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THE MILGROM OSCILLATOR

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Abstract

The Milgrom oscillator is an oscillator working in the Modified Newtonian Dynamics (MOND) setting. A brief description of MOND is given. The equation of a Milgrom oscillator is deduced. An analogy with a classical system is built and in its setting both the kinetic and potential energies are computed. The law of conservation of energy is recovered. Different regimes of the oscillator are presented. Next, its periodic character is studied through numerical simulations. A discussion of the difficulties of testing MOND with the Milgrom oscillator ends the paper.

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Key words: Modified Newtonian Dynamics, Milgrom oscillator.

1 Introduction to the MOND hypothesis

The Modified Newtonian Dynamics (MOND) hypothesis was built by Mordehai Milgrom in 1981 in order to explain the movement of stars in spiral galaxies. Initially, it was used to fit the observational data of stars moving on circular orbits around the galactic center at some distance from it. Later, the MOND hypothesis was developed into a relativistic theory, called the TeVeS (Tensor -Vector - Scalar) theory. MOND is a counterpart to the dark matter theory which is also used to explain the movement of stars in galaxies.

According to classical (Newtonian) mechanics, if the mass M of a galaxy is concentrated toward its center, the speed of rotation of a star lying at the edge of the galaxy is

$$v = \sqrt{\frac{KM}{r}} \propto \frac{1}{\sqrt{r}}, \tag{1}$$

where r is the distance from the center of the galaxy and K is the universal constant of gravitational attraction. Thus, in classical mechanics, the farther the star is from the center of the galaxy, the slower it will move around it. Astronomical observations disprove this theoretical result. Indeed, for circular orbits, the speed of rotation of a star tends towards a constant value as r increases.

In the MOND hypothesis it is assumed that the fundamental law of dynamics $F = ma$ relating acceleration a to force F through body mass m stops working for very small values

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of a. A universal constant a_0 with units of acceleration is introduced. MOND adopts the following fundamental law of dynamics:

$$F = \mu(a)ma, \tag{2}$$

where $\mu(a)$ is a coefficient depending on the acceleration a. This coefficient has the following properties:

$$\lim_{a\to\infty} \mu(a) = 1, \tag{3}$$

$$\lim_{a\to 0} \mu(a) = \frac{a}{a_0}. \tag{4}$$

Thus, for $a \gg a_0$ Newton's second law is recovered, while for $a \ll a_0$ (2) becomes:

$$F = m\frac{a^2}{a_0}. \tag{5}$$

In the vector form, it is required that the force $\vec{F}$ have the same direction as $\vec{a}$. Thus, the law (5) becomes:

$$\vec{F} = m\frac{a^2}{a_0}\frac{\vec{a}}{a}. \tag{6}$$

In one dimension, it becomes

$$F = \mathrm{sgn}(a)m\frac{a^2}{a_0}, \tag{7}$$

where sgn is the sign function.

It should be noted that there is no clear consensus on whether the MOND hypothesis works in any system of reference or just in the inertial systems of reference. We tend toward the second possibility. If there is a non-inertial variant of MOND, it must take into consideration virtual forces appearing in such systems of reference and work only when such forces give accelerations at most comparable to a_0.

Next, the rotational movement in a gravitational field in the MOND setting is analysed. For this, a body of mass m on a circular orbit with radius r around a central body of mass M is considered. We take $a \ll a_0$. The central body M acts upon the body m with a gravitational force $F_g = KMm/r^2$. This force moves m with an acceleration a which may be extracted from $F_g = ma^2/a_0$. Next, a is identified with a centripetal acceleration $a = v^2/r$, where v is the orbital velocity of m. It follows that v is given by:

$$v = \sqrt[4]{KMa_0}. \tag{8}$$

Thus, in the MOND hypothesis, the orbital velocity is a constant relative to the orbital radius r. From observations, the value of a_0 was deduced. It was found to be $a_0 \approx 1.2 \cdot 10^{-10} m/s^2$. Applying MOND to study the movement of stars in galaxies is justified by the fact that accelerations at the edges of galaxies are much smaller than a_0.

