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CATMULL-ROM SPLINE APPROACH AND THE ORDER OF CONVERGENCE OF GREEN'S FUNCTION METHOD FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

The purpose of this work is to investigate the convergence properties of Green's function method applied to boundary value problems for functional differential equations. Recently, involving Picard and Mann iterations, a Green's function technique was developed (in Int. J. Computer Math. 95, no. 10 (2018) 1937-1949) for third order functional differential equations, but without specifying the order of convergence of the proposed method. In order to improve this aspect, here we establish the maximal order of convergence of Green's function method applied to two-point boundary value problems associated to second and third order functional differential equations. In this context, by using suitable quadrature rule and appropriate spline interpolation procedure, the Picard iterations are approximated by a sequence of cubic splines on uniform mesh. Some numerical experiments are presented in order to test the theoretical results and to illustrate the accuracy of the method

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1 Introduction

The main task of this work is to construct an effective iterative algorithm within the framework of Green's function method such that the order of convergence to be maximal for second and third order two point boundary value problems associated to functional differential equations.

The study of functional differential equations is motivated by their applications in electrodynamics, astrophysics, quantum mechanics, cell growth, electrical

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networks, engineering (see [10] and [14]) and third order differential equations are important for various applications including the behavior of three-layer beams, draining flows over solid surfaces, electromagnetic waves, the transport of viscoelastic fluids and others (see [5] and [17]). An extensive study of functional differential equations is presented in [14] and the numerical methods developed for two-point boundary value problems associated to functional differential equations can be found in [7], [8], [18], [19], [20], [21] and references therein.

The existence of solutions for two point boundary value problems associated to third order differential equations was investigated in [2], [3], [13] and the Green function method for third order differential equations with various type of iterative approximations was developed in [1], [4], and [16].

The numerical solution of two-point boundary value problems for third-order functional differential equations is investigated in few works and we can mention the iterative schemes based on Green's function method with Picard and Mann's iterations developed in [15] and the novel iterative technique proposed in [11]. Unfortunately, the order of convergence of Green's function method is not specified in [15] and the corresponding order of convergence obtained in [11] is not the best possible. Therefore we try to respond to this question in the present work specifying that the order of convergence of Picard-Green's function method applied to the following two-point boundary value problem

$$\begin{cases} x'''(t) = f(t, x(t), x(\varphi(t))), \quad t \in [0,T] \\ x(0) = c, \quad x(T) = d, \quad x'(T) = r \end{cases} \tag{1}$$

with $\varphi : [0,T] \to [0,T]$, $0 \le \varphi(t) \le t$, $\forall t \in [0,T]$, and to related third order boundary value problems, is $O\left(h^3\right)$. On the other hand, we improve the order of convergence of Green's function method specified in [11] for a class of boundary value problems that includes (1). More precisely, the authors from [11] have obtained an order of convergence $O\left(h^2\right)$ by using the trapezoidal quadrature rule and piecewise linear interpolation, while here we prove that this order is better and the order $O\left(h^3\right)$, obtained by us, cannot be improved. In order to prove that the maximal order is $O\left(h^3\right)$, we use suitable quadrature rule and Catmull-Rom splines as interpolation procedure. The Catmull-Rom splines will be used both for second and third order boundary value problems. The paper is organized as follows: in Section 2 we present the convergence properties of the Catmull-Rom spline interpolation procedure, while Section 3 is devoted to the convergence analysis of the iterative method generated by the combination of Green's function technique with Catmull-Rom splines applied to second order two-point boundary value problems with deviating argument. Section 4 contains the main result regarding the order of convergence of Green's function method that involves the corrected trapezoidal rule and Catmull-Rom splines applied to the boundary value problem (1). Some numerical examples are presented in the last section in order to test the obtained theoretical results and to illustrate the accuracy of this method.

2 The Catmull-Rom spline interpolation

The Catmull-Rom splines were introduced in [9] for parametric curves regarding shape preserving properties in the context of computer aided geometric design, but can be used for uniform approximation, too, as will be viewed in the following. A Catmull-Rom cubic spline is based on interpolating piecewise Hermite cubic polynomials and on a partition Δ of an interval $[a, b]$,

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

has in each subinterval $[x_{i-1}, x_i]$, $i = \overline{1, n}$, the following expression:

$$S(x) = (1-t)^2 (2t+1) y_{i-1} + t^2 (3-2t) y_i + h_i t (1-t)^2 m_{i-1} - h_i t^2 (1-t) m_i. \tag{2}$$

Here $h_i = x_i - x_{i-1}$, $i = \overline{1, n}$, $y_i = S(x_i)$, $i = \overline{0, n}$ are given, $t = \frac{x - x_{i-1}}{h_i}$, and the derivatives on the knots $m_i = S'(x_i)$, $i = \overline{0, n}$ are obtained by using formula

$$m_i = \frac{(x_{i+1} - x_i)^2 (y_i - y_{i-1}) + (x_i - x_{i-1})^2 (y_{i+1} - y_i)}{(x_{i+1} - x_i)(x_i - x_{i-1})(x_{i+1} - x_{i-1})}, \quad i = \overline{1, n-1}. \tag{3}$$

We will use the Catmull-Rom splines as interpolation procedure on uniform partition, that is $h_i = h = \frac{b-a}{n}$, $\forall i = \overline{1, n}$, and in this case formula (3) becomes $m_i = \frac{y_{i+1} - y_{i-1}}{2h}$, $i = \overline{1, n-1}$, being completed with special treatment for m_0 and m_n at endpoints. More precisely, if S interpolates a function f on the knots x_i, $i = \overline{0, n}$, that is $S(x_i) = y_i = f(x_i)$, $i = \overline{0, n}$, then if $f'(a)$ and $f'(b)$ are known we can take $m_0 = f'(a)$, $m_n = f'(b)$, otherwise we can propose

$$m_0 = \frac{-3y_0 + 4y_1 - y_2}{2h}, \quad m_n = \frac{y_{n-2} - 4y_{n-1} + 3y_n}{2h} \tag{4}$$

inspired from numerical differentiation generated by quadratic Lagrange interpolation.

In order to estimate the interpolation error, if $f \in C^3[a, b]$ we see that Taylor expansion on $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ roud about x_i gives us $\left| f'(x_i) - \frac{y_{i+1} - y_{i-1}}{2h} \right| \le \frac{h^2}{6} \cdot \|f'''\|_\infty$, for $i = \overline{1, n-1}$, where $\|f'''\|_\infty = \max\{|f'''(x)| : x \in [a, b]\}$. On the other hand, the usual error estimate of Lagrange interpolation provides

$$\left| f'(x_0) - \frac{-3y_0 + 4y_1 - y_2}{2h} \right| \le \frac{h^2 \|f'''\|_\infty}{3}, \quad \left| f'(x_n) - \frac{y_{n-2} - 4y_{n-1} + 3y_n}{2h} \right|$$

$\le \frac{h^2 \|f'''\|_\infty}{3}$.

Considering $H(f)$ be the piecewise two-point cubic Hermite polynomial interpolation on each interval $[x_{i-1}, x_i]$, $i = \overline{1, n}$, if $f'(a)$ and $f'(b)$ are known then

$$\max_{x \in [a,b]} |S(x) - H(f)(x)| \le \max_{x \in [x_{i-1}, x_i], i = \overline{1,n}} \Big(\frac{(x_i - x)^2 (x - x_{i-1})}{h_i^2} |m_{i-1} - f'(x_{i-1})| +$$

$$+\frac{(x-x_{i-1})^2(x_i-x)}{h_i^2}\left|m_i-f'(x_i)\right|)\leq\frac{4h^3\left\|f'''\right\|_\infty}{81}$$

and if at least one of $f'(a)$ and $f'(b)$ are unknown it obtains $\max_{x\in[a,b]}|S(x)-H(f)(x)|$ $\leq\frac{2h^3}{27}\cdot\|f'''\|_\infty$. Since in the case $f\in C^4[a,b]$, the error estimate for piecewise Hermite cubic polynomial interpolation is $\max_{x\in[a,b]}|H(f)(x)-f(x)|\leq\frac{h^4}{384}\cdot\left\|f^{(4)}\right\|_\infty$, we can consider the Catmull-Rom spline operator $CR:C^4[a,b]\to C^1[a,b]$, given by $CR(f)=S(f)$ and obtain the following result.

