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COMMUTATIVITY OF PRIME RINGS WITH GENERALIZED $(\alpha, \beta)-$ REVERSE DERIVATIONS SATISFYING CERTAIN IDENTITIES

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Abstract

In this note we investigated some conditions related to the commutativity of a prime ring R that satisfies certain identities and possesses a generalized (α, β)-reverse derivation. A few examples and counterexamples are also studied.

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1 Introduction

Throughout we assume that R is an associative ring with the center $Z(R)$. For each $x, y \in R$, denote the commutator $xy - yx$ by $[x, y]$ and the anti-commutator $xy + yx$ by $x \circ y$. We will make extensive use of the following basic identities:

$$\begin{gathered}
[xy, z] = x[y, z] + [x, z]y, \\
[x, yz] = y[x, z] + [x, y]z, \\
x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z, \\
(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].
\end{gathered}$$

A ring R is prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$ for any $a, b \in R$, it is called semiprime if $aRa = \{0\}$ implies that $a = 0$ for any $a \in R$.

By a derivation on a ring R we means the most natural derivation $d : R \longrightarrow R$ which is additive as well as satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. While an additive mapping F: $R \to R$ is a generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ holds $\forall\ x, y \in R$. Suppose that α, β are endomorphisms on R, by an $(\alpha, \beta)-$derivation we mean an additive function

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$d : R \longrightarrow R$ satisfying $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$. While an additive mapping F: $R \to R$ is a generalized $(\alpha, \beta)-$derivation associated with an $(\alpha, \beta)-$derivation d if $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds $\forall\ x, y \in R$.

Herstein in [6] introduced the concept of reverse derivation as an additive mapping $d : R \longrightarrow R$ satisfying $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$ and Bresar, Vukman [5] and Samman, Alyamani [13] studied the reverse derivations. Further, Abuabakar and Gonzales [1] generalized the notion of reverse derivation by introducing the generalized reverse derivation F associated with reverse derivation d as an additive mapping $F : R \longrightarrow R$ satisfying $F(xy) = F(y)x + yd(x)$ for all $x, y \in R$. The concepts of reverse derivations and generalized reverse derivations have relations with some generalizations of derivations and generalized derivations. It is clear if R is commutative, then both of reverse derivations (derivations) and genralized reverse derivations (generalizrd derivations) are the same.

Several authors have proved commutativity theorems for prime and semiprime rings that admitting some special mappings, particularly in the form of automorphisms, derivations, generalized derivations, reverse derivations and generalized reverse derivations on an appropriate subset of R. Some well-known results concerning prime rings have been extended for prime rings (see [3], [2], [4], [9], [11], [12] and [7], for partial bibliography).

In 2018 Ozge, Atay [10] and S. Huang [14] studied the commutativity of prime rings R admitting a generalized reverse derivation F associated with reverse derivation d satisfying several identities on an appropriate subset of R. In 2018, Merve and Aydm [8] studied some properties of (α, β)-reverse derivations on prime and semiprime rings.

Motivated by above results, in the following, we investigated the commutativity of a prime ring admitting a generalized (α, β)- reverse derivation F associated with an (α, β)- reverse derivation d satisfying one of the following identities: $\forall$ $x, y \in I$,

(i) $F([x, y]) - F(x) \circ \alpha(y) - [d(y), x] \in Z(R)$,
(ii) $F([x, y]) - [F(x), \alpha(y)] - [d(y), x] \in Z(R)$,
(iii) $F([x, y]) - F(x) \circ \alpha(y) - d(y) \circ x \in Z(R)$, and
(iv) $F([x, y]) + F(x) \circ \alpha(y) - d(x) \circ d(y) \in Z(R)$.

2 Preliminary results

Let us define couple of more items:

Definition 1. *An additive mapping d: $R \to R$ is an $(\alpha, \beta)-$reverse derivation if $d(xy) = d(y)\alpha(x) + \beta(y)d(x)$ holds for all $x, y \in R$.*

Definition 2. *An additive mapping F: $R \to R$ is a generalized $(\alpha, \beta)-$reverse derivation associated with an $(\alpha, \beta)-$reverse derivation d if $F(xy) = F(y)\alpha(x) + \beta(y)d(x)$ holds for all $x, y \in R$.*