The function $\mu(a)$ describes the transition from small accelerations to big accelerations. Different forms have been proposed for μ (see e.g. [3]). In the following, we will use the form

$$\mu(a) = \frac{\frac{a}{a_0}}{\sqrt{1 + (\frac{a}{a_0})^2}} = \frac{a}{\sqrt{a^2 + a_0^2}}. \tag{9}$$

Thus, the general expression of the fundamental law becomes:

$$\vec{F} = \frac{ma^2}{\sqrt{a^2 + a_0^2}} \frac{\vec{a}}{a}, \tag{10}$$

or, in one dimension

$$F = \operatorname{sgn}(a) \frac{ma^2}{\sqrt{a^2 + a_0^2}}. \tag{11}$$

2 The Milgrom oscillator

A Milgrom oscillator is a one-dimensional oscillator working under the influence of an elastic force and which obeys the MOND expression of the fundamental law of dynamics. The oscillator describes a periodic motion around a point of equilibrium. We label its mass with m and the distance from the equilibrium point with x. The elastic force acting upon it is $F = -kx$, where k is an elastic constant. The oscillator's acceleration is obtained from (11) with the above force:

$$\frac{ma^2}{\sqrt{a^2 + a_0^2}} \operatorname{sgn}(a) = -kx. \tag{12}$$

We label $\omega_0 = \sqrt{k/m}$. By raising the above equation to square (which elliminates the sign function) and working it out, the following equation is obtained:

$$a^4 - \omega_0^4 x^2 a^2 - \omega_0^4 x^2 a_0^2 = 0. \tag{13}$$

This equation can be solved as a second degree equation in a^2. The positive solution is retained:

$$a^2 = \frac{1}{2} \left(\omega_0^4 x^2 + \sqrt{\omega_0^8 x^4 + 4 a_0^2 \omega_0^4 x^2} \right). \tag{14}$$

From the expression of the elastic force it follows that $\operatorname{sgn}(F) = \operatorname{sgn}(a) = -\operatorname{sgn}(x)$. Then, the following expression of the acceleration is obtained:

$$a = -\operatorname{sgn}(x) \sqrt{\frac{1}{2} \left[(\omega_0^2 x)^2 + \sqrt{(\omega_0^2 x)^4 + 4 a_0^2 (\omega_0^2 x)^2} \right]}. \tag{15}$$

Since $a = \ddot{x}$, the above equation is a second-order differential equation in the position $x(t)$. By a change of variable $\xi = \omega_0^2 x$, this equation takes the equivalent forms:

$$\ddot{\xi} = -\operatorname{sgn}(\xi)\frac{\omega_0^2}{\sqrt{2}}\sqrt{\xi^2+\sqrt{\xi^4+4a_0^2\xi^2}}, \tag{16}$$

$$\ddot{\xi} = -\frac{\omega_0^2\xi}{\sqrt{2}}\sqrt{1+\sqrt{1+4\frac{a_0^2}{\xi^2}}}. \tag{17}$$

For practical purposes, the two functions appearing in (16) and (17) can be expanded in Taylor series. We take a function

$$f(\xi) = \sqrt{\xi^2+\sqrt{\xi^4+4a_0^2\xi^2}}. \tag{18}$$

Then, the acceleration is given by

$$a(x) = -\operatorname{sgn}(x)\frac{1}{\sqrt{2}}f(\xi(x)). \tag{19}$$

We consider two cases. In the first one, we consider $|\xi| \ll a_0$. We make the substitution $z = \frac{\xi}{2a_0}$ in $f(\xi)$, where $|z| \ll 1$. Further, we take a function $\varphi(z) = \sqrt{z^2+\sqrt{z^4+z^2}}$. With the above substitution, it can be seen that $f(\xi) = 2a_0\varphi(z(\xi))$. The Taylor series of $\varphi(z)$ around $z_0 = 0$ has been obtained with the Maxima software:

$$\varphi(z) = \sqrt{|z|}\left(1+\frac{1}{2}|z|+\frac{1}{8}|z|^2-\frac{1}{16}|z|^3-\frac{5}{128}|z|^4+\frac{7}{256}|z|^5+...\right), z \in (-1,1). \tag{20}$$

In the second case, we take $|\xi| \gg a_0$. We make the substitution $z = \frac{2a_0}{\xi}$ in $f(\xi)$; here, $|z| \gg 1$. We take a function $\psi(z) = \sqrt{1+\sqrt{1+z^2}}$. Then, it can be seen that $f(\xi) = \frac{2a_0}{|z(\xi)|}\psi(z(\xi))$. The Taylor series of $\psi(z)$ around $z_0 = 0$ is:

$$\psi(z) = \sqrt{2}\left(1+\frac{1}{8}z^2-\frac{5}{128}z^4+\frac{21}{1024}z^6-\frac{429}{32768}z^8+\frac{2431}{262144}z^{10}-...\right), z \in (-1,1). \tag{21}$$