Lemma 1. *If $f\in C^4[a,b]$, then the error estimate of the Catmull-Rom spline operator is*

$$\|CR(f)-f\|_\infty\leq\frac{4h^3\cdot\|f'''\|_\infty}{81}+\frac{h^4}{384}\cdot\left\|f^{(4)}\right\|_\infty=O\left(h^3\right)\tag{5}$$

if $f'(a)$ and $f'(b)$ are known, and

$$\|CR(f)-f\|_\infty\leq\frac{2h^3\cdot\|f'''\|_\infty}{27}+\frac{h^4}{384}\cdot\left\|f^{(4)}\right\|_\infty=O\left(h^3\right)\tag{6}$$

otherwise.

In that follows, we will use the error estimate (6) in the convergence analysis of Green's function method applied to second and third order two-point boundary value problems with deviating argument.

3 Second order two-point boundary value problem with deviating argument

Under the construction of Green's function method, the second order two-point boundary value problem with deviating argument

$$\begin{cases} x''(t)=f(t,x(t),x(\varphi(t))), & t\in[0,T]\\ x(0)=c, \quad x(T)=d \end{cases}\tag{7}$$

is equivalent with the following Fredholm integral equation $x(t)=A(x)(t)$, $t\in[0,T]$, where the integral operator $A:C[0,T]\to C[0,T]$ is given by

$$A(x)(t)=\frac{(T-t)c}{T}+\frac{td}{T}-\int_0^T G(t,s)\cdot f(s,x(s),x(\varphi(s)))\,ds,\ t\in[0,T]\tag{8}$$

and $G:[0,T]\times[0,T]\to\mathbb{R}$, is the corresponding Green function. In this context, (8) generates a fixed point problem and the sequence of Picard iterations is

$$x_k(t)=g(t)-\int_0^T G(t,s)\cdot f(s,x_{k-1}(s),x_{k-1}(\varphi(s)))\,ds,\ t\in[0,T],\ k\in\mathbb{N}^*\tag{9}$$

with $g(t) = \frac{(T-t)c}{T} + \frac{td}{T}$ and $x_0(t) = g(t)$, $\forall t \in [0,T]$.

Since

$$G(t,s) = \begin{cases} \frac{s(T-t)}{T}, & s \le t \\ \frac{t(T-s)}{T}, & t \le s \end{cases}, \quad \frac{\partial G}{\partial t}(t,s) = \begin{cases} \frac{-s}{T}, & s \le t \\ \frac{T-s}{T}, & t \le s \end{cases}$$

we see that $\max_{t\in[0,T]} \int_0^T |G(t,s)|\, ds = \frac{T^2}{8}$, $\|G\|_\infty = \max_{(t,s)\in[0,T]^2} |G(t,s)| = \frac{T}{4}$, $\left\|\frac{\partial G}{\partial t}\right\|_\infty = 1$, $\left\|\frac{\partial G}{\partial s}\right\|_\infty = 1$, and $G(t,0) = G(t,T) = G(0,s) = G(T,s) = 0$. By applying the Banach's fixed point principle to the integral operator (8) it obtains the following result.

Theorem 1. *If $\varphi \in C[0,T]$, $f \in C([0,T]\times\mathbb{R}\times\mathbb{R})$ is Lipschitzian with respect to the second and to the third argument with corresponding Lipschitz constants α and β, and if $\frac{T^2}{8}(\alpha+\beta) < 1$, then the boundary value problem (7) has unique solution $x^* \in C^2[0,T]$ and the sequence of Picard iterations uniformly converges to x^* on $[0,T]$. Moreover this sequence and the sequences of their first two derivatives are uniformly bounded, and the following estimates hold:*

$$|x^*(t) - x_k(t)| \le \frac{\left(\frac{T^2}{8}(\alpha+\beta)\right)^k \frac{T^2}{8} M_0}{1 - \frac{T^2}{8}(\alpha+\beta)}, \quad \forall t \in [0,T],\ k \in \mathbb{N}^* \tag{10}$$

$$|x^*(t) - x_k(t)| \le \frac{\frac{T^2}{8}(\alpha+\beta)}{1 - \frac{T^2}{8}(\alpha+\beta)} |x_k(t) - x_{k-1}(t)|, \quad \forall t \in [0,T],\ k \in \mathbb{N}^* \tag{11}$$

where $x_0 = g$ and $M_0 \ge 0$ is such that $|f(t, g(t), g(\varphi(t)))| \le M_0$ for all $t \in [0,T]$.

Proof. The Banach's fixed point principle ensures the existence and uniqueness of the solution $x^* \in C[0,T]$ of (7) and the validity of the estimates (10) and (11). After two times differentiation in the equality $x^*(t) = A(x^*)(t)$ we have $x^* \in C^2[0,T]$. Denoting $\lambda = \frac{T^2}{8}(\alpha+\beta)$, by induction we get

$$|x_k(t) - x_{k-1}(t)| \le \int_0^T (\alpha+\beta)|G(t,s)| \cdot \|x_{k-1} - x_{k-2}\|_\infty\, ds \le \lambda^{k-1} \cdot \|x_1 - x_0\|_\infty$$

and

$$|x_k(t) - x_0(t)| \le \sum_{j=1}^{k} |x_j(t) - x_{j-1}(t)| \le \sum_{j=0}^{k-1} \lambda^j \cdot \|x_1 - x_0\|_\infty \le \frac{\|x_1 - x_0\|_\infty}{1-\lambda}.$$

Now, by considering $M_g = \max_{t\in[0,T]} |x_0(t)| = \max_{t\in[0,T]} |g(t)|$, it obtains

$$|x_k(t)| \le |x_k(t) - x_0(t)| + |x_0(t)| \le \frac{T^2 M_0}{8(1-\lambda)} + M_g = R,\ \forall t \in [0,T],\ k \in \mathbb{N}^*$$

that is the uniform boundedness of the sequence $(x_k)_{k\in\mathbb{N}}$. Denoting $F_k(t) = f(t, x_k(t), x_k(\varphi(t)))$ for $t \in [0,T]$, $k \in \mathbb{N}$, we infer that $(t, x_k(t), x_k(\varphi(t))) \in [0,T] \times [-R,R] \times [-R,R]$ for all $t \in [0,T]$, $k \in \mathbb{N}$, and based on the continuity of f, we deduce its boundedness on the compact set $[0,T] \times [-R,R] \times [-R,R]$, that is there exists $M \geq 0$ such that $|F_k(t)| \leq M$, $\forall t \in [0,T]$, $k \in \mathbb{N}$. Consequently, the sequence $(F_k)_{k\in\mathbb{N}}$ is uniformly bounded too. Moreover, since $x'_k(t) = g'(t) - \int_0^T \frac{\partial G}{\partial t}(t,s) F_{k-1}(s)\,ds$ and $x''_k(t) = F_{k-1}(t)$, we have $|x'_k(t)| \leq \frac{|d-c|}{T} + MT$ and $|x''_k(t)| \leq M$, $\forall t \in [0,T]$, $k \in \mathbb{N}^*$ and the sequences $(x'_k)_{k\in\mathbb{N}^*}$ and $(x''_k)_{k\in\mathbb{N}^*}$ are uniformly bounded. □

Under the conditions of Theorem 1, if $f \in C^2\left([0,T] \times \mathbb{R}^2\right)$, by the continuity of the partial derivatives of f on the compact set $[0,T] \times [-R,R] \times [-R,R]$ we obtain the uniform boundedness of the sequences $(F'_k)_{k\in\mathbb{N}}$ and $(F''_k)_{k\in\mathbb{N}}$, that is $|F'_k(s)| \leq M'$ and $|F''_k(s)| \leq M''$, $\forall s \in [0,T]$, $k \in \mathbb{N}^*$ for some $M', M'' \geq 0$. From these, we get $x'''_k(t) = F'_{k-1}(t)$, $x^{(4)}_k(t) = F''_{k-1}(t)$, having $|x'''_k(t)| \leq M'$ and $\left|x^{(4)}_k(t)\right| \leq M''$ for all $t \in [0,T]$, $k \in \mathbb{N}^*$.