Remark 1. *The concepts of $(\alpha, \beta)-$reverse derivations and generalized $(\alpha, \beta)-$ reverse derivations have relations with some generalizations of $(\alpha, \beta)-$derivations and generalized $(\alpha, \beta)-$derivations. It is clear if R is commutative, then both of $(\alpha, \beta)-$reverse derivations ($(\alpha, \beta)-$derivations) and generalized $(\alpha, \beta)-$ reverse derivations (generalized $(\alpha, \beta)-$derivations) are the same, but the converse may not be true in general as the following example which also justifies the existence of genenralized $(\alpha, \beta)-$reverse derivation associated with an $(\alpha, \beta)-$ reverse derivation.*

Example 1. *Let $\mathbb{Z}$ be the ring of integers and let*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Then R is a ring under usual operations. Define $F : R \to R$ by $F \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$, and $d : R \to R$ by $d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix}$, $\alpha : R \to R$ by $\alpha \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}$, and $\beta : R \to R$ by $\beta \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. It is easy to check that F is an additive mapping which is a generalized (α, β)-reverse derivation associated with an (α, β)-reverse derivation d, and α, β are epimorphisms on R. We can remark although R is noncommutative, but F is also a generalized (α, β)- derivation associate with an (α, β)- derivation d.

We begin our discussion with the following lemmas which are essential for developing the proof of our main results.

Lemma 1. *Let R be a prime ring and S an additive subgroup of R. Let $f, g : S \longrightarrow R$ be additive functions such that $f(s)Rg(s) = 0$ for all $s \in S$. Then either $f(s) = 0$ for all $s \in S$, or $g(s) = 0$ for all $s \in S$.*

Lemma 2. *Let R be a prime ring. For a nonzero element $a \in Z(R)$, if $ab \in Z(R)$, then $b \in Z(R)$.*

Both lemmas can be easily proved. In particular, the proof of Lemma (2.1) basic and is based on the fact that a group cannot be written as a specific theoretical union of its two proper subgroups.

We can extend Lemma(2), to the following.

Lemma 3. *Let R be a prime ring with center $Z(R)$ and S an additive subgroup of R. If $f, g : S \longrightarrow R$ be additive functions such that $f(s)g(s) \in Z(R)$ for all $s \in S$, where $0 \neq f(s) \in Z(R)$, then $g(s) \in Z(R)$ for all $s \in S$.*

Proof. We have $f(s)g(s) \in Z(R)$ for all $s \in S$, then for any $r \in R$ we have $[f(s)g(s), r] = 0$, i.e, $[f(s), r]g(s) + [g(s), r]f(s) = 0$, for all $s \in S$ and $r \in R$ but $0 \neq f(s) \in Z(R)$ for all $s \in S$. So the last equation becomes $[g(s), r]f(s) = 0$, for all $s \in S$ and $r \in R$. But $f(s) \neq 0$, the primeness of R forces that $[g(s), r] = 0$ for all $r \in R$ and $s \in S$. Hence $g(s) \in Z(R)$. ■

Lemma 4 ([14], Lemma 3)**.** *If a prime ring R contains a nonzero commutative right ideal, then R is commutative.*

Lemma 5. *Let R be a prime ring and I be a nonzero ideal of R. If $\beta : R \longrightarrow R$ is an automorphism on R, then for any $0 \neq c \in Z(R)$, $\beta(c) \in Z(R)$.*

Proof. We have $0 \neq c \in Z(R)$, that is $[c, r] = 0$ for all $r \in R$ and hence $\beta([c, r]) = 0$. Since β is an automorphism on R; therefore, $[\beta(c), \beta(r)] = 0$ for all $r \in R$. Now, replacing r by $\beta^{-1}(s)$, we get that $[\beta(c), s] = 0$ for all $s \in R$ and hence $\beta(c) \in Z(R)$. ■

3 Main Results

In this section we will prove the main results and will discuss some consequences. Throughout we assume that $0 \neq d(Z(R)) \subseteq Z(R)$, α, β are automorphisms on R such that $\alpha(I), \beta(I) \subseteq I$ and the pair (F, d) denotes the generalized (α, β)-reverse derivation F associated with an (α, β)-reverse derivation d.