It can be noted that in the MOND hypothesis the classical results regarding mechanical energy and work no longer apply in a general setting. Yet, in the case of the Milgrom oscillator, an analogy can be built that will allow us to apply classical formulas and derive a potential and a kinetic energy. The analogy is done by establishing an equivalence between the behaviour of the Milgrom oscillator and the behaviour of a body of the same mass m under a corresponding force in the classical setting of Newtonian mechanics [5]. For this, the coefficient μ is eliminated by dividing the force F with it. A new force $F' = \frac{F}{\mu(a)}$ is obtained, which acts upon the mass m according to the Newtonian law of dynamics $F' = ma$. Since $a = a(x)$ according to (15), the force F' is a function of x. Moreover, since $\operatorname{sgn}(F') = -\operatorname{sgn}(x)$, the body describes a periodic trajectory $x = x(t)$. Next, the potential energy of mass m can be determined from $dU = -F'(x)dx = -\frac{F'(\xi)}{\omega_0^2}d\xi$. The force F' is simply $F' = ma(x)$, with a given by (15). Thus,

$$U(\xi) = \int \operatorname{sgn}(\xi)\frac{1}{\sqrt{2}}\frac{m}{\omega_0^2}f(\xi)d\xi + C, \tag{22}$$

where C is a constant determined by the value of U in a chosen point of reference. This integral can be computed. A function $g(\xi) = \sqrt{-\xi^2 + \sqrt{\xi^4 + 4a_0^2\xi^2}}$ is introduced. Next, f is written as

$$f(\xi) = \sqrt{\frac{\xi^2 + 2ia_0\xi}{2}} + \sqrt{\frac{\xi^2 - 2ia_0\xi}{2}}. \tag{23}$$

Integration was carried out with Maxima. The primitive of f is given through the function:

$$V(\xi) = 2\left(|\xi|f(\xi) - a_0 g(\xi)\right) + \sqrt{2}a_0^2 \ln\left[4\sqrt{2}\left(|\xi|f(\xi) + a_0 g(\xi)\right) + 4\left(f^2(\xi) + a_0^2\right)\right]. \tag{24}$$

The potential energy is

$$U(\xi) = \frac{m}{4\sqrt{2}\omega_0^2}V(\xi) + C. \tag{25}$$

The corresponding kinetic energy can also be found. The kinetic energy is by definition $T = \frac{mv^2}{2}$, where v is the velocity of mass m. We will use the equation $\ddot{x} = a(x)$, where $a(x)$ is given by (19). Since $v = \dot{x}$, this equation can be written as $\dot{v} = a(x)$. Next, $\dot{v}$ can be expressed as

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{v^2}{2}\right). \tag{26}$$

Thus,

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) = a(x) = -\operatorname{sgn}(x)\frac{1}{\sqrt{2}}f(\xi(x)). \tag{27}$$

It follows that the kinetic energy is given by

$$T(\xi) = -\frac{m}{4\sqrt{2}\omega_0^2}V(\xi) + K, \tag{28}$$

where K is a constant determined by initial conditions. It should be noted that $T(x) = -U(x) + C + K$, whence $T(x) + U(x) = C + K = const.$, which is precisely the law of conservation of the total energy. Also, it should be noted that $V(0) = 4a_0^2\ln(4a_0^2) \neq 0$. Thus, the constants of integration of T and U also contain a term depending on a_0. T and U determined above correspond to the classical analogue of the Milgrom oscillator, not to its original MOND version.

3 MOND regimes

As the Milgrom oscillator moves from the point of maximum amplitude towards the equilibrium point, its acceleration decreases from a maximum value towards zero and it passes through a series of regimes. The most important are presented below. The corresponding approximations for the acceleration a and the potential energy U are derived. It should be noted that these approximations depend very much on the choice of the function $\mu(a)$. Thus, the implications of the approximations below are mostly qualitative in nature.

1. Classical regime

   If $|\xi| \gg a_0$, we will use the first term in the Taylor expansion of $\psi(x)$. Consequently, the acceleration takes the classical oscillator form:

   $$a = -\xi = -\omega_0^2 x. \tag{29}$$

   We take $|a| \gg a_0$; the factor $\mu(a) = 1$. The potential energy takes the classical form:

   $$U(\xi) = \frac{m}{2\omega_0^2}\xi^2 + C = \frac{m\omega_0^2 x^2}{2} + C. \tag{30}$$

   This regime describes the oscillator's evolution when it is far from the equilibrium point.

