$ *and*

$$\begin{aligned} \frac{\pi^2}{400} \quad \leq \quad & \frac{\sin x}{7} \max\left\{\left(\frac{\pi}{4} - x\right)^2, \frac{\pi^2}{25}, x^2\right\} \\ & + \frac{1}{9} \max\left\{\left|\left(\frac{\pi}{4} - x\right)\left(\frac{\pi}{5} - x\right)\right|, \frac{\pi^2}{20}\right\} \\ & + \frac{\pi \sin x}{24} \left|\frac{\pi}{5} - x\right|, \end{aligned}$$

4. *finally, for* $x \in (\frac{\pi}{4}, \frac{\pi}{2}]$, $y \in [0, \frac{\pi}{4}]$, *we have* $fx = \frac{2\pi}{9}$, $gy = \frac{\pi}{4}$, $hx = \frac{\pi}{2}$, $ky = y$ *and*

$$\begin{aligned} \frac{\pi^2}{1296} \quad \leq \quad & \frac{\sin(\frac{\pi}{2} - y)}{7} \max\left\{\frac{25\pi^2}{324}, \left(\frac{\pi}{4} - y\right)^2, \left(\frac{\pi}{2} - y\right)^2\right\} \\ & + \frac{1}{9} \max\left\{\frac{5\pi^2}{72}, \left|\left(\frac{\pi}{4} - y\right)\left(\frac{2\pi}{9} - y\right)\right|\right\} \\ & + \frac{\pi \sin(\frac{\pi}{2} - y)}{24} \left|\frac{2\pi}{9} - y\right|, \end{aligned}$$

so, all hypotheses of the above theorem are satisfied and $\frac{\pi}{4}$ *is the unique common fixed point of mappings* f, g, h *and* k.

2.2 A unique common fixed point theorem for a sequence of mappings

In this part, we will consider many pairs of mappings simultaneously for common fixed point. In fact, a whole sequence of mappings can be considered for this purpose.

Theorem 4. *Let h, k and $\{f_n\}_{n=1,2,\ldots}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the following inequality*

$$\begin{aligned} d^2(f_nx, f_{n+1}y) \leq\ & a\max\{d^2(f_nx, hx), d^2(f_{n+1}y, ky), d^2(hx, ky)\} \\ & +b\max\{d(f_nx, hx)d(hx, f_{n+1}y), d(f_nx, ky)d(f_{n+1}y, ky)\} \\ & +cd(hx, f_{n+1}y)d(ky, f_nx) \end{aligned} \tag{3}$$

for all x, $y \in \mathcal{X}$, where a, b, $c \geq 0$ such that $4a + 2b + c < 1$. If the pair (f_n, h) as well as (f_{n+1}, k) is occasionally weakly h-biased of type (A) and occasionally weakly k-biased of type (A), respectively, then h, k and $\{f_n\}_{n=1,2,\ldots}$ have a unique common fixed point.

Proof. Putting $n = 1$, we get that mappings f_1, f_2, h and k satisfy the hypotheses of Theorem 2, then, they have a unique common fixed point w.

Now, letting $n = 2$, we get that mappings f_2, f_3, h and k have a unique common fixed point t. Suppose that $t \neq w$, the use of inequality 3 gives

$$\begin{aligned} d^2(f_2w, f_3t) \leq\ & a\max\{d^2(f_2w, hw), d^2(f_3t, kt), d^2(hw, kt)\} \\ & +b\max\{d(f_2w, hw)d(hw, f_3t), d(f_2w, kt)d(f_3t, kt)\} \\ & +cd(hw, f_3t)d(kt, f_2w); \end{aligned}$$

i.e.,

$$\begin{aligned} d^2(w, t) &\leq (a + c)d^2(w, t) \\ &< d^2(w, t), \end{aligned}$$

which is a contradiction, hence $t = w$.

Continuing in this way, we certify that w is the required point; i.e., w is the unique common fixed point of h, k and $\{f_n\}_{n=1,2,\ldots}$. □

Theorem 5. *Let h, k and $\{f_n\}_{n=1,2,\ldots}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the following inequality*

$$\begin{aligned} d^2(f_nx, f_{n+1}y) \leq\ & a(d(hx, ky))\max\{d^2(f_nx, hx), d^2(f_{n+1}y, ky), d^2(hx, ky)\} \\ & +b(d(hx, ky))\max\{d(f_nx, hx)d(hx, f_{n+1}y), \\ & d(f_nx, ky)d(f_{n+1}y, ky)\} \\ & +c(d(hx, ky))d(hx, f_{n+1}y)d(ky, f_nx) \end{aligned} \tag{4}$$

for all x, $y \in \mathcal{X}$, where a, b, $c : [0, \infty) \to [0, 1)$ are non-decreasing functions and satisfying the next condition

$$4a(t) + 2b(t) + c(t) < 1\ \forall t > 0.$$

If f_n and h as well as f_{n+1} and k are occasionally weakly h-biased of type (A) and occasionally weakly k-biased of type (A), respectively, then h, k and $\{f_n\}_{n=1,2,\ldots}$ have a unique common fixed point.