Theorem 1. *Let R be a prime ring with center $Z(R)$ and I a nonzero ideal of R. If R admits a generalized (α, β)-reverse derivation (F, d) such that $F([x, y]) - F(x) \circ \alpha(y) - [d(y), x] \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. We have $F([x, y]) - F(x) \circ \alpha(y) - [d(y), x] \in Z(R)$ for all $x, y \in I$. If $F = 0$, then

$$[d(y), x] \in Z(R); \text{ for all } x, y \in I. \tag{1}$$

Since $d(Z(R)) \neq 0$, we can choose an element $c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in (1) and using (1), Lemma(5) we get

$$[\beta(y), x]d(c) \in Z(R) \text{ for all } x, y \in I.$$

Thus, the assumption that $0 \neq d(Z(R)) \subseteq Z(R)$ and Lemma(3) force that

$$[\beta(y), x] \in Z(R); \text{ for all } x, y \in I. \tag{2}$$

But β is an automorphism on R so we can replace y by $\beta^{-1}(m)$ in the last equation to get $[m, x] \in Z(R)$ for all $m, x \in I$. Using arguments which used in [14] Theorem 1, we get the required result.

Henceforth, we shall assume that $F \neq 0$, then we have

$$F([x, y]) - F(x) \circ \alpha(y) - [d(y), x] \in Z(R) \text{ for all } x, y \in I. \tag{3}$$

Since $d(Z(R)) \neq 0$, then there exists an element $c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in (3) and using (3), Lemma(5), we get

$$(\beta([x, y]) - [\beta(y), x])d(c) \in Z(R) \text{ for all } x, y \in I. \tag{4}$$

Since $0 \neq d(Z(R)) \subseteq Z(R)$, so applying Lemma(3), we get $\beta([x, y]) - [\beta(y), x] \in Z(R)$. Now, take $x = \beta(y)$ in the above equation, to get

$$\beta([\beta(y), y]) \in Z(R) \text{ for all } y \in I. \tag{5}$$

But β is an automorphism on R so $[\beta(y), y] \in Z(R)$ for all $y \in I$. Replacing y by $y + x$ in the last equation and using it, we get

$$[\beta(y), x] + [\beta(x), y] \in Z(R) \text{ for all } x, y \in I. \tag{6}$$

Again since $d(Z(R)) \neq 0$, then there exists an element $c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in (6) and using (6), Lemma(5), we get $[\beta(y), x]\beta(c) \in Z(R)$.

But $\beta(c) \in Z(R)$ by Lemma(5), so using Lemma(3), we get $[\beta(y), x] \in Z(R)$ for all $x, y \in I$. Using similar arguments which have been used after equation (2), we get the required result. ■

Theorem 2. *Let R be a prime ring with center $Z(R)$ and I a nonzero ideal of R. If R admits a generalized (α, β)-reverse derivation (F, d) such that $F([x, y]) - [F(x), \alpha(y)] - [d(y), x] \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. We have $F([x, y]) - [F(x), \alpha(y)] - [d(y), x] \in Z(R)$ for all $x, y \in I$. If $F = 0$, then

$$[d(y), x] \in Z(R); \text{ for all } x, y \in I.$$

Using the similar arguments which used after equation (1), we get the required result.

Henceforth, we shall assume that $F \neq 0$, then we have

$$F([x, y]) - [F(x), \alpha(y)] - [d(y), x] \in Z(R) \text{ for all } x, y \in I. \tag{7}$$

Since $d(Z(R)) \neq 0$, then there exists an element $c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in (7) and using (7), Lemma(5), we get

$$(\beta([x, y]) - [\beta(y), x])d(c) \in Z(R) \text{ for all } x, y \in I.$$

Using the similar arguments which used after equation (4), we get the required result. ■

Theorem 3. *Let R be a prime ring with center $Z(R)$ and I a nonzero ideal of R. If R admits a generalized (α, β)-reverse derivation (F, d) such that $F([x, y]) - F(x) \circ \alpha(y) - d(y) \circ x \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. We have $F([x, y]) - F(x) \circ \alpha(y) - d(y) \circ x \in Z(R)$ for all $x, y \in I$. If $F = 0$, then

$$d(y) \circ x \in Z(R); \text{ for all } x, y \in I.$$

That is

$$[d(y) \circ x, r] = 0 \text{ for all } x, y \in I, r \in R. \tag{8}$$

Replacing x by $xd(y)$ in equation (8), we get $[(d(y) \circ x)d(y), r] = 0$, i.e, $(d(y) \circ x)[d(y), r] + [d(y) \circ x, r]d(y) = 0$, using (8), we get $(d(y) \circ x)[d(y), r] = 0$ for all $x, y \in I$ and $r \in R$. Now for any $s \in R$, replacing r by rs in the last equation and using it, we get