2. Perturbed classical regime

   If $|\xi| \gg a_0$ and by taking the first two terms in the Taylor series (21) the following equation is obtained:

   $$a = -\xi\left(1 + \frac{1}{2}\frac{a_0^2}{\xi^2}\right) = -\xi - \frac{1}{2}\frac{a_0^2}{\xi}. \tag{31}$$

   The corresponding potential energy is:

   $$U(x) = \frac{m}{2\omega_0^2}\xi^2 + \frac{ma_0^2}{2\omega_0^2}\ln|\xi| + C. \tag{32}$$

   This regime describes the oscillator's evolution when the MOND effects begin to appear. The second term in the expression of U is a perturbation term with relation to expression (30) of the classical regime.

3. Deep MOND regime

   If $|\xi| \ll a_0$, we use (20) and obtain:

   $$a = -\operatorname{sgn}(x)\sqrt{a_0|\xi|} = -\operatorname{sgn}(x)\omega_0\sqrt{a_0|x|}. \tag{33}$$

   The potential energy can be computed separately for $x > 0$ and $x < 0$. It is found to take the form:

   $$U(x) = \frac{2m}{3\omega_0^2}\sqrt{a_0|\xi|^3} + C. \tag{34}$$

   This regime describes the oscillator's evolution close to the equilibrium point at $x = 0$, where the MOND effects are strongest.

Perturbed forms of this regime can be found. If we take the first two terms in the expansion (20), the formulas (33) and (34) become:

$$a = -\operatorname{sgn}(x)\sqrt{a_0|\xi|}\left(1+\frac{|\xi|}{4a_0}\right), \tag{35}$$

$$U(x) = \frac{2m}{3\omega_0^2}\sqrt{a_0|\xi|^3}+\frac{m}{10\omega_0^2}\sqrt{\frac{|\xi|^5}{a_0}}+C. \tag{36}$$

4 Periodic behaviour of the Milgrom oscillator

The behaviour of the Milgrom oscillator was analysed numerically. For this, equation (16) was integrated with the method of finite differences.

The Milgrom oscillator describes an orbit in the phase space (x, v). The shape of the orbit depends on the influence that the acceleration a has on the oscillator's movement along its path. In figure 1 phase space orbits have been represented for different values of a_0. The acceleration as a function of x has also been represented. For greater values of a_0 the oscillator stays longer in the MOND regime. The classical case can be observed for $a_0 \ll 1$. We took $\omega_0 = 1$ and the initial conditions $x(0) = 0$ and $v(0) = 1$.

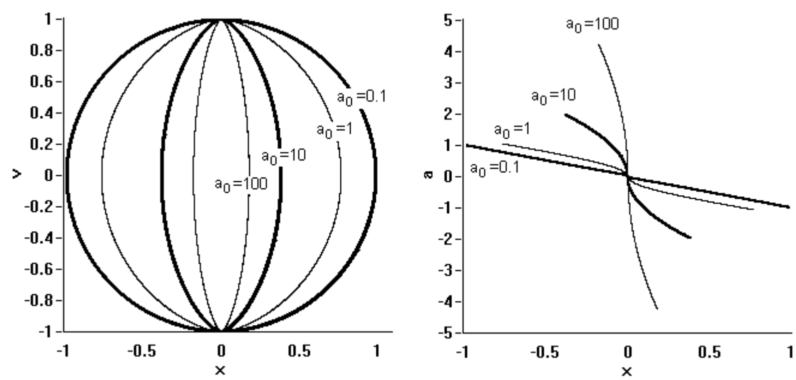

Figure 1: (Left) Velocity v as function of position x describes the phase space orbit of the Milgrom oscillator at different values of a_0. The case $a_0 = 0.1$ describes accurately the classical regime, while $a_0 = 10$ and $a_0 = 100$ are in the deep MOND regime. (Right) The acceleration a as a function of x gives another description of the oscillator's regimes.

It can be seen that MOND influences the movement of the oscillator in the sense of reducing its maximum amplitude for the same initial velocity. This is due to the fact that MOND increases the oscillator's acceleration with respect to the classical case. Since $\operatorname{sgn}(a) = -\operatorname{sgn}(x)$, a acts as a deceleration on the oscillator, keeping it closer to the equilibrium point with respect to the classical case.