Since $\lim\limits_{k\to\infty} x_k(t) = x^*(t)$, $\forall t \in [0,T]$, we have to approximate the Picard iteration terms and for this purpose, we should to approximate the integrals in (9). This will be done by using the trapezoidal quadrature rule with the error estimate established in [6]:

$$\left|\int_a^b F(t)\,dt - \frac{(b-a)}{2n}\sum_{i=1}^n \left[F(t_{i-1}) + F(t_i)\right]\right| \leq \frac{L'(b-a)^3}{12n^2} \tag{12}$$

where $t_i = a + \frac{i(b-a)}{n}$, $i = \overline{0,n}$ are the knots of a uniform partition and $L' \geq 0$ is the Lipschitz constant of the derivative F', under the hypothesis $F \in C^1[a,b]$ with Lipschitzian F'. On the other hand, the presence of a deviating argument in (9) impose the use of an interpolation procedure at each iterative step, and this is the Catmull-Rom spline presented in (2)-(4) with the error estimate (6). Therefore, we consider a uniform partition of $[0,T]$ with the knots $t_i = \frac{iT}{n} = ih$, $i = \overline{0,n}$ and stepsize $h = \frac{T}{n}$, $n \in \mathbb{N}^*$. In this way we arrive to the iterative procedure:

$$x_k(t_0) = c, \quad x_k(t_n) = d,\ k \in \mathbb{N}, \quad x_0(t_i) = g(t_i),\ i = \overline{0,n}, \tag{13}$$

and

$$x_k(t_i) = g(t_i) + \frac{T}{2n}\sum_{j=1}^n [G(t_i,t_{j-1}) \cdot F_{k-1}(t_{j-1}) + G(t_i,t_j) \cdot F_{k-1}(t_j)] + R_{k,i} \tag{14}$$

for $i = \overline{1,n-1}$, $k \in \mathbb{N}^*$, with $|R_{k,i}| \leq \frac{L'Th^2}{12}$. Denoting $F_{k-1,i}(s) = G(t_i,s) \cdot F_{k-1}(s)$, the Lipschitz constant of $\frac{\partial}{\partial s}(G(t_i,s) \cdot F_{k-1}(s)) = F'_{k-1,i}(s)$, $i = \overline{0,n}$,

$k \in \mathbb{N}^*$, is $L' = 2M' + \frac{T}{4}M''$, based on the inequality

$$\begin{aligned} & \left|F'_{k-1,i}(s) - F'_{k-1,i}(s')\right| \\ \leq & \left|F'_{k-1}(s')\right| \left|G(t_i, s) - G(t_i, s')\right| + \left|G(t_i, s)\right| \left|F'_{k-1}(s) - F'_{k-1}(s')\right| \end{aligned}$$

$$+\left|\frac{\partial G(t_i, s)}{\partial s}\right| \left|F_{k-1}(s) - F_{k-1}(s')\right| \leq 2M' \left|s - s'\right| + \frac{T}{4}M'' \left|s - s'\right|, \ \forall s, s' \in [0, T].$$

For approximating the third argument in $F_{k-1}(s) = f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))$ we use the Catmull-Rom spline interpolation procedure (2)-(4) and it obtains the following iterative algorithm:

The first iterative step is $x_0(t) = g(t)$, $\forall t \in [0, T]$, and at endpoint we have

$$x_k(t_0) = c, \qquad x_k(t_n) = d, \quad \forall k \in \mathbb{N}^* \tag{15}$$

and denote $\overline{S_k}(t_0) := c$, $\overline{S_k}(t_n) := d$ for $k \in \mathbb{N}^*$. Taking $k = 1$ in (14) we get

$$x_1(t_i) = \overline{S_1}(t_i) + R_{1,i}, \quad \forall i = \overline{1, n-1} \tag{16}$$

with

$$\overline{S_1}(t_i) = g(t_i) + \frac{T}{2n}\sum_{j=1}^{n}[G(t_i, t_{j-1}) \cdot F_0(t_{j-1}) + G(t_i, t_j) \cdot F_0(t_j)], \ i = \overline{1, n-1}$$

and construct the Catmull-Rom spline interpolating the values $\overline{S_1}(t_i)$, $i = \overline{0, n}$, where $\overline{S_1}(t_0) = c$, $\overline{S_1}(t_n) = d$. By induction for $k \in \mathbb{N}^*$, $k \geq 2$ it obtains

$$x_k(t_i) = g(t_i) + \frac{T}{2n}\sum_{j=1}^{n}[G(t_i, t_{j-1}) \cdot F_{k-1}(t_{j-1}) + G(t_i, t_j) \cdot F_{k-1}(t_j)] + R_{k,i} =$$

$$= g(t_i) + \frac{T}{2n}\sum_{j=1}^{n}[G(t_i, t_{j-1}) \cdot f\left(t_{j-1}, \overline{S_{k-1}}(t_{j-1}), \overline{S_{k-1}}(\varphi(t_{j-1}))\right) +$$

$$+G(t_i, t_j) \cdot f\left(t_j, \overline{S_{k-1}}(t_j), \overline{S_{k-1}}(\varphi(t_j))\right)] + \overline{R_{k,i}} = \overline{S_k}(t_i) + \overline{R_{k,i}}, \quad i = \overline{1, n-1}, \tag{17}$$

where $\overline{S_{k-1}}$ is the Catmull-Rom spline interpolating the values $\overline{S_{k-1}}(t_i)$, $i = \overline{0, n}$, computed at the previous iterative step. This spline is described by the expression

$$\overline{S_{k-1}}(\tau) = (1-\tau)^2(2\tau + 1)\overline{S_{k-1}}(t_{i-1}) + \tau^2(3 - 2\tau)\overline{S_{k-1}}(t_i) +$$

$$+h\tau(1-\tau)^2\overline{m_{k-1}}(i-1) - h\tau^2(1-\tau)\overline{m_{k-1}}(i), \quad t \in [t_{i-1}, t_i], \ i = \overline{1, n} \tag{18}$$

under the notation $\tau = \frac{t - t_{i-1}}{h}$, where

$$\begin{aligned} \overline{m_{k-1}}(0) &= \frac{-3c + 4\overline{S_{k-1}}(t_1) - \overline{S_{k-1}}(t_2)}{2h}, \\ \overline{m_{k-1}}(n) &= \frac{\overline{S_{k-1}}(t_{n-2}) - 4\overline{S_{k-1}}(t_{n-1}) + 3d}{2h} \\ \overline{m_{k-1}}(i) &= \frac{\overline{S_{k-1}}(t_{i+1}) - \overline{S_{k-1}}(t_{i-1})}{2h}, \quad i = \overline{1, n-1}. \end{aligned} \tag{19}$$

The last iteration "k" is determined such that $\left|\overline{S_k}(t_i) - \overline{S_{k-1}}(t_i)\right| < \varepsilon$, $\forall i = \overline{1, n-1}$, for previously given $\varepsilon > 0$.

Concerning the convergence of the iterative algorithm (14)-(19) we obtain the following result.

Theorem 2. *Under of hypotheses of Theorem 1, if* $f \in C^2\left([0,T] \times \mathbb{R}^2\right)$ *and* $\frac{T^2}{4}\left(\alpha + \frac{47}{27}\beta\right) < 1$, *then the sequence of Catmull-Rom splines* $\left(\overline{S_k}\right)_{k\in\mathbb{N}^*}$ *approximates the solution of (7) and the error estimates in the discrete and continuous approximation are:*

$$\left|x^*(t_i) - \overline{S_k}(t_i)\right| \leq \frac{\left(\frac{T^2}{8}(\alpha+\beta)\right)^k \frac{T^2}{8} M_0}{1 - \frac{T^2}{8}(\alpha+\beta)} +$$

$$+\frac{L'Th^2}{12[1 - \frac{T^2}{4}\left(\alpha + \frac{47}{27}\beta\right)]} + \frac{\beta T^2\left(\frac{2M'}{27}h^3 + \frac{M''h^4}{384}\right)}{4[1 - \frac{T^2}{4}\left(\alpha + \frac{47}{27}\beta\right)]}, \quad i = \overline{1, n-1}, \ k \in \mathbb{N}^* \tag{20}$$

and

$$\left|x^*(t) - \overline{S_k}(t)\right| \leq \frac{\left(\frac{T^2}{8}(\alpha+\beta)\right)^k \frac{T^2}{8} M_0}{1 - \frac{T^2}{8}(\alpha+\beta)} + \frac{47L'Th^2}{324[1 - \frac{T^2}{4}\left(\alpha + \frac{47}{27}\beta\right)]} +$$

$$+\frac{47\beta T^2\left(\frac{2M'}{27}h^3 + \frac{M''h^4}{384}\right)}{108[1 - \frac{T^2}{4}\left(\alpha + \frac{47}{27}\beta\right)]} + \frac{2M'}{27}h^3 + \frac{M''h^4}{384} = \frac{\lambda^{k+1}M_0}{1-\lambda} + O\left(h^2\right) \tag{21}$$

for all $t \in [0,T]$, $k \in \mathbb{N}^*$, *respectively.*

Proof. Since $\left|x^*(t) - \overline{S_k}(t)\right| \leq |x^*(t) - x_k(t)| + \left|x_k(t) - \overline{S_k}(t)\right|$, we have to estimate $\left|x_k(t) - \overline{S_k}(t)\right|$. For this purpose, in the discrete case, by (17) we have