$$(d(y) \circ x)R[d(y), s] = 0; \text{ for all } x, y \in I, s \in R.$$

Since R is prime then by Lemma(1) either $d(y) \circ x = 0$ for all $x, y \in I$ or $[d(y), s] = 0$ for all $y \in I$ and $s \in R$.
If $d(y) \circ x = 0$ for all $x, y \in I$, then replacing x by xm in the last equation and using it, we get $x[d(y), m] = 0$ for all $x, , y, m \in I$, i.e, $xR[d(y), m] = 0$ for all $x, y, m \in I$. But I is nonzero ideal of prime ring R, so that $[d(y), m] = 0$ for all $y, m \in I$ by Lemma(1).

Since $d(Z(R)) \neq 0$, we can choose an element $c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in the last equation and using it, we get $[\beta(y), m]d(c) = 0$, but $0 \neq d(Z(R)) \subseteq Z(R)$, so using Lemma(3), we get $[\beta(y), m] = 0$ for all $y, m \in I$. Since β is an automorphism on R, replace y by $\beta^{-1}(z)$ in the last equation, to get $[z, m] = 0$ for all $z, m \in I$ i.e, I is commutative by Lemma(4) and so is R.

On the other hand, if $[d(y), s] = 0$ for all $y \in I$ and $s \in R$. Since $d(Z(R)) \neq 0$, we can choose an element $c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in the last equation and using it, $[\beta(y), r]d(c) = 0$ for all $y \in I$, $r \in R$ and $c \in Z(R)$, now $0 \neq d(Z(R)) \subseteq Z(R)$, so by Lemma(3) we get $[\beta(y), r] = 0$ for all $y \in I$ and $r \in R$ again since β is an automorphism, replacing y by $\beta^{-1}(z)$, we get $[z, r] = 0$ for all $z \in I$ and $r \in R$. Hence, $I \subseteq Z(R)$ i.e, I is commutative by Lemma(4) and so is R.

Henceforth we shall assume that $F \neq 0$, then we have

$$[F(x), \alpha(y)] - d(y) \circ x \in Z(R) \text{ for all } x, y \in I. \tag{9}$$

Since $d(Z(R)) \neq 0$, then there exists an element $c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by $yc = cy$ in (9) and using (9), we get

$$(\beta(y) \circ x)d(c) \in Z(R) \text{ for all } x, y \in I.$$

Since $0 \neq d(Z(R)) \subseteq Z(R)$, applying Lemma(3), we get $\beta(y) \circ x \in Z(R)$,

that is $[\beta(y) \circ x, r] = 0$, for any $x, y \in I$ and $r \in R$, replacing x by xy, we get $(\beta(y) \circ x)[\beta(y), r] = 0$. Now replacing r by rs in the last equation and using it, we get $(\beta(y) \circ x)R[\beta(y), s] = 0$ for any $x, y \in I$ and $s \in R$. So, by primeness of R and Lemma(1) we get either $\beta(y) \circ x = 0$ for all $x, y \in I$ or $[\beta(y), s] = 0$ for all $y \in I$ and $s \in R$.

If $\beta(y) \circ x = 0$ for all $x, y \in I$. Replacing x by xm in the last equation and using it, we get $x[\beta(y), m] = 0$ for all $x, y, m \in I$, i.e, $xR[\beta(y), m] = 0$. But I is nonzero ideal of R, so primeness of R and Lemma(1) leads that $[\beta(y), m] = 0$ for all $y,m\in I$. Since β is an automorphism, take $\beta(y)=z$, to get $[z,m]=0$ for all $z,m\in I$. By Lemma(4), I is commutative, yields that R is commutative also. On the other hand, $[\beta(y),s]=0$ for all $y\in I$ and $s\in R$, since β is an automorphism in R, take $\beta(y)=z$, we get $[z,s]=0$ for all $z\in I$ and $s\in R$. Hence $I\subseteq Z(R)$ i.e, I is commutative by Lemma(4) and so is R.