The oscillator's movement has a characteristic period T and an angular velocity ω. Generally, $\omega \neq \omega_0$. From the oscillator's movement, its angular velocity can be computed.

We may try to analytically compute the angular velocity in the following way: we take $A = x_{max}$ as the maximum amplitude of the oscillator. Next, we equate the acceleration

$a(x)$ with its classical form $\omega^2 x$ in $x = A$. From the resulting equality, the angular velocity can be computed as $\omega = \sqrt{a(A)/A}$. For the general form (19), this yields:

$$\omega = \sqrt{\frac{f(\omega_0^2 A)}{A\sqrt{2}}}. \tag{37}$$

In the MOND regime this becomes:

$$\omega = \omega_0 \sqrt{\frac{a_0}{A}}. \tag{38}$$

Paper [4] cites a method in [1] which yields in the MOND regime:

$$\omega = 1.051\omega_0 \sqrt{\frac{a_0}{A}}. \tag{39}$$

The last result is more precise that the first, as it can be seen from numerical simulations. Nevertheless, as the oscillator spends less time in the MOND regime, the first formula becomes a better approximation for ω.

In figure 2 a graph of ω as a function of the maximum amplitude A is presented. The values were obtained from numerical simulations. Both axes have a logarithmic scale. Here we took $a_0 = 1$, $\omega_0 = 1$ and $x(0) = 0$. The value of A was controlled through $v(0)$.

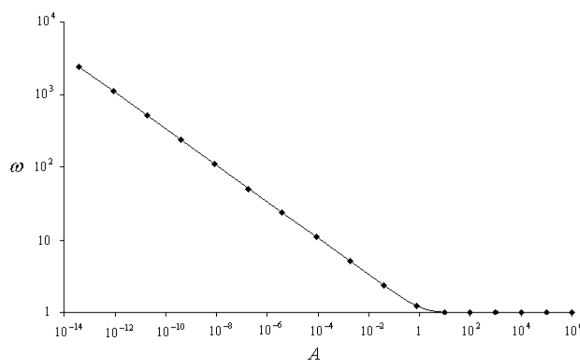

Figure 2: The angular velocity ω as a function of the maximum amplitude A for fixed a_0. The left portion corresponds to the MOND regime, while the right one corresponds to the classical regime. The scales are logarithmic, so the linear behaviour at left actually represents a power law.

If the oscillator stays mostly in the deep MOND regime, formula (39) is obeyed very well. If it stays mostly in the classical regime, then $\omega \approx \omega_0$ as expected. The transition from one situation to the other takes place from $A \approx 1$ to $A \approx 10$. Thus, with respect to the orders of magnitude of A, the oscillator is either in the MOND regime (slow oscillator) or in the classical regime (fast oscillator).

The nonharmonic character of the Milgrom oscillator can be seen from the Fourier transform $\mathcal{F}$ of its position. In figure 3 $|\mathcal{F}|$ is presented as a function of ω for different values of a_0. It can be seen that for $a_0 = 0$ (no MOND regime) the oscillator is harmonic, as it reduces to the classical oscillator. As a_0 increases, the MOND regime becomes more dominant and other harmonics besides the fundamental appear.

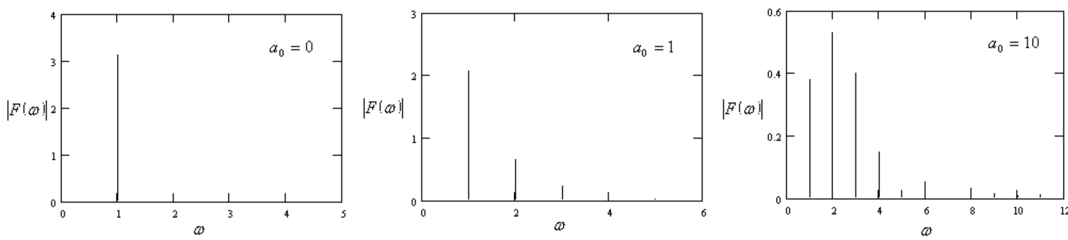

Figure 3: The absolute value of the Fourier transform of $x(t)$ as a function of ω.