$$\left|x_k(t_i) - \overline{S_k}(t_i)\right| = \left|\overline{R_{k,i}}\right| \leq |R_{k,i}| + \frac{T^2}{4}\left(\alpha\left|\overline{R_{k-1}}\right| + \beta\left\|x_{k-1} - \overline{S_{k-1}}\right\|_\infty\right) \tag{22}$$

for all $i = \overline{1, n-1}$, where $\left|\overline{R_{k-1}}\right| = \max\{\left|\overline{R_{k-1,i}}\right| : i = \overline{1,n}\}$, and for estimating $\left\|x_{k-1} - \overline{S_{k-1}}\right\|_\infty$ we consider the Catmull-Rom spline $S_{k-1} = CR(x_{k-1})$ interpolating $x_{k-1}(t_i)$, $i = \overline{0,n}$ and the piecewise Hermite cubic polynomial H_{k-1} interpolating $x_{k-1}(t_i)$ and $x'_{k-1}(t_i)$, $i = \overline{0,n}$, and we use the scheme $\overline{S_{k-1}} \to S_{k-1} \to H_{k-1} \to x_{k-1}$ and Lemma 1 applied to the pair (H_{k-1}, S_{k-1}). By (6) we obtain the estimate

$$\|x_{k-1} - S_{k-1}\|_\infty \leq \|x_{k-1} - H_{k-1}\|_\infty + \|H_{k-1} - S_{k-1}\|_\infty \leq$$

$$\leq \frac{2h^3}{27} \cdot \left\|x'''_{k-1}\right\|_\infty + \frac{h^4}{384} \cdot \left\|x^{(4)}_{k-1}\right\| \leq \frac{2M'}{27}h^3 + \frac{M''h^4}{384} \tag{23}$$

and similarly to (18) we have

$$S_{k-1}(t) = (1-\tau)^2(2\tau+1)x_{k-1}(t_{i-1}) + \tau^2(3-2\tau)x_{k-1}(t_i) +$$

$$+h\tau(1-\tau)^2 m_{k-1}(i-1) - h\tau^2(1-\tau) m_{k-1}(i)$$

where $m_{k-1}(i)$, $i=\overline{0,n}$ are obtained analogous to (19). Consequently, it obtains

$$\left|S_{k-1}(t)-\overline{S_{k-1}}(t)\right| \le (1-\tau)^2(2\tau+1)\left|x_{k-1}(t_{i-1})-\overline{S_{k-1}}(t_{i-1})\right| + \tau^2(3-2\tau)\cdot$$

$$\cdot\left|x_{k-1}(t_i)-\overline{S_{k-1}}(t_i)\right| + \tau(1-\tau)^2\left|m_{k-1}(i-1)-\overline{m_{k-1}}(i-1)\right| + \tau^2(1-\tau)\cdot$$

$$\cdot\left|m_{k-1}(i)-\overline{m_{k-1}}(i)\right| \le \left|\overline{R_{k-1}}\right| + \frac{8h}{27}\max_{i=\overline{0,n}}\left|m_{k-1}(i)-\overline{m_{k-1}}(i)\right| \le \frac{47}{27}\left|\overline{R_{k-1}}\right|$$

and by (23) we get

$$\left\|x_{k-1}-\overline{S_{k-1}}\right\|_\infty \le \frac{47}{27}\left|\overline{R_{k-1}}\right| + \frac{2M'}{27}h^3 + \frac{M''h^4}{384} \tag{24}$$

remaining to estimate $\left|\overline{R_{k-1}}\right|$. By (22) in inductive manner it obtains, $\left|\overline{R_{1,i}}\right| = |R_{1,i}| \le \frac{L'Th^2}{12}$

$$\left|\overline{R_{2,i}}\right| \le \left[1+\frac{T^2}{4}\left(\alpha+\frac{47}{27}\beta\right)\right]\frac{L'Th^2}{12} + \frac{\beta T^2}{4}\left(\frac{2M'}{27}h^3+\frac{M''h^4}{384}\right),\ i=\overline{1,n-1}$$

and

$$\left|\overline{R_{k,i}}\right| \le \left(1+\omega+...+\omega^{k-1}\right)\frac{L'Th^2}{12} + \frac{\beta T^2}{4}\left(1+\omega+...+\omega^{k-2}\right)\left(\frac{2M'}{27}h^3+\right.$$

$$\left.+\frac{M''h^4}{384}\right) \le \frac{L'Th^2}{12(1-\omega)} + \frac{\beta T^2\left(\frac{2M'}{27}h^3+\frac{M''h^4}{384}\right)}{4(1-\omega)} = O\left(h^2\right),\ i=\overline{1,n-1} \tag{25}$$

for all $k\in\mathbb{N}^*$, where $\omega = \frac{T^2}{4}\left(\alpha+\frac{47}{27}\beta\right)$. Now, by using (24) and (25) we obtain (20) and (21). □

We see that $\left\|x_k-\overline{S_k}\right\|_\infty = O\left(h^2\right)$ and therefore, the order of convergence of Green's function method applied to (7) is $O\left(h^2\right)$. This order is maximal because $\frac{\partial G}{\partial t}$ and $\frac{\partial G}{\partial s}$ have discontinuity on the line "$t=s$". Since in (9) we have

$$x_k(t_i) = g(t_i) - \int_0^{t_i}\frac{s(T-t_i)}{T}F_{k-1}(s)\,ds - \int_{t_i}^{T}\frac{t_i(T-s)}{T}F_{k-1}(s)\,ds,\ i=\overline{0,n}$$

the quadrature rule (12) is applied separately on the intervals $[0,t_i]$ and $[t_i,T]$. The kernel functions in these integrals have Lipschitzian first order derivative and the second order derivative does not exists. Of course, by (21) we have $\lim_{k\to\infty,h\to 0}\left|x^*(t)-\overline{S_k}(t)\right| = 0$ for all $t\in[0,T]$, that is the convergence of the iterative method (13)-(19).

4 Third order two-point boundary value problems with deviating argument

4.1 Green's function iterative method

In this section we present the performances of the Green function method, that involves Catmull-Rom splines as interpolation procedure, when it is applied to two-point boundary value problems associated with third order differential equations with retarded argument. In this context we consider the boundary value problem (1) which is equivalent with the following Fredholm integral equation $x(t) = A(x)(t)$, $t \in [0,T]$. Here the integral operator $A : C[0,T] \to C[0,T]$ is given as

$$A(x)(t) = g(t) + \int_0^T G(t,s) \cdot f(s, x(s), x(\varphi(s)))\, ds,\ t \in [0,T] \qquad (26)$$

with the Green function G,

$$G(t,s) = \begin{cases} H(t,s) = \frac{s^2(T-t)^2}{2T^2},\ s \le t \\ K(t,s) = \frac{s^2(T-t)^2}{2T^2} - \frac{(s-t)^2}{2},\ t \le s \end{cases},$$

$$\frac{\partial G(t,s)}{\partial s} = \begin{cases} \frac{s(T-t)^2}{T^2},\ s \le t \\ \frac{s(T-t)^2}{T^2} - (s-t),\ t \le s \end{cases}$$

and $g(t) = \frac{(rT+c-d)t^2}{T^2} + \frac{(2(d-c)-rT)t}{T} + c$. After elementary calculus we deduce $G(t,0) = G(t,T) = G(0,s) = G(T,s) = 0$,

$$\|G\|_\infty = \max_{(t,s)\in[0,T]^2} |G(t,s)| = \frac{\left(5\sqrt{5}-11\right)T^2}{4} = CT^2,\ \max_{t\in[0,T]} \int_0^T |G(t,s)|\, ds = \frac{2T^3}{81} \qquad (27)$$

$\max_{t\in[0,T]} \int_0^T \left|\frac{\partial G}{\partial t}(t,s)\right| ds = \frac{T^2}{6}$, $\max_{t\in[0,T]} \int_0^T \left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| ds = \frac{2T}{3}$, $\left\|\frac{\partial G}{\partial s}\right\|_\infty = \frac{23T}{27}$, $\left\|\frac{\partial^2 G}{\partial t^2}\right\|_\infty = 1$, $\left\|\frac{\partial^2 G}{\partial s^2}\right\|_\infty = 1$, where we have denoted $C = \frac{5\sqrt{5}-11}{4}$ and observe $CT^2 \le \frac{T^2}{22}$. It is easy to see that $G \in C^1([0,T]\times[0,T])$, the second order partial derivatives are discontinuous on the line "$t = s$", and therefore third order partial derivatives of G does not exist. Moreover, we get $\frac{\partial G}{\partial t}(0,s) = s - \frac{s^2}{T}$, $\frac{\partial G}{\partial s}(0,s) = \frac{\partial G}{\partial s}(T,s) = 0$

$$\frac{\partial G}{\partial s}(t,T) = \frac{(T-t)^2}{T} - (T-t), \quad \frac{\partial G}{\partial s}(t,0) = 0 \qquad (28)$$

and $\frac{\partial G}{\partial t}(t,0) = \frac{\partial G}{\partial t}(t,T) = \frac{\partial G}{\partial t}(T,s) = 0$. By applying the Banach fixed point principle to the integral operator (26) it obtains the following result.