Proof. As in the proof of Theorem 4, putting $n = 1$, we get that mappings f_1, f_2, h and k satisfy the hypotheses of Theorem 3, then they have a unique common fixed point w.

Now, letting $n = 2$, we get that mappings f_2, f_3, h and k have a unique common fixed point t. Assume that $t \neq w$, using inequality 4 we get

$$\begin{aligned} d^2(f_2w, f_3t) \leq\ & a(d(hw, kt)) \max\{d^2(f_2w, hw), d^2(f_3t, kt), d^2(hw, kt)\} \\ & +b(d(hw, kt)) \max\{d(f_2w, hw)d(hw, f_3t), \\ & d(f_2w, kt)d(f_3t, kt)\} \\ & +c(d(hw, kt))d(hw, f_3t)d(kt, f_2w); \end{aligned}$$

i.e.,

$$\begin{aligned} d^2(w, t) & \leq [a(d(w, t)) + c(d(w, t))]d^2(w, t) \\ & < d^2(w, t), \end{aligned}$$

which is a contradiction, hence $t = w$.

Continuing in this manner, we clearly see that w is the unique common fixed point of mappings h, k and $\{f_n\}_{n=1,2,\ldots}$. □

3 Conclusion

In this work, we could extend and improve the main result of [12] by removing some conditions, extending the constants to functions, and increasing the number of mappings using our new concept of occasionally weakly biased of type (A) mappings. Moreover, we could provide some examples to illustrate the usability of the obtained results.

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References

[1] Agarwal, R.P., Bisht, R.K. and Shahzad, N., *A comparison of various non-commuting conditions in metric fixed point theory and their applications*, Fixed Point Theory Appl. **48** (2014), 1-33.

[2] Al-Thagafi, M.A. and Shahzad, N., *Generalized I-nonexpansive selfmaps and invariant approximations*, Acta Math. Sin. (Engl. Ser.) **24** (2008), no. 5, 867-876.

[3] Banarjee, A. and Singh Thakur, B., *A Note on a theorem of Tas, Telci and Fisher*, Appl. Math. Mech. **19** (1998), no. 4, 333-334.

[4] Bouhadjera, H., *Fixed points for occasionally weakly biased of type* (A) *mappings*, Submitted in 2021.

[5] Bouhadjera, H. and Djoudi, A., *Fixed point for occasionally weakly biased maps*, Southeast Asian. Bull. Math. **36** (2012), no. 4, 489-500.

[6] Jungck, G., *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. **9** (1986), no. 4, 771-779.

[7] Jungck, G., *Common fixed points for noncontinuous nonself maps on non-metric spaces*, Far East J. Math. Sci. **4** (1996), no. 2, 199-215.

[8] Jungck, G., Murthy, P.P. and Cho, Y.J., *Compatible mappings of type* (A) *and common fixed points*, Math. Japon. **38** (1993), no. 2, 381-390.

[9] Jungck, G. and Pathak, H.K., *Fixed points via biased maps*, Proc. Amer. Math. Soc. **123** (1995), no. 7, 2049-2060.

[10] Jungck, G. and Rhoades, B.E., *Fixed point theorems for occasionally weakly compatible mappings*, Fixed Point Theory **7** (2006), no. 2, 287-296 (Fixed Point Theory **9** (2008), no. 1, 383-384 (erratum)).

[11] Pathak, H.K., Cho, Y.J. and Kang, S.M., *Common fixed points of biased maps of type* (A) *and applications*, Internat. J. Math. Math. Sci. **21** (1995), no. 4, 681-694.

[12] Phaneendra, T. and Swatmaram, T., *A generalization of a theorem of Banarjee and Thakur to two weakly compatible pairs of self-maps*, Bull. Math. Anal. Appl. **4** (2012), no. 4, 1-5.

[13] Popa, V., *On unique common fixed point for compatible mappings of type* (A), Demonstratio Math. **30** (2003), no. 4, 931-936.

[14] Sessa, S. *On a weak commutativity condition in fixed point considerations*, Publ. Inst. Math. (Beograd) (N.S.) **32** (1982), no. 46, 149-153.