Theorem 4. *Let R be a prime ring with center $Z(R)$ and I a nonzero ideal of R. If R admits a generalized (α,β)-reverse derivation (F,d) such that $F([x,y])+F(x)\circ\alpha(y)-d(x)\circ d(y)\in Z(R)$ for all $x,y\in I$, then R is commutative.*

Proof. If $F=0$, then

$$d(x)\circ d(y)\in Z(R);\ \text{for all } x,y\in I. \tag{10}$$

Since $d(Z(R))\neq 0$, we can choose an element $c\in Z(R)$ such that $0\neq d(c)\in Z(R)$. Replacing y by $yc=cy$ in (10) and using (10), we get

$$(d(x)\circ\beta(y))d(c)\in Z(R)\ \text{for all } x,y\in I.$$

Since $0\neq d(Z(R))\subseteq Z(R)$ so applying Lemma(3), we get $d(x)\circ\beta(y)\in Z(R)$, replacing x by $xc=cx$ in the last equation and using it, we get $(\beta(x)\circ\beta(y))d(c)\in Z(R)$, again $0\neq d(Z(R))\subseteq Z(R)$ and Lemma(3) yields that $\beta(x)\circ\beta(y)\in Z(R)$ for all $x,y\in I$. Since β is an automorphism on R, so $x\circ y\in Z(R)$ for all $x,y\in I$. Using arguments which were used in [14] Theorem 4, we get the required result.

Therefore we shall assume that $F\neq 0$, so for any $x,y\in I$ we have

$$F([x,y])+F(x)\circ\alpha(y)-d(x)\circ d(y)\in Z(R)\ \text{for all } x,y\in I. \tag{11}$$

Since $d(Z(R))\neq 0$, then there exists an element $c\in Z(R)$ such that $0\neq d(c)\in Z(R)$. Replacing y by $yc=cy$ in (11) and using (11), we get

$$(\beta([x,y])-d(x)\circ\beta(y))d(c)\in Z(R)\ \text{for all } x,y\in I.$$

Since, $0\neq d(Z(R))\subseteq Z(R)$, so Lemma(3) gives us $\beta([x,y])-d(x)\circ\beta(y)\in Z(R)$. Again replacing y by yx in the last equation and using it, we get

$$\beta(y)[d(x),\beta(x)]\in Z(R)\ \text{for all } x,y\in I. \tag{12}$$

Replacing x by $x+y$ in (12) and using (12), we get

$$\beta(y)[d(x),\beta(y)]+\beta(y)[d(y),\beta(x)]\in Z(R)\ \text{for all } x,y\in I. \tag{13}$$

Now, $d(Z(R))\neq 0$, then there exists an element $c\in Z(R)$ such that $0\neq d(c)\in Z(R)$. Replacing x by $xc=cx$ in (13) and using (13), we get

$$(\beta(y)[\beta(x),\beta(y)])d(c)\in Z(R)\ \text{for all } x,y\in I.$$

Assumption that $0 \neq d(Z(R)) \subseteq Z(R)$ and Lemma(3), yields that $\beta(y)[\beta(x),\beta(y)] \in Z(R)$ for all $x,y \in I$, but β is an automorphism on R, so

$$y[x,y] \in Z(R) \text{ for all } x,y \in I. \tag{14}$$

That is for any $r \in R$ $[y[x,y],r] = 0$, which equals to $y[[x,y],r] + [y,r][x,y] = 0$ for any $x,y \in I$, $r \in R$. Replacing x by yx in the last equation, we get $y[y[x,y],r]+[y,r]y[x,y] = 0$. Now usning the equation (14), we get $[y,r]y[x,y] = 0$ for all $x,y \in I$ and $r \in R$. Replacing r by sr in the last equation and using it, we get

$$[y,s]Ry[x,y] = 0; \text{ for all } x,y \in I, s \in R.$$

Now primeness of R and Lemma(1) yield that either $[y,s] = 0$ for all $y \in I$, $s \in R$ or $y[x,y] = 0$ for all $x,y \in I$. If $[y,s] = 0$ for all $y \in I$ and $s \in R$, then $I \subseteq Z(R)$ i.e, I is commutative by Lemma(4), so is R. On the other hand, if $y[x,y] = 0$ for all $x,y \in I$, then $yR[x,y] = 0$ for all $x,y \in I$. But I is a nonzero ideal of R, so the primeness of R and Lemma(1) force that $[x,y] = 0$ for all $x,y \in I$. Hence I is a commutative ideal of R by Lemma(4), and so R is commutative.$\blacksquare$

The following Corollary is an immediate consequence of the Theorems 1, 2, 3 and 4 by taking $\alpha = \beta = 1$ where $1 : R \longrightarrow R$ is the identity map.