Theorem 3. *If $\varphi \in C[0,T]$, $0 \leq \varphi(t) \leq t$, $\forall t \in [0,T]$, $f \in C([0,T] \times \mathbb{R} \times \mathbb{R})$ is Lipschitzian with respect to the second and to the third argument with corresponding Lipschitz constants α and β, and if $\frac{2T^3}{81}(\alpha+\beta) < 1$, then the boundary value problem (1) has unique solution $x^* \in C^3[0,T]$. Moreover, the sequence of Picard iterations $(x_k)_{k\in\mathbb{N}}$, given by $x_{k+1} = A(x_k)$ with $x_0 = g$, converges to x^* on $C[0,T]$, that is $\lim_{k\to\infty} x_k(t) = x^*(t)$ (uniformly for $t \in [0,T]$). In the approximation of x^* by the sequence $(x_k)_{k\in\mathbb{N}}$, the error estimate is*

$$|x^*(t) - x_k(t)| \leq \frac{\left(\frac{2T^3}{81}(\alpha+\beta)\right)^k \frac{2T^3}{81} M_0}{1 - \frac{2T^3}{81}(\alpha+\beta)}, \quad \forall t \in [0,T],\ k \in \mathbb{N}^*, \tag{29}$$

where $x_0 = g$ and $M_0 \geq 0$ is such that $|f(t, g(t), g(\varphi(t)))| \leq M_0$ for all $t \in [0,T]$. In addition to this, the sequence of Picard iterations is uniformly bounded, and an "a posteriori" error estimate holds, too:

$$|x^*(t) - x_k(t)| \leq \frac{\frac{2T^3}{81}(\alpha+\beta)}{1 - \frac{2T^3}{81}(\alpha+\beta)} |x_k(t) - x_{k-1}(t)|, \quad \forall t \in [0,T],\ k \in \mathbb{N}^*. \tag{30}$$

Proof. The Banach fixed point principle applied to the integral operator (26) leads to the existence and uniqueness of the solution $x^* \in C[0,T]$ and based on $\|x_1 - x_0\| \leq \frac{2T^3}{81} M_0$, and (27) the error estimates (29) and (30) are obtained. After three times differentiation of the equality $x^* = A(x^*)$ we get $x^* \in C^3[0,T]$ and since the sequence of Picard iterations is written as

$$x_k(t) = g(t) + \int_0^T G(t,s) \cdot f(s, x_{k-1}(s), x_{k-1}(\varphi(s)))\, ds,\ t \in [0,T],\ k \in \mathbb{N}^* \tag{31}$$

it obtains $x_k \in C^3[0,T]$, $\forall k \in \mathbb{N}$ and $x_k'''(t) = f(t, x_{k-1}(t), x_{k-1}(\varphi(t)))$, $\forall t \in [0,T]$, $k \in \mathbb{N}^*$. Similarly to the proof of Theorem 1 we get $|x_k(t) - x_0(t)| \leq \frac{\|x_1 - x_0\|_\infty}{1 - \frac{2T^3}{81}(\alpha+\beta)}$ and

$$|x_k(t)| \leq \frac{2T^3 M_0}{81\left(1 - \frac{2T^3}{81}(\alpha+\beta)\right)} + M_g = R,\ \forall t \in [0,T],\ k \in \mathbb{N}^*$$

that is the uniform boundedness of the sequence $(x_k)_{k\in\mathbb{N}}$.
Consequently, $(t, x_k(t), x_k(\varphi(t))) \in [0,T] \times [-R,R] \times [-R,R]$ for all $t \in [0,T]$ and $k \in \mathbb{N}^*$, and therefore, with the notation $F_k(t) = f(t, x_k(t), x_k(\varphi(t)))$, it obtains $|F_k(t)| \leq M$, $\forall t \in [0,T]$, $k \in \mathbb{N}^*$ for some $M \geq 0$, that is the uniform boundedness of the sequence $(F_k)_{k\in\mathbb{N}}$. After three times differentiation in (31), together with $\max_{t\in[0,T]} \int_0^T \left|\frac{\partial G}{\partial t}(t,s)\right| ds = \frac{T^2}{6}$ and $\max_{t\in[0,T]} \int_0^T \left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| ds = \frac{2T}{3}$, we obtain $|x_k'(t)| \leq \frac{2|rT+c-d| + |2(d-c)-rT|}{T} + \frac{MT^2}{6}$, $|x_k''(t)| \leq \frac{2|rT+c-d|}{T^2} + \frac{2MT}{3}$ and $|x_k'''(t)| = |f(t, x_{k-1}(t), x_{k-1}(\varphi(t)))| \leq M$, $\forall t \in [0,T]$, $k \in \mathbb{N}^*$. So, the sequences $(x_k')_{k\in\mathbb{N}}$, $(x_k'')_{k\in\mathbb{N}}$, $(x_k''')_{k\in\mathbb{N}}$ are uniformly bounded, too. □

Under the conditions of Theorem 3, if $f \in C^3\left([0,T]\times\mathbb{R}^2\right)$, by the continuity of the partial derivatives of f on the compact set $[0,T]\times[-R,R]\times[-R,R]$ we get the uniform boundedness of the sequences $(F_k')_{k\in\mathbb{N}}$, $(F_k'')_{k\in\mathbb{N}}$, and $(F_k''')_{k\in\mathbb{N}}$, that implies the existence of $M_1, M_2, M_3 \geq 0$ such that

$$\left|F_k'(s)\right| \leq M_1, \quad \left|F_k''(s)\right| \leq M_2, \quad \left|F_k'''(s)\right| \leq M_3. \tag{32}$$

Now, by considering a uniform partition of $[0,T]$ with stepsize $h = \frac{T}{n}$ and knots $t_i = ih$, $i = \overline{0,n}$, we are able to describe the iterative method provided by Green's function technique and Catmull-Rom splines. In order to approximate the integrals in (31) we use the corrected trapezoidal rule with the error estimate deduced by Lemma 2 from [12]. Since on the knots t_i, $i = \overline{0,n}$, the iterations (31) can be written as

$$x_k(t_i) = g(t_i) + \int_0^{t_i} H(t_i,s)\cdot F_{k-1}(s)\,ds + \int_{t_i}^{T} K(t_i,s)\cdot F_{k-1}(s)\,ds \tag{33}$$

and $\frac{\partial^2 H(t_i,s)}{\partial s^2} = \frac{(T-t_i)^2}{T^2} - 1$, $\frac{\partial^2 K(t_i,s)}{\partial s^2} = \frac{(T-t_i)^2}{T^2}$, under the condition $f \in C^3\left([0,T]\times\mathbb{R}^2\right)$ we infer that the second order derivative (with respect by "s") of the kernel functions from the integrals in (33) are Lipschitzian. Remembering that the inequality in Lemma 2 from [12] is

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)(f(a)+f(b))}{2} + \frac{(b-a)^2}{12}\left(f'(b)-f'(a)\right)\right| \leq$$

$$\leq \frac{(b-a)^3(M-m)}{32}$$

with $m = \min\limits_{x\in[a,b]} f''(x)$, $M = \max\limits_{x\in[a,b]} f''(x)$, in the case of L-Lipschitzian f'' we get