5 Testing the MOND hypothesis

The behaviour of the Milgrom oscillator can be used to test the MOND hypothesis. As it follows from (33), an oscillator with a small ω_0 must be used in order to reduce its acceleration. This can be accomplished by increasing its mass m. For example, with a maximum amplitude $A = 1cm$, the maximum acceleration is $a = a_0$ if $\omega_0 \approx 10^{-4}s$, according to (33). This would yield, through (39), an angular velocity of $\omega \approx 1.3 \cdot 10^{-8}rad/s$ or, equivalently, a period of oscillation $T \approx 5 \cdot 10^8 s$ or almost 16 years, which is not very practical. Further, if we consider the oscillator to be a mass hung by a spring with elastic constant $k = 1$, its mass would have to be $m \approx 6 \cdot 10^{15} kg$, or about the mass of a cube of water with side length of around $18km$. These computations have been carried out to show how well hidden the MOND regime is from an experimental point of view. This does not mean it is impossible to be reached. With a better choice of parameters, a practical Milgrom oscillator could be built.

If the MOND hypothesis holds in any system of reference (which is doubtful), this test can be carried out anywhere on the Earth's surface. However, if MOND applies only in inertial systems of reference, the test will not work for most of the time no matter where it is done. Indeed, the Earth's surface is not an inertial system of reference. Moreover, the accelerations acting at the surface of the Earth are much stronger than a_0 (see [7]). On one side, there is the gravitational acceleration of the Earth $g = 9.8m/s^2$. For a horizontal oscillator it doesn't pose a problem, since $\vec{g}$ is always vertical. More troubling may be horizontal gravitational accelerations that are generated by bodies on the Earth's surface. For example, a mountain $1km$ high and with a base $2km$ wide at a distancce of $1km$ exerts an acceleration of about $2 \cdot 10^{-4}m/s^2$. Next, the Earth's centrifugal acceleration has a maximum value of $a_{cf} \approx 3.4 \cdot 10^{-2}m/s^2$. Its orientation is parallel to the equatorial plane. In conjunction with g, it has a detrimental effect on a horizontal oscillator: only one of the two accelerations can be canceled out. The only place where both can be canceled is on the Equator. The Moon exerts a gravitational acceleration on the Earth of about $3 \cdot 10^{-5}m/s^2$, while the Sun exerts an acceleration of about $6 \cdot 10^{-3}m/s^2$. Both of these accelerations have a variable direction in time, which make them hard to control. To give a sense of the influence of these accelerations, we mention that, in order for the Sun to act with a gravitational acceleration equal to a_0, it would have to be located at about 0.1 light-years away from the Earth.

One way to circumvent these problems is to place the oscillator in outer space in a region with very small gravitational accelerations. This is hard to accomplish (it would have to be at least 0.1 light-years away from the Sun). Another solution is to wait fo a moment when the perturbing accelerations acting upon the oscillator cancel each other out. Then, for a short time, the oscillator may go through a MOND regime. This is incompatible with the goal of having a small ω, as it would lead to big periods of oscillation. Thus, the oscillator will not describe a complete oscillation in the given time and it will have to be observed as a simple dynamical body in movement under the MOND constraints. Yet, one can hope that further refinements of these two possibilities may bring a practical test for the MOND hypothesis.

References

[1] Weaver, W., Timoshenko, S. P., Young, D. H., *Vibration Problems in Engineering*, Wiley-Interscience, New York, 1990.

[2] Milgrom, M., *Dynamics with a non-standard inertia-acceleration relation: an alternative to dark matter in galactic systems*, e-print: arXiv:astro-ph/9303012v1, 1993.

[3] Zhao, H. S., Famaey, B., *Refining the MOND interpolating function and TeVeS Lagrangian*, The Astrophysical Journal, 638:L9L12, 2006.

[4] Irvine, T., *The Mathematics of a Milgrom Oscillator*, Vibrationdata Newsletter, e-print: www.vibrationdata.com/Newsletters/July2006_NL.pdf, July 2006.

[5] Shariati, A., Jafari, N., *MOND, dark matter, and conservation of energy*, e-print: arXiv:0710.1411v1, 2007.

[6] Milgrom, M., *The MOND paradigm*, e-print: arXiv:0802.3133v2, 2008.

[7] Ignatiev, A. Yu., *Newtons second law versus modified-inertia MOND: a test using the high-latitude effect*, e-print: arXiv:0802.1599v1, 2008.

[8] Amit, R., *A possible solution to the Dark Matter, Dark Energy, and Pioneer Anomaly problems via a VSL approach*, e-print: arxiv:/0802.2123, 2008.