Corollary 1. *Let R be a prime ring with center $Z(R)$ and I a nonzero ideal of R. If R admits a generalized reverse derivation (F,d) such that*
(i) $F([x,y]) - F(x) \circ y - [d(y),x] \in Z(R)$ for all $x,y \in I$, or
(ii) $F([x,y]) - [F(x),y] - [d(y),x] \in Z(R)$ for all $x,y \in I$, or
(iii) $F([x,y]) - F(x) \circ y - d(y) \circ x \in Z(R)$ for all $x,y \in I$, or
(iv) $F([x,y]) + F(x) \circ y - d(x) \circ d(y) \in Z(R)$ for all $x,y \in I$,

then R is commutative.

The following examples show that the restrictions imposed on the hypotheses of the various results are not superfluous. But before we explore the examples we start with the following definition

Definition 3. *Define the set*

$$K_{2^n} := \langle x_1, ..., x_n |\ x_i x_j = x_i,\ 2x_i = 0,\ \forall i = 1,2,...n \rangle .$$

Clearly, K_{2^n} is a non commutative ring with $Char(K_{2^n}) = 2$.

In particular, for $n = 2$, we can write $K_{2^2} = \langle a,b \rangle = \{0,a,b,c\}$ with the following relations:

$$2a = 2b = 0,\ c = a+b,\ a^2 = ab = a,\ b^2 = ba = b.$$

The additive and multiplicative tables of this ring are given by

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

;

.	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	b	b	0
c	0	c	c	0

.

Note that the concrete forms of K_{2^2} and $(K_{2^2})^{op}$ in matrix notation are

$$K_{2^2} = \left\{0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ 0,1 \in \mathbb{Z}_2\right\}$$

and

$$(K_{2^2})^{op} = \left\{0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 0,1 \in \mathbb{Z}_2\right\},$$

respectively.

Example 2. *In case of the ring $K_{2^2} = \langle a, b\rangle = \{0, a, b, c\}$, we define*

$$I = \{0, c\},$$

$$F(x) = \begin{cases} 0 & \text{if } x = 0, c \\ b & \text{if } \quad x = a, b \end{cases},$$

$$d(x) = \begin{cases} 0 & \text{if } x = 0, c \\ c & \text{if } \quad x = a, b \end{cases},$$

$$\alpha(x) = \begin{cases} 0 & \text{if } x = 0, c \\ b & \text{if } \quad x = a, b \end{cases} \quad \text{and}$$

$$\beta(x) = \begin{cases} 0 & \text{if } x = 0, c \\ a & \text{if } \quad x = a, b \end{cases}$$

Then it is easy to check that I is an ideal of K_{2^2}, F is an additive mapping which is a generalized (α,β)-reverse derivation associate with an (α,β)-reverse derivation d, and α and β are epimorphisms on K_{2^2}, satisfying either (i) $F([x,y]) - F(x)\circ\alpha(y) - [d(y),x] \in Z(R)$ or (ii) $F([x,y]) - [F(x),\alpha(y)] - [d(y),x] \in Z(R)$, or (iii) $F([x,y]) - F(x)\circ\alpha(y) - d(y)\circ x \in Z(R)$ or (iv) $F([x,y]) + F(x)\circ\alpha(y) - d(x)\circ d(y) \in Z(R)$, for all $x,y \in I$, but K_{2^2} non commutative. Hence, in Theorems 1, 2, 3 and 4, the hypothesis of primeness cannot be omitted.

Example 3. *Let $\mathbb{Z}$, R, F, d, α, and β as in Example (2.1) and $I = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z}\right\}$. Then I is an ideal of R. It is easy to check that F is an additive mapping which is a generalized (α,β)-reverse derivation associated* *with an (α,β)-reverse derivation d. We can also check that α and β are epimorphisms on R satisfying either (i) $F([x,y]) - F(x)\circ\alpha(y) - [d(y),x] \in Z(R)$ or (ii) $F([x,y]) - [F(x),\alpha(y)] - [d(y),x] \in Z(R)$, or (iii) $F([x,y]) - F(x)\circ\alpha(y) - d(y)\circ x \in Z(R)$ or (iv) $F([x,y]) + F(x)\circ\alpha(y) - d(x)\circ d(y) \in Z(R)$, for all $x,y \in I$, but R is non commutative. Hence, in Theorems 1, 2, 3 and 4, the hypothesis of primeness cannot be omitted.*

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