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)(f(a)+f(b))}{2} + \frac{(b-a)^2}{12}\left(f'(b)-f'(a)\right)\right| \leq \frac{L(b-a)^4}{32} \tag{34}$$

which can be applied to uniform partitions. Now, we use (34) on the uniform partition of $[0,T]$ in the approximation of the integrals in (33). By using (28) and the fact $G(t,0) = G(t,T) = 0$, we get:

$$x_k(t_0) = c, \quad x_k(t_n) = d, \ k\in\mathbb{N}, \quad x_0(t_i) = g(t_i), \ i = \overline{0,n},$$

$$x_k(t_i) = g(t_i) + \frac{T}{2n}\sum_{j=1}^{n}[G(t_i,t_{j-1})\cdot F_{k-1}(t_{j-1}) + G(t_i,t_j)\cdot F_{k-1}(t_j)] -$$

$$-\frac{T^2}{12n^2}\left(\frac{(T-t_i)^2}{T} - (T-t_i)\right) f(t_n, d, x_{k-1}(\varphi(t_n))) + R_{k,i}, \quad i = \overline{1,n-1}, \ k\in\mathbb{N}^* \tag{35}$$

with $|R_{k,i}| \le \frac{L''T^4}{32n^3}$, $\forall i=\overline{1,n-1}$, $k \in \mathbb{N}^*$, where $L'' \ge 0$ is the greatest Lipschitz constant of $\frac{\partial^2}{\partial s^2}\left(H\left(t_i,s\right)\cdot F_{k-1}\left(s\right)\right)$ and $\frac{\partial^2}{\partial s^2}\left(K\left(t_i,s\right)\cdot F_{k-1}\left(s\right)\right)$, $i=\overline{0,n}$, $k\in\mathbb{N}^*$. Since for approximating the third argument in $F_{k-1}(s)=f(s,x_{k-1}(s),x_{k-1}(\varphi(s)))$ we use the Catmull-Rom spline with $m_n = r$, by (35) we obtain the following iterative algorithm:

The initial iterative step is $x_0(t)=g(t)$, $\forall t\in[0,T]$, and

$$x_k\left(t_0\right)=c,\qquad x_k\left(t_n\right)=d,\quad \forall k\in\mathbb{N}^* \tag{36}$$

and denote $\overline{S_k}\left(t_0\right):=c$, $\overline{S_k}\left(t_n\right):=d$ for $k\in\mathbb{N}^*$. With $k=1$ in (35) we get $x_1\left(t_i\right)=\overline{S_1\left(t_i\right)}+R_{1,i}$, $\forall i=\overline{1,n-1}$. By induction for $k\in\mathbb{N}^*$, $k\ge 2$, it obtains

$$\begin{aligned}x_k\left(t_i\right)&=g\left(t_i\right)+\frac{T}{2n}\sum_{j=1}^{n}[G\left(t_i,t_{j-1}\right)\cdot f\left(t_{j-1},\overline{S_{k-1}}\left(t_{j-1}\right),\overline{S_{k-1}}\left(\varphi\left(t_{j-1}\right)\right)\right)+\\&+G\left(t_i,t_j\right)\cdot f\left(t_j,\overline{S_{k-1}}\left(t_j\right),\overline{S_{k-1}}\left(\varphi\left(t_j\right)\right)\right)]-\frac{T^2}{12n^2}\left(\frac{\left(T-t_i\right)^2}{T}-\left(T-t_i\right)\right)\cdot\\&\cdot f\left(t_n,d,\overline{S_{k-1}}\left(\varphi\left(t_n\right)\right)\right)+\overline{R_{k,i}}=\overline{S_k}\left(t_i\right)+\overline{R_{k,i}},\quad i=\overline{1,n-1}\end{aligned} \tag{37}$$

where $\overline{S_{k-1}}$ is the Catmull-Rom spline interpolating the values $\overline{S_{k-1}}\left(t_i\right)$, $i=\overline{0,n}$, computed at the previous iterative step, with the expression

$$\begin{aligned}\overline{S_{k-1}}\left(\tau\right)&=\left(1-\tau\right)^2\left(2\tau+1\right)\overline{S_{k-1}}\left(t_{i-1}\right)+\tau^2\left(3-2\tau\right)\overline{S_{k-1}}\left(t_i\right)+\\&+h\tau\left(1-\tau\right)^2\overline{m_{k-1}}\left(i-1\right)-h\tau^2\left(1-\tau\right)\overline{m_{k-1}}\left(i\right),\quad t\in\left[t_{i-1},t_i\right],\ i=\overline{1,n}\end{aligned} \tag{38}$$

and notation $\tau=\frac{t-t_{i-1}}{h}$, such that $\overline{m_{k-1}}\left(n\right)=r$,

$$\overline{m_{k-1}}\left(0\right)=\frac{-3c+4\overline{S_{k-1}}\left(t_1\right)-\overline{S_{k-1}}\left(t_2\right)}{2h},\ \overline{m_{k-1}}\left(i\right)=\frac{\overline{S_{k-1}}\left(t_{i+1}\right)-\overline{S_{k-1}}\left(t_{i-1}\right)}{2h} \tag{39}$$

for $i=\overline{1,n-1}$.

The algorithm is stopped at the iteration "k" determined such that $\left|\overline{S_k}\left(t_i\right)-\right.$ $\left.-\overline{S_{k-1}}\left(t_i\right)\right|<\varepsilon$, $\forall i=\overline{1,n-1}$, with previously given $\varepsilon>0$.

4.2 Convergence analysis

Concerning the convergence of the iterative method (35)-(39) we obtain the main result of this work, as follows.

Theorem 4. *Under the conditions of Theorem 3, if* $f\in C^3\left([0,T]\times\mathbb{R}\times\mathbb{R}\right)$, $CT^3\left(\alpha+\frac{41}{27}\beta\right)<1$, *and* $n\in\mathbb{N}^*$ *is such that* $n>\sqrt{\frac{T^3\left(\alpha+\frac{41}{27}\beta\right)}{12\left(1-CT^3\left(\alpha+\frac{41}{27}\beta\right)\right)}}$, *then the Catmull-Rom spline* $\overline{S_k}$, *given in (38), approximates the solution of (1) and the error estimates in the discrete and continuous approximation are:*

$$\left|x^*\left(t_i\right)-\overline{S_k}\left(t_i\right)\right|\le\frac{\left(\frac{2T^3}{81}\left(\alpha+\beta\right)\right)^k\frac{2T^3}{81}M_0}{1-\frac{2T^3}{81}\left(\alpha+\beta\right)}+$$

$$+\frac{L''Th^3}{32\left[1-\left(\alpha+\frac{41}{27}\beta\right)\left(CT^3+\frac{T^3}{12n^2}\right)\right]}+\frac{\frac{2Mh^3}{9}+\frac{M_1h^4}{128}}{3\left[1-\left(\alpha+\frac{41}{27}\beta\right)\left(CT^3+\frac{T^3}{12n^2}\right)\right]} \quad (40)$$

for $i=\overline{1,n-1}$, $k\in\mathbb{N}^*$, *and*

$$\left|x^*(t)-\overline{S_k}(t)\right|\le\frac{\left(\frac{2T^3}{81}(\alpha+\beta)\right)^k\frac{2T^3}{81}M_0}{1-\frac{2T^3}{81}(\alpha+\beta)}+\frac{2Mh^3}{27}+\frac{M_1h^4}{384}+$$

$$+\frac{41L''Th^3}{864\left[1-\left(\alpha+\frac{41}{27}\beta\right)\left(CT^3+\frac{T^3}{12n^2}\right)\right]}+\frac{41\left(256Mh^3+9M_1h^4\right)}{93312\left[1-\left(\alpha+\frac{41}{27}\beta\right)\left(CT^3+\frac{T^3}{12n^2}\right)\right]} \quad (41)$$

for all $t\in[0,T]$, $k\in\mathbb{N}^*$.

Proof. Based on the inequality (29) we have to estimate $\left|\overline{R_{k,i}}\right|=\left|x_k(t_i)-\overline{S_k}(t_i)\right|$, $i=\overline{1,n-1}$. Then $\left\|x_k-\overline{S_k}\right\|_\infty$ follows using a similar method as in Section 3, which leads to formula (24). For this purpose, firstly we determine the Lipschitz constant L'' and denoting $G_{k,i}(s)=G(t_i,s)\cdot F_{k-1}(s)$ we get

$$\begin{aligned}
&\left|G''_{k,i}(s)-G''_{k,i}(s')\right|\\
\le\ &\left\|\frac{\partial^2G}{\partial s^2}\right\|_\infty\left|F_{k-1}(s)-F_{k-1}(s')\right|+2\left|F'_{k-1}(s)\right|\cdot\left|\frac{\partial G}{\partial s}(t_i,s)-\frac{\partial G}{\partial s}(t_i,s')\right|\\
&+2\left\|\frac{\partial G}{\partial s}\right\|_\infty\left|F'_{k-1}(s)-F'_{k-1}(s')\right|+\|G\|_\infty\left|F''_{k-1}(s)-F''_{k-1}(s')\right|\\
&+\left|F''_{k-1}(s')\right|\left|G(t_i,s)-G(t_i,s')\right|\\
\le\ &\left(3M_1+\frac{23T}{9}M_2+CT^2M_3\right)\left|s-s'\right|=L''\left|s-s'\right|
\end{aligned}$$

for all $s,s'\in[0,T]$, $k\in\mathbb{N}^*$, $i=\overline{0,n}$, where M_1,M_2,M_3 are given in (32), obtaining $L''=3M_1+\frac{23T}{9}M_2+CT^2M_3$. Now, let S_k be the Catmull-Rom spline interpolating the values $x_k(t_i)$, $i=\overline{0,n}$ and based on (6) we need to estimate $\left\|x_k^{(4)}\right\|_\infty$. Elementary calculus lead us to $\left\|x_k^{(4)}\right\|_\infty\le M_1$ and denoting $\left|\overline{R_k}\right|=\max\{\left|\overline{R_{k,i}}\right|:i=\overline{0,n}\}$, by (6) we get

$$\left\|x_{k-1}-\overline{S_{k-1}}\right\|_\infty\le\frac{41}{27}\left|\overline{R_{k-1}}\right|+\frac{2Mh^3}{27}+\frac{M_1h^4}{384} \quad (42)$$

remaining to estimate $\left|\overline{R_{k-1}}\right|$. In this purpose, in inductive manner, by (35) and (37) it obtains $\left|\overline{R_{1,i}}\right|=|R_{1,i}|\le\frac{L''Th^3}{32}$ and

$$\left|\overline{R_{k,i}}\right|\le|R_{k,i}|+\frac{T}{2n}\sum_{j=1}^{n}[G(t_i,t_{j-1})(\alpha\left|x_{k-1}(t_{j-1})-\overline{S_{k-1}}(t_{j-1})\right|+$$

$$+\beta\left|x_{k-1}(\varphi(t_{j-1}))-\overline{S_{k-1}}(\varphi(t_{j-1}))\right|)+G(t_i,t_j)(\alpha\left|x_{k-1}(t_j)-\overline{S_{k-1}}(t_j)\right|+$$

$$+\beta\left|x_{k-1}\left(\varphi\left(t_j\right)\right)-\overline{S_{k-1}}\left(\varphi\left(t_j\right)\right)\right|)]+\frac{T^2}{12n^2}\left|\frac{\left(T-t_i\right)^2}{T}-\left(T-t_i\right)\right|\beta\times$$

$$\times\left|x_{k-1}\left(\varphi\left(t_n\right)\right)-\overline{S_{k-1}}\left(\varphi\left(t_n\right)\right)\right|,\quad \forall k\in\mathbb{N}^*,\ k\geq 2,\ i=\overline{1,n-1}.$$

In the case $k=2$, by this recurrent inequality we get

$$\left|\overline{R_{2,i}}\right|\leq\frac{L''Th^3}{32}+\frac{T}{2n}\sum_{j=1}^{n}2\left\|G\right\|_\infty\left(\alpha\frac{L''Th^3}{32}+\beta\left\|x_1-\overline{S_1}\right\|_\infty\right)+\frac{T^3\beta}{12n^2}\left\|x_1-\overline{S_1}\right\|_\infty\leq$$

$$\leq\frac{L''Th^3}{32}+CT^3\left(\alpha\frac{L''Th^3}{32}+\beta(\frac{41}{27}\cdot\frac{L''Th^3}{32}+\frac{2Mh^3}{27}+\frac{M_1h^4}{384})\right)+$$

$$+\frac{T^3\beta}{12n^2}\left(\frac{41}{27}\cdot\frac{L''Th^3}{32}+\frac{2Mh^3}{27}+\frac{M_1h^4}{384}\right)\leq$$

$$\leq[1+\left(\alpha+\frac{41}{27}\beta\right)CT^3+\frac{41}{27}\beta\frac{T^3}{12n^2}]\frac{L''Th^3}{32}+CT^3\beta\left(\frac{2Mh^3}{27}+\frac{M_1h^4}{384}\right)+$$

$$+\frac{T^3\beta}{12n^2}\left(\frac{2Mh^3}{27}+\frac{M_1h^4}{384}\right)\leq\left[1+\left(\alpha+\frac{41}{27}\beta\right)\left(CT^3+\frac{T^3}{12n^2}\right)\right]\frac{L''Th^3}{32}+$$

$$+\frac{41}{27}\beta\left(CT^3+\frac{T^3}{12n^2}\right)\left(\frac{2Mh^3}{27}+\frac{M_1h^4}{384}\right),\ \forall i=\overline{1,n-1}.$$

Now, denoting $\omega=\left(\alpha+\frac{41}{27}\beta\right)\left(CT^3+\frac{T^3}{12n^2}\right)$ with $n>\sqrt{\frac{T^3\left(\alpha+\frac{41}{27}\beta\right)}{12\left(1-CT^3\left(\alpha+\frac{41}{27}\beta\right)\right)}}$ we have $\omega<1$, and for $k\geq 3$ by induction we obtain

$$\left|\overline{R_{k,i}}\right|\leq\left(1+\omega+...+\omega^{k-1}\right)\frac{L''Th^3}{32}+\omega\left(1+\omega+...+\omega^{k-2}\right)\cdot$$

$$\cdot\left(\frac{2Mh^3}{27}+\frac{M_1h^4}{384}\right)\leq\frac{L''Th^3}{32\left(1-\omega\right)}+\frac{\omega\left(\frac{2Mh^3}{27}+\frac{M_1h^4}{384}\right)}{1-\omega}=O\left(h^3\right),\ \forall i=\overline{1,n-1}. \tag{43}$$

By (43) and (29) the estimate (40) follows, and using (42) we obtain the estimate (41). □

By (42) and (43) we see that $\left\|x_k-\overline{S_k}\right\|_\infty=O\left(h^3\right)$, $\forall k\in\mathbb{N}^*$ and therefore we conclude that the order of convergence of Green's function method is $O\left(h^3\right)$. Due to the discontinuity of $\frac{\partial^2G}{\partial t^2}$, $\frac{\partial^2G}{\partial s^2}$, $\frac{\partial^2G}{\partial t\partial s}$ on the line "$t=s$", this order cannot be improved being maximal. By (40) and (41) we have $\lim\limits_{k\to\infty,h\to 0}\left|x^*\left(t\right)-\overline{S_k}\left(t\right)\right|=0$ for all $t\in[0,T]$, which means the convergence of the iterative method (35)-(39).

Remark 1. *Analogous to (35)-(39), the following boundary value problems associated to the differential equation* $x'''\left(t\right)=f(t,x\left(t\right),x\left(\varphi\left(t\right)\right))$, *that involve one of the endpoint conditions:*

$$\begin{array}{c} x\left(0\right)=c,\quad x'\left(0\right)=w,\quad x'\left(T\right)=r,\ or\\ x\left(T\right)=d,\quad x'\left(0\right)=w,\quad x'\left(T\right)=r,\ or\\ x\left(0\right)=c,\quad x'\left(0\right)=w,\quad x\left(T\right)=d \end{array} \tag{44}$$

could be approached, and similarly to Theorem 4 it obtains the order of convergence $\left\|x_k - \overline{S_k}\right\|_\infty = O\left(h^3\right)$, $\forall k \in \mathbb{N}^*$.

5 Numerical experiments and concluding remarks

In order to test the obtained theoretical results in Theorem 2 and Theorem 4 and to illustrate the accuracy of the Green function method, applied to the boundary value problems (7) and (1), we present some numerical examples as follows.

Example 1. *The second order two-point boundary value problem*

$$\begin{cases} x''(t) = 4e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) x\left(\frac{t}{2}\right), \quad t \in [0, \frac{\pi}{4}] \\ x(0) = 1, \quad x\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} e^{-\frac{\pi}{4}} \end{cases} \tag{45}$$

has the exact solution $x^*(t) = e^{-t}\cos t$ *and with* $\varepsilon = 10^{-22}$ *the algorithm (35)-(39) is stopped after* $k = 14$ *iterations. The convergence is tested by taking* $n = 10$, $n = 100$, $n = 1000$, *and the numerical results* $e_i = \left|x^*(t_i) - \overline{S_{14}(t_i)}\right|$, $i = \overline{0, n}$, *are presented in Table 1. We see that for stepsize* $h = 0.1$, $h = 0.01$, $h = 0.001$, *the accuracy is* $O\left(10^{-4}\right)$, $O\left(10^{-6}\right)$, $O\left(10^{-8}\right)$, *respectively, and the order of convergence* $O\left(h^2\right)$ *stated in Theorem 2 is confirmed.*

$t_i \backslash e_i$	e_i, $n = 10$	e_i, $n = 100$	e_i, $n = 1000$
$\frac{\pi}{20}$	6.93e-05	6.93e-07	6.93e-09
$\frac{\pi}{10}$	9.61e-05	9.62e-07	9.61e-09
$\frac{3\pi}{20}$	8.85e-05	8.85e-07	8.85e-09
$\frac{\pi}{5}$	5.42e-05	5.41e-07	5.42e-09
$\frac{\pi}{4}$	0	0	0

Table 1. Numerical results for (45)

Example 2. *The exact solution of the boundary value problem*

$$\begin{cases} x'''(t) = -\frac{4}{(t+1)^4} - \left([x(t)]^4 + [x(t)]^3\right) \cdot x\left(\frac{t}{2}\right), \quad t \in [0, 1] \\ x(0) = 1, \quad x(1) = \frac{1}{2}, \quad x'(1) = -\frac{1}{4} \end{cases} \tag{46}$$

is $x^*(t) = \frac{1}{t+1}$ *and applying the algorithm (35)-(39) with* $n = 10$, $n = 100$, $n = 1000$, *and* $\varepsilon = 10^{-22}$ *we get* $k = 13$ *iterations. The obtained numerical results* $e_i = \left|x^*(t_i) - \overline{S_{13}(t_i)}\right|$, $i = \overline{0, n}$, *are presented in Table 2 for stepsize values* $h = 0.1$, $h = 0.01$, *and* $h = 0.001$.

$t_i \backslash e_i$	e_i, $h = 0.1$	e_i, $h = 0.01$	e_i, $h = 0.001$
0.2	2.187770e-06	2.402558e-10	2.398e-14
0.4	1.650722e-06	1.783128e-10	1.788e-14
0.6	6.793574e-07	7.391376e-11	7.439e-15
0.8	3.709491e-08	5.125900e-12	5.552e-16

Table 2. Numerical results for (46)

Example 3. *The solution of the linear third order two-point boundary value problem*

$$\begin{cases} x'''(t) = -\frac{2}{3}x(t) - \frac{1}{3}e^{-0.5t}x\left(\frac{t}{2}\right), & t \in [0,1] \\ x(0) = 1, \quad x(1) = \frac{1}{e}, \quad x'(1) = -\frac{1}{e} \end{cases} \tag{47}$$

is $x^*(t) = e^{-t}$ *and with* $\varepsilon = 10^{-22}$ *the number of iterations will be* $k = 11$. *The numerical results* $e_i = \left|x^*(t_i) - \overline{S_{11}(t_i)}\right|$, $i = \overline{0,n}$ *are presented in Table 3.*

$t_i \backslash e_i$	e_i, $h = 0.1$	e_i, $h = 0.01$	e_i, $h = 0.001$
0.2	4.378843e-08	3.793854e-12	3.331e-16
0.4	6.480995e-08	6.160406e-12	6.662e-16
0.6	6.732228e-08	6.596057e-12	7.772e-16
0.8	4.693276e-08	4.663381e-12	3.331e-16

Table 3. Numerical results for (47)

Example 4. *In this numerical experiment we present some examples that correspond to the cases mentioned in Remark 1, by considering the following third order two-point boundary value problems:*

$$\begin{cases} x'''(t) = e^{-t}\left[x(t)\right]^{\frac{3}{2}} \cdot x\left(\frac{t}{2}\right), & t \in [0,1] \\ x(0) = 1, \quad x'(0) = 1, \quad x(1) = e \end{cases} \tag{48}$$

$$\begin{cases} x'''(t) = 24t - \frac{1}{3}x\left(t^2\right) + \frac{1}{3}[x(t)]^2, & t \in [0,1] \\ x(0) = 0, \quad x'(0) = 0, \quad x'(1) = 4. \end{cases} \tag{49}$$

$$\begin{cases} x'''(t) = \frac{1}{2}x(t) + \frac{1}{2}e^{(1-q)t}x(qt), & t \in [0,1],\ q \in (0,1) \\ x'(0) = 1, \quad x(1) = e, \quad x'(1) = e \end{cases} \tag{50}$$

where the exact solution of (48) and (50) is $x^*(t) = e^t$. *In the case of (49), the deviating argument is* $\varphi(t) = t^2$ *and the exact solution will be* $x^*(t) = t^4$. *For (48), taking* $\varepsilon = 10^{-22}$ *we get* $k = 11$ *iterations and the numerical results are presented in Table 4. Similarly, by considering* $\varepsilon = 10^{-22}$, *the iterative algorithm applied to (49) provides* $k = 8$ *iterations and the numerical results at the last iteration* $e_i = \left|x^*(t_i) - \overline{S_8(t_i)}\right|$, $i = \overline{0,n}$, *can be found in Table 5. The boundary value problem (50) was considered for* $q \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ *obtaining the number of iterations* $k = 18$, $k = 16$ *and* $k = 15$, *respectively, and the numerical results* $e_i = \left|x^*(t_i) - \overline{S_k(t_i)}\right|$, $i = \overline{0,n}$ *are presented in Table 6.*

$t_i \backslash e_i$	e_i, $h = 0.1$	e_i, $h = 0.01$	e_i, $h = 0.001$
0.2	5.403e-07	4.614e-11	4.441e-15
0.4	7.053e-07	6.703e-11	6.884e-15
0.6	6.776e-07	6.490e-11	6.439e-15
0.8	4.301e-07	4.181e-11	4.441e-15

Table 4. Numerical results for (48)

$t_i\backslash e_i$	e_i, $h=0.1$	e_i, $h=0.01$	e_i, $h=0.001$
0.2	1.071654e-08	9.116789e-13	1.697861e-16
0.4	3.821559e-08	3.055615e-12	4.649059e-16
0.6	7.282348e-08	5.561190e-12	1.026956e-15
0.8	1.003167e-07	7.704226e-12	6.661338e-16
1	1.110156e-07	8.615109e-12	1.665335e-15

Table 5. Numerical results for (49)

$e_i\backslash t_i$	0	0.2	0.4	0.6	0.8
$q\backslash n : \frac{1}{4}\backslash 10$	1.32e-06	1.17e-06	9.94e-07	7.39e-07	4.21e-07
$n=10^2$	1.50e-10	1.32e-10	1.08e-10	7.75e-11	4.19e-11
$n=10^3$	1.51e-14	1.34e-14	1.11e-14	8.22e-15	4.44e-15
$q\backslash n : \frac{1}{2}\backslash 10$	1.46e-06	1.28e-06	1.05e-06	7.62e-07	4.15e-07
$n=100$	1.51e-10	1.32e-10	1.07e-10	7.72e-11	4.18e-11
$n=1000$	1.52e-14	1.29e-14	1.09e-14	7.99e-15	4.01e-15
$q\backslash n : \frac{3}{4}\backslash 10$	1.46e-06	1.28e-06	1.04e-06	7.57e-07	4.04e-07
$n=10^2$	1.50e-10	1.30e-10	1.06e-10	7.64e-11	4.15e-11
$n=10^3$	1.49e-14	1.29e-14	1.07e-14	7.78e-15	4.01e-15

Table 6. Numerical results for (50)

Concerning the order of convergence of Green's function method applied to third order two-point boundary value problems with deviating argument, the results presented in Tables 2-6 can be summarized in Table 7 where we see that the order $O\left(h^3\right)$ stated in Theorem 4 is confirmed.

$h\backslash e_i$	eq. (46)	eq. (47)	eq. (48)	eq. (49)	eq. (50)
10^{-1}	$O\left(10^{-5}\right)$	$O\left(10^{-7}\right)$	$O\left(10^{-6}\right)$	$O\left(10^{-6}\right)$	$O\left(10^{-5}\right)$
10^{-2}	$O\left(10^{-9}\right)$	$O\left(10^{-11}\right)$	$O\left(10^{-10}\right)$	$O\left(10^{-10}\right)$	$O\left(10^{-9}\right)$
10^{-3}	$O\left(10^{-13}\right)$	$O\left(10^{-15}\right)$	$O\left(10^{-14}\right)$	$O\left(10^{-14}\right)$	$O\left(10^{-13}\right)$

Table 7. The accuracy for third order BVP's

In Theorem 4 we proved that the maximal order of convergence of Green's function method applied to third order two-point boundary value problems with deviating argument is $O\left(h^3\right)$ and this was realized by choosing the Catmull-Rom cubic spline as suitable interpolation procedure and by using the corrected trapezoidal quadrature rule. The corrected-trapezoidal quadrature rule has the error estimate described in terms of Lipschitz constants for the second order derivative of the kernel function. For the boundary value problems corresponding to the first two cases in (44), the complete cubic spline interpolation procedure provides the same order of convergence $O\left(h^3\right)$, but the computational cost is increased because a tridiagonal linear system should be solved at each iterative step. Therefore, the approach of using Catmull-Rom spline is better from computational cost point of view for third order two-point boundary value problems with deviating argument.

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